Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 8 (2026) 3611

Journal of the Nigerian Society of Physical Sciences

Kaluza–Klein cosmological model with Tsallis holographic dark energy in $f(R,T)$ gravity

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Abstract

In this study, we analyze a cosmological model based on Tsallis Holographic Dark Energy (THDE) in Kaluza–Klein (KK) spacetime within $f(R,T)$ gravity. Exact solutions to the field equations are derived, and the evolution of the deceleration parameter, energy density, pressure, and equation-of-state (EoS) parameter as functions of redshift is investigated. The model exhibits continuous accelerated expansion, characterized by a negative deceleration parameter throughout the considered redshift range. Here, λ is the constant associated with the adopted deceleration-parameter parametrization and controls the evolution of the Hubble and deceleration parameters, thereby determining the expansion dynamics of the model. The statefinder analysis shows that the model evolves toward the Lambda cold dark matter (ΛCDM) fixed point and approaches it asymptotically in the future. Furthermore, the null, weak, and dominant energy conditions (NEC, WEC, and DEC) are satisfied, whereas the strong energy condition (SEC) is violated, consistent with the accelerated expansion of the Universe. The stability of the model is assessed independently using the effective squared speed of sound, which indicates classical instability for the considered parameter set. Overall, the model provides an analytical framework for investigating late-time cosmic acceleration in $f(R,T)$ gravity, while its perturbative and observational viability requires further investigation.

DOI: 10.46481/jnsps.2026.3611

Keywords: Kaluza–Klein metric, Tsallis holographic dark energy, Cosmology, Dark energy, $f(R,T)$ gravity

Article History:
Received: 15 June 2026
Received in revised form: 13 August 2026
Accepted for publication: 14 August 2026
Available online: 17 September 2026

Communicated by: B. J. Falaye

1. Introduction

Observations over the past few decades, particularly measurements of Type Ia supernovae, the cosmic microwave background (CMB), and baryon acoustic oscillations (BAO), have firmly established that the Universe is undergoing accelerated expansion. This unexpected behaviour has led to the conclusion that a dominant portion of the cosmic energy budget, roughly 70%, is made up of dark energy, an exotic component characterised by negative pressure, as demonstrated by Riess *et al.* [1], Perlmutter *et al.* [2], Aghanim *et al.* [3], and Eisenstein *et al.* [4]. While the standard Lambda cold dark matter (ΛCDM) framework successfully reproduces a wide range of observational data, it is not free from conceptual difficulties. In particular, unresolved issues such as the fine-tuning problem and the cosmic coincidence problem have prompted researchers to explore alternative theoretical models that can provide a more satisfactory explanation of cosmic acceleration, as discussed by Weinberg [5], 't Hooft [6], and Susskind [7].

The holographic principle, originally introduced by Gerard 't Hooft [6] and later expanded by Leonard Susskind [7], suggests that the fundamental information describing a physical system is determined by its boundary surface, rather than

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the volume it occupies. Based on this concept, HDE models were introduced by Cohen *et al.* [8]. However, standard HDE relies on classical entropy, which may not be valid at quantum scales. To address this limitation, Tsallis entropy provides a generalised non-extensive framework that leads to THDE, which has been shown to successfully describe complex cosmic behaviour, including stable late-time acceleration and dynamical dark energy evolution by Tsallis [9], Tavayef *et al.* [10], and Barrow [11].

In parallel, HDE models based on generalised entropy formalisms have attracted considerable attention. Among these, THDE provides a natural extension of the standard holographic framework by incorporating non-extensive entropy effects. Recent investigations have demonstrated that THDE models can describe a wide range of cosmological behaviours, including quintessence and phantom regimes, depending on the choice of model parameters. Recent observational studies of THDE using datasets such as Pantheon+ by Brout *et al.* [12], BAO by the DESI Collaboration [13], and Planck data by the Planck Collaboration [3] have investigated its ability to reproduce the observed expansion history and constrain its model parameters. In addition, extensions of THDE within modified gravity frameworks have further improved agreement with cosmological observations by Satyanarayana *et al.* [14], Li *et al.* [15] and Imdiker *et al.* [16] .

The framework of $f(R,T)$ gravity has been considerably developed in recent years, and a broad range of cosmological models beyond the conventional formulation have been obtained. Several subsequent studies have investigated more generalised forms such as $f(R,T,L_m)$, $f(R,\Sigma,T)$, and metric-affine extensions, which give a more detailed understanding of matter–geometry coupling. These models introduce additional degrees of freedom and allow for richer cosmological dynamics, including variable gravitational coupling and non-conservation of the energy–momentum tensor (EMT). Such advancements have significantly improved the capability of $f(R,T)$ gravity to interpret both early-time inflation and late-time cosmic acceleration within a unified theoretical framework.

Modifying the theory of gravity itself is another line of action. In this context, models such as $f(R,T)$ gravity are extensions of general relativity (GR) that incorporate a direct connection between matter and spacetime geometry as proposed by Harko *et al.* [17]. One of the most important consequences of $f(R,T)$ gravity is that the rapid expansion of the Universe can be explained through the additional dynamical effects generated by the matter–geometry interaction, without necessarily invoking exotic or unknown forms of energy.

Moreover, multidimensional theories such as KK spacetime proposed by Overduin and Wesson [18], introduce additional spatial dimensions and thus provide broader modelling scope for cosmological theories. They enable researchers to explore the evolution of the Universe at both the earlier and later stages of its existence. Despite the vast literature on the topic of $f(R,T)$ gravity and holographic dark energy theories, there are very few papers that have explored the joint effect of these models in the cosmology of multidimensional and anisotropic spaces. To the best of my knowledge, only one study considers the role of THDE in the multidimensional space within $f(R,T)$ theory of gravity, and this research area is relatively unexplored yet as discussed by Trivedi *et al.* [19] and Mahanta and Das [20].

The aim of the current study is to examine a THDE model that is employed along with the $f(R,T)$ gravity in a Kaluza-Klein cosmological scenario. Exact solutions of the modified field equations are derived and analysed in terms of important cosmological variables. It is observed that the Universe is expanding with acceleration, which is expected for the dark-energy-dominated cosmology. Here, λ is the constant associated with the adopted deceleration-parameter parametrization and controls the evolution of the expansion dynamics in the interaction process between dark energy and matter, hence determining the dynamics of the model. Further, the statefinder analysis shows that the model converges to the ΛCDM model in the present age but can deviate in the future as reported by Reddy and Santhi Kumar [21], Santhi Kumar and Satyanarayana [22], Krishna *et al.* [23], and Ramanamurty *et al.* [24].

The NEC, WEC, and DEC are satisfied, whereas the SEC is violated, while the squared sound-speed analysis indicates classical instability for the considered parameter set. All things considered, the model proves to be a viable option for an alternative to the traditional cosmological constant paradigm. It should be highlighted that the present study is entirely theoretical and aims to derive analytical solutions under the assumption of $f(R,T)$ gravity theory. From the analytical nature of the solution obtained, useful information can be derived regarding matter–geometry coupling and higher dimensions in cosmology.

The remainder of the article is organized as follows. In Section 2, the Kaluza-Klein metric is considered, and the field equations for the proposed metric in $f(R,T)$ gravity are found. The third part is dedicated to obtaining analytical solutions to the obtained field equations under certain physical assumptions and studying the physical characteristics of the resulting cosmological model, taking into account restrictions on energy, stability, and statefinder. In Section 4, the behaviour of parameters from the derived model is compared with parameters of the ΛCDM cosmological model. Section 5 is devoted to the physical interpretation of the obtained results. Finally, Section 6 summarizes the results.

2. Mathematical formulation of $f(R,T)$ gravitational theory

This section provides a concise overview of the theoretical framework of $f(R,T)$ gravity and derives corresponding field equations for the current field of study. This theory generalises Einstein's General Relativity by extending the gravitational Lagrangian to incorporate an arbitrary function of the Ricci scalar R and the trace of the energy–momentum tensor T. This change establishes a direct connection between matter and geometry, resulting in more complex dynamical behaviour than conventional gravitational theories.

The $f(R,T)$ theory was presented as a generalisation of $f(R)$ gravity by Harko *et al.* [17], with the aim of incorporating matter–geometry interaction at the level of the action. One of the most important consequences of such a coupling is the lack of conservation of the energy–momentum tensor, which can be regarded as energy and momentum exchange between matter and the gravitational force. Such a phenomenon may arise from quantum effects or a more fundamental interaction between the matter and spacetime geometry.

The action for *f (R,T)* gravity is given by

$$S = \int \left[\frac{1}{16\pi} f(R,T) + \mathcal{L}_m \right] \sqrt{-g}\; d^5x. \tag{1}$$

Since the present work is formulated in a five-dimensional Kaluza–Klein spacetime, the action is correspondingly written over the five-dimensional manifold using d^5x.

In this expression, $\mathcal{L}_m$ corresponds to the Lagrangian density associated with the matter content, whereas g denotes the determinant of the spacetime metric tensor $g_{\mu\nu}$.

In this manner, the $f(R,T)$ gravity provides a natural and physically acceptable extension of GR, offering an alternative explanation for cosmic acceleration through matter–geometry coupling rather than invoking unknown dark energy components. f(R,T) permits minimum coupling along with $\mathcal{L}_m$. In this case, T is the trace, and R is the Ricci scalar.

The matter EMT is defined by

$$\begin{aligned} T_{\mu\vartheta} &= -\frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}L_m)}{\partial g^{\mu\vartheta}}, \\ \Theta_{\mu\vartheta} &= -2T_{\mu\vartheta} - p g_{\mu\vartheta}, \end{aligned} \tag{2}$$

where p represents the pressure of the cosmic fluid.

In the present work, the matter Lagrangian is chosen as $L_m = -p$, following the standard formulation of $f(R,T)$ gravity. With this choice, the tensor $\Theta_{\mu\nu}$ is obtained accordingly and the corresponding field equations are derived consistently.

The field equations of $f(R,T)$ gravity are obtained by varying the modified action in the form of Eq. (1), in terms of the metric $g_{\mu\nu}$. This derivation is done in units of gravity, with the constants normalised ($8\pi G = c = 1$).

$$\begin{aligned} f_R R_{\mu\vartheta} - \frac{1}{2} f g_{\mu\vartheta} + \left(g_{\mu\vartheta}\nabla^\kappa\nabla_\kappa - \nabla_\mu\nabla_\vartheta\right) f_R \\ = 8\pi T_{\mu\vartheta} - f_T T_{\mu\vartheta} - f_T \Theta_{\mu\vartheta}, \end{aligned} \tag{3}$$

where $f_R = \frac{\partial f(R,T)}{\partial R}$, $f_T = \frac{\partial f(R,T)}{\partial T}$ & $\Theta_{\mu\vartheta} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\vartheta}}$.

The notation here is $T_{\mu\vartheta}$ ordinary matter EMT, which is obtained from L_m which is the Lagrangian, and ∇_μ is the covariant derivative. For the present work, we use the following functional form $f(R,\ T) = R + 2f(T)$, where $f(T) = \gamma T$. Substituting $\Theta_{\mu\vartheta} = -2T_{\mu\vartheta} - p g_{\mu\vartheta}$ into Eq. (3), together with the choice $f(R,T) = R + 2\gamma T$, yields the reduced field equations given in Eq. (4).

$$R_{\mu\vartheta} - \frac{1}{2} R\, g_{\mu\vartheta} = \left(8\pi + 2f'(T)\right) T_{\mu\vartheta} + \left(2p\, f'(T) + f(T)\right)\, g_{\mu\vartheta}, \tag{4}$$

where $T_{\mu\nu}$ is the EMT of a perfect fluid.

This simplified form of the field equations facilitates the derivation of exact analytical solutions in the subsequent sections and highlights the role of matter–geometry coupling in modifying the gravitational dynamics.

3. Kaluza–Klein cosmological model

In this section, we construct the cosmological model within the KK framework and derive accurate solutions to the modified field equations. The higher-dimensional metric provides a natural extension of the standard four-dimensional cosmological model and allows a richer description of the Universe's dynamical evolution. By imposing suitable physical assumptions, we obtain explicit expressions for the scale factors and analyse the behaviour of key cosmological parameters.

3.1. Metric and field equations

We now adopt a five-dimensional Kaluza–Klein framework, in which the spacetime geometry is characterised by the following line element:

$$ds^2 = dt^2 - l^2(t)\left(dx^2 + dy^2 + dz^2\right) - m^2(t) d\psi^2 \tag{5}$$

The EMT for a perfect fluid can be written in the following form.

$$T_{\mu\vartheta} = (\rho + p)\, u_\mu u_\vartheta - p\, g_{\mu\vartheta}. \tag{6}$$

Since, $T_1^1 = T_2^2 = T_3^3 = T_5^5 = -p$ and $T_4^4 = \rho$.
Therefore,

$$T = \rho - 4p. \tag{7}$$

Substituting $f(T)$ and $f'(T)$ into Eq. (4) yields

$$R_{\mu\vartheta} - \frac{1}{2} R\, g_{\mu\vartheta} = (8\pi + 2\gamma) T_{\mu\vartheta} + (2\gamma p + \gamma T)\, g_{\mu\vartheta}. \tag{8}$$

Here, p corresponds to the pressure of the perfect fluid, while T denotes the trace of the matter stress–energy tensor.

By substituting Eqs. (5), (6), and (8), the resulting system of nonlinear and homogeneous field equations can be obtained as follows.

$$\left(\frac{2\ddot{l}}{l} + \frac{\ddot{m}}{m} + \frac{2\dot{l}\dot{m}}{lm} + \frac{\dot{l}^2}{l^2}\right) = (8\pi + 4\gamma)p - \gamma\rho, \tag{9}$$

$$\left(\frac{3\ddot{l}}{l} + \frac{3\dot{l}^2}{l^2}\right) = (8\pi + 4\gamma)p - \gamma\rho, \tag{10}$$

$$\left(\frac{3\dot{l}^2}{l^2} + \frac{3\dot{l}\dot{m}}{lm}\right) = -(8\pi + 3\gamma)\rho + 2p\gamma. \tag{11}$$

From Eq. (8), using the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, we obtain

$$(8\pi + 2\gamma)\nabla^\mu T_{\mu\nu} + \gamma g_{\mu\nu}\nabla^\mu(2p + T) = 0. \tag{12}$$

Therefore, $\nabla^\mu T_{\mu\nu} = -\frac{\gamma}{8\pi+2\gamma} g_{\mu\nu}\nabla^\mu(2p + T)$.

This relation shows that, in general, $\nabla^\mu T_{\mu\nu} \neq 0$, which arises due to the explicit dependence of the gravitational action on the trace T of the energy–momentum tensor. Thus, $f(R,T)$ gravity allows an effective exchange of energy between the matter sector and the gravitational field.

However, in the present work, we restrict our analysis to the non-interacting THDE scenario by assuming that there is no direct energy transfer between the matter and dark energy components, i.e., $Q = 0$. It should be emphasized that $Q = 0$ only excludes direct energy transfer between the matter and THDE sectors and does not, by itself, eliminate the matter–geometry exchange permitted by $f(R,T)$ gravity. In addition to $Q = 0$, we adopt the separate conservation of the matter and THDE components as an explicit phenomenological assumption for the cosmological branch considered here. Thus, the individual components satisfy $\dot{\rho}_m + 4H\rho_m = 0$, and $\dot{\rho}_T + 4H(\rho_T + p_T) = 0$. Adding the above equations gives the total conservation equation,

$$\dot{\rho}_m + \dot{\rho}_T + 4H(\rho_m + \rho_T + p_T) = 0. \tag{13}$$

Accordingly, the relation obtained $\rho_m = k_1(a(t))^{-4}$used subsequently follows from the explicitly imposed matter-conservation condition and should not be interpreted as a consequence of $Q = 0$ alone.

3.2. Exact solutions

By solving the nonlinear gravitational field equations under physically motivated assumptions, we construct anisotropic cosmological models driven by THDE. To obtain a deterministic solution of the system, we impose the following physically reasonable condition.

We assume a proportionality relation between the expansion scalar θ and the shear scalar σ. This assumption introduces a constraint among the directional scale factors, leading to a power-law relation.

$$l = m^n, \ n > 0, \tag{14}$$

where the constant n quantifies the degree of anisotropy of the spacetime. This condition, first proposed by Collins *et al.* [25], ensures that the ratio $\frac{\sigma}{\theta}$ remains constant during cosmic evolution. Such an assumption is frequently employed in spatially homogeneous, anisotropic cosmological models, as it yields mathematically consistent solutions while preserving physically realistic anisotropic behaviour.

The deceleration parameter q is assumed to vary linearly with the Hubble parameter H, and is expressed as follows.

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \alpha + \lambda H, \tag{15}$$

where α and λ are arbitrary constants.

Here, λ is an arbitrary constant introduced through the adopted deceleration-parameter parametrization. It controls the time evolution of the Hubble parameter and the deceleration parameter.

The above parametrization is adopted as a mathematical ansatz to obtain exact analytical solutions of the modified field equations. Since the Hubble parameter evolves with cosmic time, this choice naturally gives a time-dependent deceleration parameter and provides a simple framework for studying the late-time dynamics of the Universe. Moreover, this parametrization establishes a direct relationship between the expansion rate and the deceleration parameter, allowing the cosmological dynamics to be expressed in a simple analytical form. This considerably simplifies the integration of the modified field equations while preserving the essential features required to investigate late-time cosmic acceleration.

Similar variable deceleration parameter parametrizations have been adopted in several cosmological models to obtain exact analytical solutions and to study the evolution of the Universe (see, for example, Pradhan *et al.* [26], Bolotin *et al.* [27], Koussour *et al.* [28], and Tiwari *et al.* [29]).

For $\alpha = -1$, we get a solution of the above equation, which is

$$a(t) = \exp\left(\frac{1}{\lambda}\sqrt{2\lambda t + k}\right), \tag{16}$$

where k is an integration constant. For $\alpha = -1$, the adopted parametrization becomes $q = -1 + \lambda H$. Since the Hubble parameter H evolves with cosmic time, the deceleration parameter also remains time-dependent. Thus, the choice $\alpha = -1$ does not correspond to a constant deceleration parameter but describes a dynamically evolving cosmological model. The present work is primarily concerned with cosmological models possessing a time-dependent deceleration parameter, as such models are more suitable for investigating the late-time accelerated expansion of the Universe and the influence of matter–geometry coupling within the framework of $f(R,T)$ gravity. Accordingly, the adopted parametrization is used to obtain exact analytical solutions and to study the cosmological evolution of the proposed model.

Solving Eq. (15) yields the metric potentials

$$m = \exp\left(\frac{4}{\lambda(3n+1)}\sqrt{2\lambda t + k}\right), \tag{17}$$

$$l = m^n = \exp\left(\frac{4n}{\lambda(3n+1)}\sqrt{2\lambda t + k}\right), \tag{18}$$

With this setup, the metric given in Eq. (5) can be reformulated in the following form:

$$\begin{aligned} ds^2 = dt^2 & \\ & - \exp\left[\frac{8n}{\lambda(3n+1)}\sqrt{2\lambda t + k}\right](dx^2 + dy^2 + dz^2) \\ & - \exp\left[\frac{8}{\lambda(3n+1)}\sqrt{2\lambda t + k}\right]d\psi^2. \end{aligned} \tag{19}$$

The metric given by Eq. (19) represents a five-dimensional Kaluza–Klein cosmological model characterised by anisotropic expansion between the three-dimensional spatial section and the extra spatial dimension. The exponential dependence on $\sqrt{2\lambda t + k}$ leads to a nonlinear evolution of the scale factors, governed by the parameters n and λ.

It is evident that the scale factors corresponding to the three spatial dimensions evolve proportionally, ensuring isotropy within the observable spatial section, whereas the extra-dimensional scale factor evolves differently, reflecting the higher-dimensional anisotropic nature of the model.

The parameter λ enters the metric functions through the adopted deceleration-parameter parametrization and consequently controls the temporal evolution of the scale factors and the analysis highlights the influence of matter–geometry coupling, expansion rate. The matter–geometry interaction in the f(R,T) framework, on the other hand, is governed by the coupling parameter γ. Thus, the effects associated with λ should be interpreted as consequences of the assumed expansion parametrization rather than as direct effects of matter–geometry coupling.

Therefore, the obtained solution provides a consistently higher-dimensional cosmological model, which serves as a suitable foundation for analysing the dynamical behaviour of key physical parameters in the subsequent sections.

3.2.1. Redshift

To study the cosmological evolution in terms of the observable redshift, we use the standard relation $1+z=\frac{a_0}{a(t)}$, where $a_0 = a(t_0) = \exp\left(\frac{1}{\lambda}\sqrt{2\lambda t_0 + k}\right)$, where $t_0 = 13.8$ Gyr denotes the present cosmic time and $a(t_0)$ is the present value of the scale factor .Then $1+z=\frac{a(t_0)}{a(t)}$. Where the scale factor is $a(t)=\exp\left(\frac{1}{\lambda}\sqrt{2\lambda t+k}\right)$. The redshift becomes

$$1+z=\frac{\exp\left(\frac{1}{\lambda}\sqrt{2\lambda t_0+k}\right)}{\exp\left(\frac{1}{\lambda}\sqrt{2\lambda t+k}\right)}.$$

Consequently, cosmic time as a function of redshift is

$$t(z)=\frac{\left[\sqrt{2\lambda t_0+k}-\lambda ln(1+z)\right]^2-k}{2\lambda}.$$

At the present epoch $z=0$, this relation gives ($t=t_0$), ensuring that the redshift transformation is regular at present. The positive branch of the square root is retained throughout the transformation.

In the numerical analysis, the cosmological quantities are first evaluated as functions of cosmic time ‘t’, and the corresponding redshift is obtained from $z=\frac{a(t_0)}{a(t)}-1$. Accordingly, the redshift coordinates in Figures 1–10 are obtained directly from the scale-factor ratio $\frac{a(t_0)}{a(t)}$, rather than by substituting the analytical $t(z)$ relation into the corresponding expressions. The graphical analysis is carried out using the parameter values $k=0.05, \delta=1.2, \beta=1, n=2, \gamma=0.5, k_1=1$, while the parameter λ is varied as $\lambda=1.2,\ 1.4,\ 1.6$, to study its influence on the cosmological evolution.

3.3. Cosmological parameters

The key cosmological parameters are derived from the obtained solutions. The volume of space is given by

$$V=l^3m=\exp\left(\frac{4}{\lambda}\sqrt{2\lambda t+k}\right). \tag{20}$$

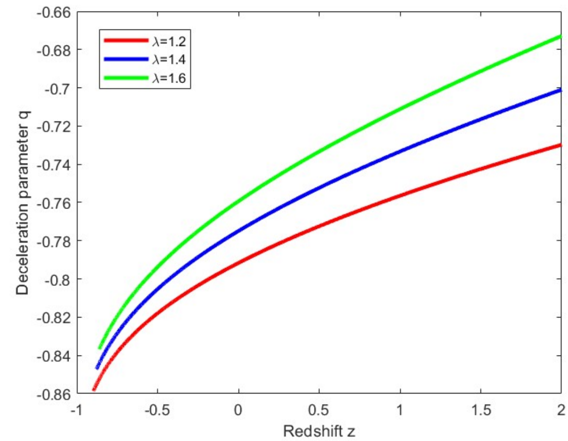

Figure 1: Evolution of the deceleration parameter q with redshift z.

The average scale factor is

$$a(t)=\exp\left(\frac{1}{\lambda}\sqrt{2\lambda t+k}\right). \tag{21}$$

The mean Hubble parameter

$$H(t)=\frac{1}{\sqrt{2\lambda t+k}}. \tag{22}$$

The expansion scalar is

$$\Theta=4H=\frac{4}{\sqrt{2\lambda t+k}}. \tag{23}$$

The shear scalar can be expressed as follows:

$$\sigma^2=\frac{6(n-1)^2}{(3n+1)^2}\left(\frac{1}{2\lambda t+k}\right). \tag{24}$$

The anisotropic parameter A_m is given by

$$A_m=\frac{4(n-1)^2}{(3n+1)^2}. \tag{25}$$

Eq. (25) shows that the anisotropy parameter A_m depends only on the constant parameter n and is therefore independent of cosmic time. Hence, the degree of anisotropy remains constant throughout the cosmic evolution for fixed n. The model becomes isotropic only for the special case $n=1$, for which $A_m=0$. Therefore, for $n\neq1$, the present model describes a Universe with a constant level of anisotropy rather than a dynamically isotropizing one.

The deceleration parameter is defined as follows:

$$q=-1+\frac{\lambda}{\sqrt{2\lambda t+k}}. \tag{26}$$

Figure 1 depicts the evolution of the deceleration parameter q with redshift in the THDE model formulated within $f(R,T)$ gravity for different values of the parameter λ. The results show that q remains negative over the entire interval $(-1 \leq z \leq 2)$, indicating that the Universe undergoes accelerated expansion throughout the considered redshift range. Although q increases with redshift, it does not become positive, implying that the model does not exhibit a transition from a decelerating to an accelerating phase within the chosen parameter range. The separation of the curves for different values of λ indicates that the cosmic expansion is sensitive to this parameter, with smaller values of λ producing stronger acceleration.

The obtained solution therefore represents a continuously accelerating Universe for the selected model parameters. Hence, the present model is primarily intended to describe the late-time evolution of a dark energy-dominated Universe rather than the complete cosmic history. We acknowledge that, unlike the standard ΛCDM model, which predicts an early matter-dominated decelerating epoch followed by the present accelerated expansion, the adopted parametrization of the deceleration parameter does not reproduce this transition for the chosen parameter values. Instead, the parametrization is adopted to obtain exact analytical solutions and to investigate the late-time behaviour of the proposed cosmological model within the framework of $f(R, T)$ gravity. A more general parametrization capable of describing both the early decelerating and late accelerating epochs, including the transition point $q = 0$, will be considered in future work.

3.4. Tsallis holographic dark energy with the Hubble horizon as the IR cutoff

According to the Tsallis entropy formalism, the holographic dark energy density is given by $\rho_T = \beta L^{2\delta-4}$ where L denotes the infrared (IR) cutoff, β is a positive model constant with appropriate dimensions, and δ is the Tsallis non-extensive parameter that characterizes the deviation from the standard Boltzmann–Gibbs entropy. In the present work, we choose the Hubble horizon as the infrared cutoff, i.e., $L = H^{-1}$, where H is the Hubble parameter. Consequently, the THDE density takes the form $\rho_T = \beta H^{4-2\delta}$. Throughout this work, natural units $(8\pi G = c = 1)$ are adopted.

The parameter δ plays a crucial role in determining the behaviour of the THDE model. For $\delta = 1$, the Tsallis holographic dark energy reduces to the standard holographic dark energy model. For $\delta = 2$, the THDE behaves like a cosmological constant. Furthermore, when $\delta < 1$, the gravitational field is sufficiently strong that the contributions of dark matter and dark energy become less significant in constructing the cosmological model, whereas for $\delta > 1$, the gravitational field becomes relatively weaker, requiring larger contributions from dark matter and dark energy to describe the observed Universe Riess *et al.* [30] and Astier *et al.* [31].

Substituting the Hubble parameter obtained from the exact solution of the field equations into the above expression yields the explicit form of the THDE density used in the subsequent analysis. The energy density of THDE is

$$\rho_T = \beta\left(\frac{1}{\sqrt{2\lambda t + k}}\right)^{4-2\delta}. \tag{27}$$

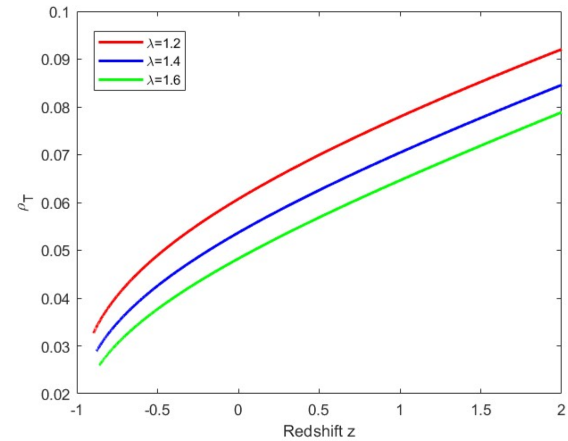

Figure 2: Evolution of the THDE energy density ρ_T with redshift z.

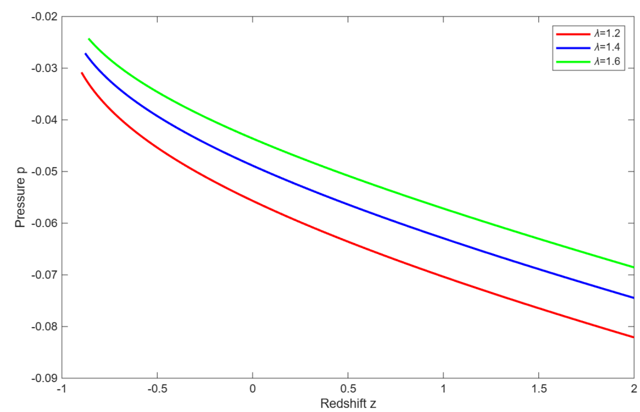

Figure 3: Evolution of the THDE pressure p_T with redshift z.

The behaviour of the energy density, $\rho_T(z)$, at varying redshift values within the framework of $f(R, T)$ gravity for various choices of the λ as shown in Figure 2. It indicates that the energy density is significantly higher at earlier times (larger redshifts) and declines progressively as the Universe expands, reflecting its evolution toward later cosmic epochs. The distinct separation of curves reflects the influence of λ, where smaller values lead to higher ρ_T, showing a stronger contribution of the THDE component. The smooth and regular behaviour of $\rho_T(z)$ suggests that the model is physically consistent and provides a reasonable description of dark energy, while supporting late-time accelerated expansion.

The pressure of THDE is

$$\begin{aligned} p_T &= \omega_T \rho_T \\ &= \left(-1 + \frac{(4-2\delta)\lambda}{4\sqrt{2\lambda t + k}}\right)\left[\beta\left(\frac{1}{\sqrt{2\lambda t + k}}\right)^{4-2\delta}\right]. \end{aligned} \tag{28}$$

Figure 3 shows the evolution of the THDE pressure p_T for different values of parameter λ . The pressure remains negative throughout the considered redshift range, indicating the negative-pressure behaviour associated with the THDE component. The magnitude of the pressure increases towards the

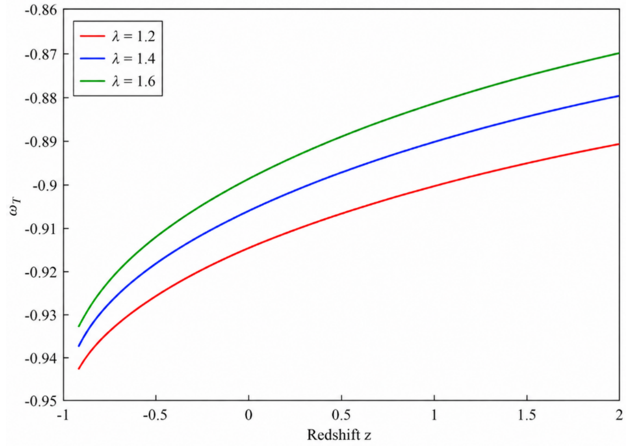

Figure 4: Evolution of the EoS parameter ω_T with redshift z.

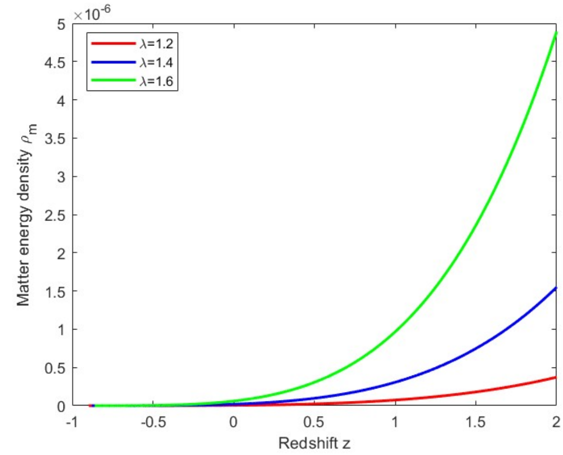

Figure 5: Evolution of the matter energy density ρ_m with redshift z.

present epoch, corresponding to the increasing influence of the dark-energy component at late times. It is also observed that increasing λ, makes the pressure less negative, while the qualitative behaviour remains unchanged. The obtained pressure is consistently related to the THDE density and EoS parameter through $p_T = \omega_T \rho_T$, ensuring consistency among the thermodynamic quantities of the model.

The separation between the curves shows that the cosmological evolution is sensitive to the parameter λ. As the value of λ increases from 1.2 to 1.6, the pressure becomes less negative, while smaller values of λ correspond to relatively more negative pressure throughout the evolution.

The smooth and continuous behaviour of all three curves demonstrates that the pressure evolves regularly without any abrupt changes or singularities over the chosen redshift range. Since the present model is intended to describe the late-time evolution of the Universe, the emergence of negative pressure at later stages is consistent with the dark-energy-driven accelerated expansion predicted by the model.

The EoS parameter corresponding to the THDE, denoted by ω_T, can be expressed as follows:

$$\omega_T = -1 + \frac{(4-2\delta)\lambda}{4\ \sqrt{2\lambda t + k}} \tag{29}$$

Figure 4 indicates the evolution of the total equation of state parameter ω_T for the THDE model in $f(R,T)$ gravity for different values of the λ. Throughout the redshift range $-1 \leq z \leq 2$, ω_T remains negative, indicating the dominance of dark energy and the occurrence of accelerated cosmic expansion. The parameter ω_T evolves smoothly from lower negative values at late times towards values closer to zero at higher redshifts, showing a gradual weakening of the dark energy effect in the past. The curves exhibit a clear dependence on λ, where larger values of λ result in comparatively higher values of ω_T. Since ω_T remains above the phantom divide line ($\omega_T = -1$), the model exhibits a quintessence-like behaviour and provides a consistent description of the late-time accelerated expansion of the Universe.

To obtain exact analytical solutions, we assume that the matter energy–momentum tensor satisfies the standard conservation law. Under this assumption, the conservation equation for the five-dimensional Kaluza–Klein spacetime is given by Eq. (13), where the factor 4 arises from the four spatial dimensions of the Kaluza–Klein Universe. Integrating the above equation, we obtain $\rho_m = k_1(a(t))^{-4}$, where k_1 is an integration constant.

The energy density of matter is

$$\rho_m = k_1 \exp\left(\left(-\frac{4}{\lambda}\right)\ \sqrt{2\lambda t + k}\right) \tag{30}$$

Figure 5 depicts how the matter energy density, $\rho_m(z)$, varies with redshift in the THDE model under the framework of $f(R,T)$ gravity for different choices of the λ. The trend clearly shows that matter density increases with redshift, indicating that matter was more abundant in the earlier phases of cosmic evolution. Higher values of λ lead to a stronger growth of ρ_m, reflecting the sensitivity of the matter-density evolution to the adopted deceleration-parameter parametrization. Here, λ controls the expansion dynamics and should not be interpreted as the matter–geometry coupling parameter. At low redshift ($z \approx 0$), ρ_m becomes small, consistent with a dark energy–dominated Universe. This behaviour shows the role of λ in shaping the cosmic expansion and matter-density evolution.

3.5. Stability analysis

The model's stability is examined by the squared sound speed calculated as $v_s^2 = \frac{dp}{d\rho}$. The sign of v_s^2 determines the stability of perturbations. The behaviour of this quantity is examined for the obtained solutions.

The stability analysis is given by

$$v_s^2 = -1 + \frac{5-2\delta}{4}\frac{\lambda}{\sqrt{2\lambda t + k}} \tag{31}$$

The evolution of the squared sound speed, (v_s^2), with redshift for different values of λ in the THDE model within $f(R,T)$ gravity as shown in Figure 6. It is evident that v_s^2 remains negative throughout the considered redshift range, indicating that

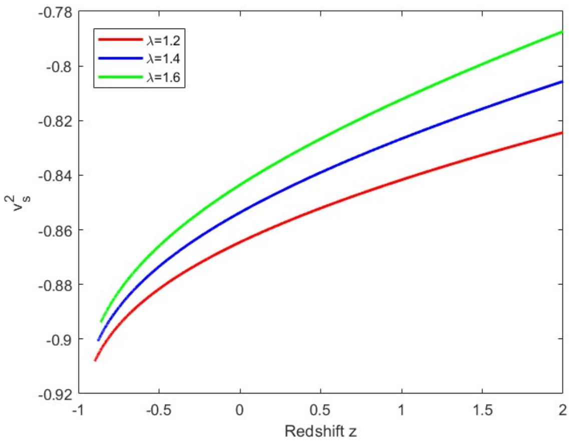

Figure 6: Evolution of the squared sound speed v_s^2 with redshift z.

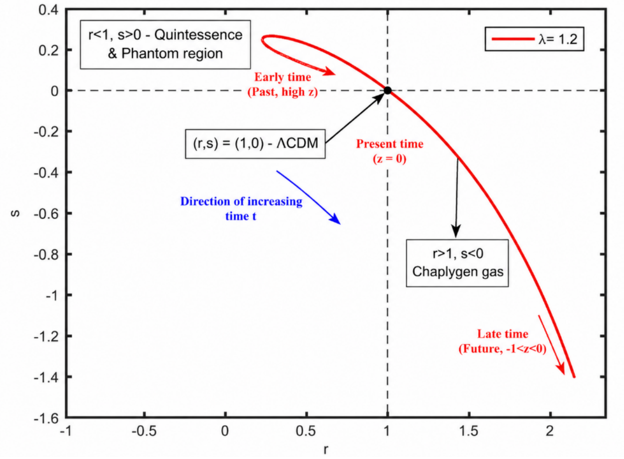

Figure 7: Statefinder trajectory in the (r, s) plane.

the model is classically unstable against adiabatic perturbations for the chosen parameter values. Similar behaviour has been reported in several holographic and Tsallis holographic dark energy models and is therefore not uncommon within this class of cosmological models.

The figure further shows that the instability becomes more pronounced as the Universe evolves towards the present epoch $z \to 0$, where dark energy dominates, whereas it is comparatively weaker at higher redshifts, where the relative contribution of the matter component increases within the adopted theoretical solution. The obtained values of v_s^2 remain negative, throughout the interval, indicating classical instability at the perturbative level. It is also observed that increasing the value of λ shifts the v_s^2 curves upward, thereby reducing the magnitude of the instability; however, v_s^2 remains negative, and the model does not attain classical stability for the considered parameter values.

The present work is primarily devoted to obtaining exact analytical solutions and investigating the background cosmological evolution of the proposed THDE model within the framework of $f(R, T)$ gravity. Since no detailed cosmological perturbation analysis has been carried out, the negative squared sound speed should be regarded as indicating perturbative instability of the model over the considered domain. Consequently, the present model should be interpreted as a background cosmological solution, while a comprehensive investigation including scalar perturbations, structure formation, non-adiabatic effects, or other possible stabilization mechanisms remains an important direction for future research. The observed behaviour may also be influenced by the non-extensive nature of Tsallis entropy together with the matter–geometry coupling in $f(R, T)$ gravity.

3.6. Statefinder diagnostics (SFD)

We use the statefinder parameters (r, s), which are described as follows, to further describe the model $r = \frac{\dddot{a}}{aH^3}, s = \frac{r-1}{3(q-\frac{1}{2})}$. The trajectory of the model in the $r - s$ plane is analysed and compared with the standard ΛCDM model.

The statefinder parameters set is

$$\begin{aligned} r &= 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \\ &= 1 - \frac{3\lambda}{\sqrt{2\lambda t + k}} + \frac{3\lambda^2}{2\lambda t + k} \end{aligned} \tag{32}$$

$$s = \frac{-3\lambda/\sqrt{2\lambda t + k} + 3\lambda^2/(2\lambda t + k)}{3\left(\lambda/\sqrt{2\lambda t + k} - 3/2\right)} \tag{33}$$

Figure 7 presents the statefinder trajectory (r, s) of the proposed THDE model for $\lambda = 1.2$. The arrow indicates the direction of increasing cosmic time. The trajectory begins in the region $r > 1,\ s < 0$ at earlier cosmic times and evolves toward the ΛCDM fixed point $(r, s) = (1, 0)$. With further evolution, the trajectory enters the region $(r < 1,\ s > 0)$, which is associated with a quintessence-like diagnostic behaviour. In the asymptotic future, the analytical solution satisfies $(r, s) \to (1, 0)$. Thus, the model exhibits a transition through different regions of the statefinder plane while approaching the ΛCDM fixed point asymptotically.

The dimensionless matter and Tsallis holographic dark energy density parameters are defined with respect to the critical density $\rho_c = 3H^2$,as $\Omega_M = \frac{\rho_m}{3H^2}$ & $\Omega_T = \frac{\rho_T}{3H^2}$. Here $3H^2$ is adopted as a normalization scale for expressing the relative density contributions with respect to the Hubble expansion. Since the present model is formulated in five-dimensional anisotropic $f(R, T)$ gravity, these quantities are not interpreted as the conventional four-dimensional GR critical-density fractions.

Substituting the expressions for the matter energy density (ρ_m), the THDE density (ρ_T), and the Hubble parameter (H) obtained from Eqs. (22), (27), and (30), we obtain the overall density parameters

$$\Omega_M = \frac{k_1}{3}(2\lambda t + k)\exp\left(-\frac{4}{\lambda}\sqrt{2\lambda t + k}\right) \tag{34}$$

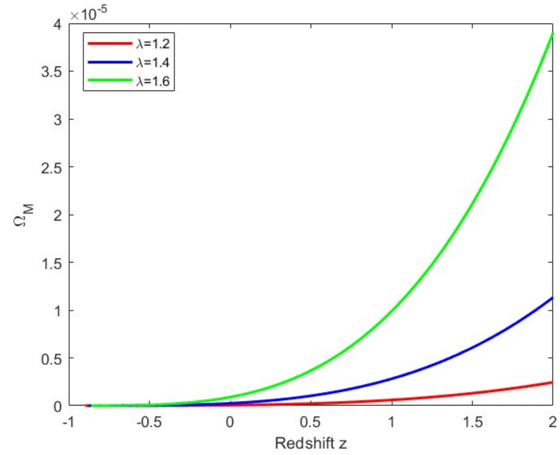

Figure 8: Evolution of the matter density parameter Ω_M with redshift z.

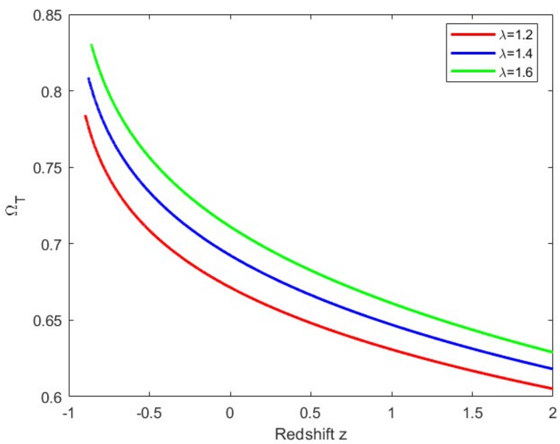

Figure 9: Evolution of the THDE density parameter Ω_T with redshift z.

$$\Omega_T = \frac{\beta}{3}(2\lambda t + k)^{\delta - 1} \tag{35}$$

Figure 8 illustrates the evolution of the matter density parameter, Ω_M , for different values of λ. The matter density parameter $\Omega_M$increases with increasing redshift; however, its magnitude remains very small throughout the displayed redshift range. Even at higher redshifts, $\Omega_M$remains of order 10^{-5} or smaller for the adopted parameter values, while its value near the present epoch is substantially below the conventional matter fraction $\Omega_M \approx 0.3$. Therefore, the present numerical solution does not reproduce a standard matter-dominated epoch or a conventional transition from matter domination to THDE domination. Instead, the results indicate that the particular exact solution and parameter set considered here describe a predominantly THDE-dominated cosmological branch. A parameter calibration capable of reproducing the standard matter fraction would require a separate analysis and is beyond the scope of the present theoretical investigation.

Figure 9 illustrates the evolution of the THDE density parameter,Ω_T, for different values of λ. The results show that Ω_T decreases with increasing redshift, indicating that the THDE contribution becomes more significant as the Universe evolves toward the present epoch. For the adopted parameter values,Ω_T at $z = 0$ is approximately $0.67, 0.69$, and 0.71 for $\lambda = 1.2, 1.4$ & 1.6 respectively. It is also observed that larger values of λ, produce higher values of Ω_T, indicating that it enhances the THDE contribution within the adopted theoretical framework. The figure, therefore, demonstrates the dependence of the THDE density parameter on λ and its evolution during the late-time cosmological regime. Since the present work is based on exact analytical solutions and does not involve observational parameter fitting, these results are presented as theoretical predictions rather than observational constraints.

3.7. *Energy conditions*

The physical viability of the model is examined using the standard energy conditions. These are given by

$$\text{NEC: } \rho + p \geq 0, \qquad \text{WEC: } \rho \geq 0,\ \rho + p \geq 0,$$
$$\text{DEC: } \rho \geq 0,\ \rho - |p| \geq 0, \qquad \text{SEC: } \rho + p \geq 0,\ \rho + 2p \geq 0.$$

The obtained solutions are analysed under these conditions to assess their physical consistency. $(\rho_T + p_T)$, (ρ_T), $(\rho_T - |p_T|)$, and $(\rho_T + 2p_T)$.

The evolution of the energy conditions, namely NEC, WEC, SEC, and DEC, is shown in Figure 10 for different values of λ. It is observed that the quantities corresponding to $\rho_T + p_T$, ρ_T, $\rho_T + 2p_T$, and $\rho_T - |p_T|$ exhibit distinct behaviours over the considered redshift range. The NEC, WEC, and DEC quantities remain positive, indicating that these three energy conditions are satisfied. In contrast, $\rho + 2p$ remains negative, demonstrating the violation of the SEC. The SEC violation is consistent with the accelerated expansion exhibited by the model. Although variations in modify the magnitude of the energy-condition quantities, they do not change their qualitative behaviour. Thus, the model satisfies the NEC, WEC, and DEC while violating the SEC over the considered redshift range.

4. Comparison of model parameters with standard cosmological expectations

To assess the physical behaviour of the proposed model, we compare the evolution of important cosmological parameters with commonly accepted values and qualitative expectations reported in the literature. The comparison is based on well-established observational results, including those from the Planck Collaboration, and focuses on key quantities such as the deceleration parameter, EoS parameter, matter and dark energy density parameters, and other relevant physical properties.

Since the present study does not involve observational parameter estimation, the values reported in the comparison are not intended as direct observational constraints on the five-dimensional model. Instead, they are used as reference values to examine the qualitative behaviour of the proposed cosmological scenario. In particular, the comparison helps assess whether

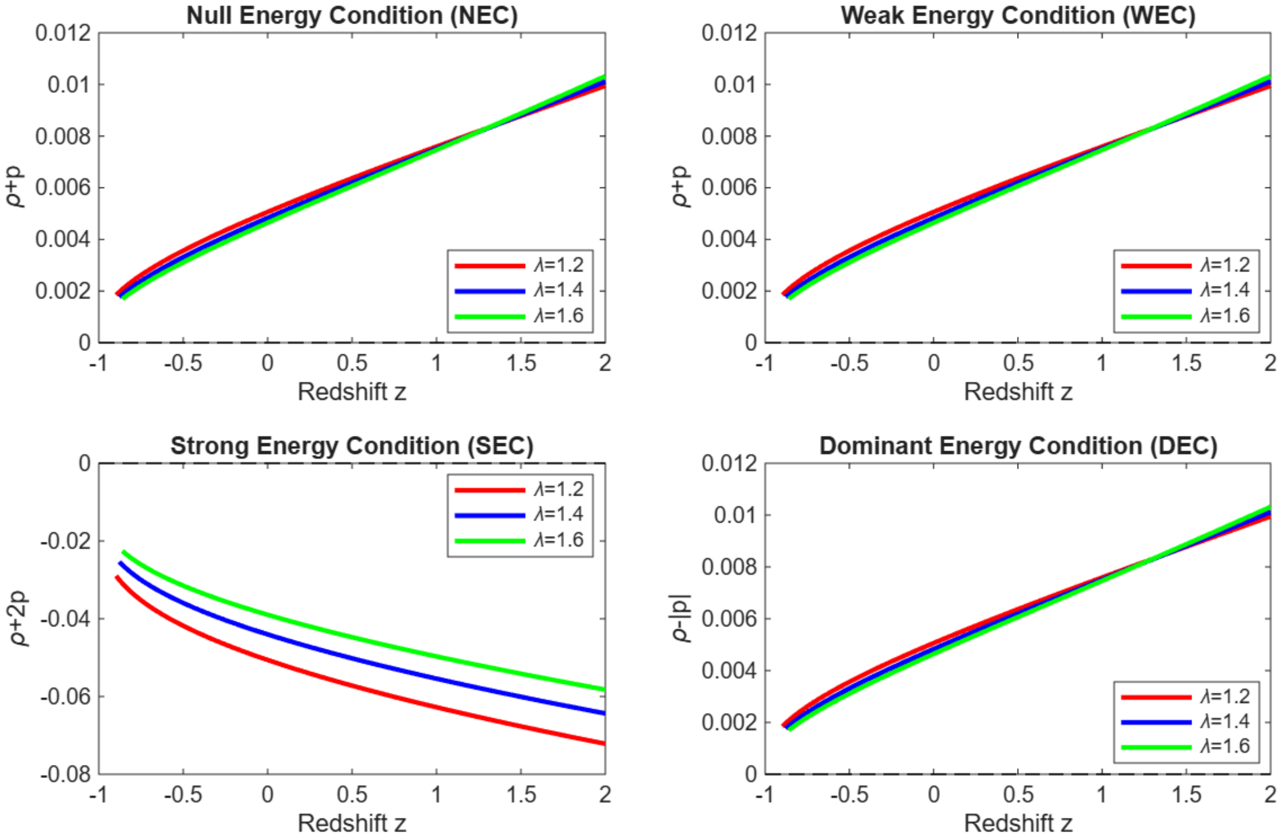

Figure 10: Evolution of the energy conditions with redshift z.

the model captures important features of late-time cosmic evolution, such as accelerated expansion and the dominance of the dark-energy component. At the same time, the comparison highlights the limitations of the adopted exact solution, particularly the very small matter contribution and the absence of a conventional matter-dominated epoch for the chosen parameter set. The results of this comparative analysis are presented in the following table.

It is evident from the revised comparison that the proposed model captures several qualitative features associated with late-time accelerated cosmological evolution. The deceleration parameter remains negative over the considered evolution, while the EoS parameter exhibits DE-like behaviour. The density parameter analysis, however, does not reproduce the conventional present-day matter fraction for the adopted parameter set. Instead, the matter contribution remains very small, whereas the THDE contribution is the dominant component according to the definitions adopted in Eqs. (34) and (35). For comparison, the Planck values quoted in the table are used only as reference values corresponding to the standard four-dimensional ΛCDM cosmology. They are not treated as direct observational constraints on the present five-dimensional theoretical model. Therefore, the present solution should not be interpreted as reproducing the standard matter-dominated epoch or a conventional matter-to-DE transition. A quantitative comparison with observational density fractions would require a separate parameter calibration and a consistent definition of the critical density appropriate to the five-dimensional framework.

5. Physical interpretation and discussion

In the current investigation, a cosmological model is constructed by introducing THDE into a five-dimensional Kaluza–Klein Universe within the framework of $f(R,T)$ gravity. Unlike many existing models based on general relativity and other modified gravity theories, the present model incorporates matter–geometry coupling, which provides an alternative approach to understanding the accelerated expansion of the Universe.

Most previous THDE models have been developed in isotropic cosmological backgrounds, whereas the present study considers an anisotropic Universe. This allows us to examine how anisotropy evolves with cosmic expansion. The obtained results show that the shear scalar decreases gradually with cosmic time, indicating a reduction in the magnitude of anisotropic expansion. However, since the anisotropy parameter A_m and equivalently the ratio $\frac{\sigma}{\theta}$ remains constant for a fixed value of n, the model does not evolve towards an isotropic state in the general case. Complete isotropy is achieved only in the special case $n = 1$, for which $A_m = 0$.

The statefinder diagnostic provides additional insight into the evolution of dark energy. The results indicate that the proposed model approaches the ΛCDM model at late times while

Table 1: Comparison of the present model with standard cosmological expectations.

Parameter	Symbol	Present model behaviour	Observational/expected range	Reference	Physical interpretation
Hubble parameter	H_0	Finite, positive	$\sim$ 67–74 km s^{-1} Mpc^{-1}	Refs. [3, 32]	Current expansion rate
Deceleration parameter	q_0	$q < 0$, ≈ -0.5 to -0.7	-0.5 to -0.7	Refs. [3, 12]	Accelerating Universe
Equation-of-state parameter	ω_T	$-1 < \omega_T < -0.8$	≈ -1	Refs. [3, 12, 13]	Quintessence/dark-energy-dominated phase
Matter density parameter	Ω_M	5.08×10^{-8}–8.90×10^{-7} at $z = 0$ for $\lambda = 1.2, 1.4, 1.6$	≈ 0.3	Ref. [3]	Matter contribution is very small for the adopted parameter set
Dark-energy density parameter	Ω_T	0.6715–07112 at $z = 0$ for $\lambda = 1.2, 1.4, 1.6$	≈ 0.7	Ref. [3]	THDE is the dominant component according to the adopted density-parameter definition
THDE energy density	ρ_T	Decreasing with time	High at early times and decreases with expansion	Refs. [33, 34]	Drives accelerated expansion
Matter energy density	ρ_m	Increasing with redshift	Very small matter contribution for the adopted parameter set	Ref. [3]	Structure formation
Pressure (THDE)	p	Negative	Negative	Refs. [1, 2]	Causes accelerated expansion
Transition redshift	z_t	No transition (z_t does not exist)	0.5–1	Refs. [12, 35]	Accelerated expansion throughout the considered redshift range
Statefinder pair	(r, s)	Approaches $(1, 0)$	$(1, 0)$ for the ΛCDM model	Refs. [36, 37]	Consistency with the ΛCDM model
Anisotropy parameter	A_m	Constant for fixed n	$A_m = 0$ only when $n = 1$	Ref. [25]	Anisotropization
Shear scalar	σ^2	Decreases with time	Tends to 0 at late times	Ref. [38]	The shear magnitude decreases during cosmic evolution
Expansion scalar	θ	Positive, decreasing	Positive	Ref. [39]	The Universe undergoes continuous expansion
Squared sound speed	v_s^2	Negative	$v_s^2 > 0$ for classical stability	Ref. [40]	Indicates classical stability/instability
Energy conditions	NEC, WEC, DEC	Satisfied	Expected to be satisfied	Ref. [41]	Physical viability of the model
Strong energy condition	SEC	Violated	Expected violation in an accelerating Universe	Refs. [42, 43]	Necessary for cosmic acceleration

still allowing deviations due to modified gravity and higher-dimensional geometry. The arbitrary constant λ, introduced through the deceleration parameter parametrization, influences the evolution of the cosmological parameters by controlling the behaviour of the deceleration parameter. It is independent of the matter–geometry coupling parameter γ in the adopted $f(R, T)$gravity model. Overall, the present model extends earlier THDE studies by combining modified gravity, anisotropy, and higher-dimensional geometry within a single theoretical framework.

The present work is primarily an analytical investigation of the proposed cosmological model and does not include parameter estimation using observational datasets. Nevertheless, the predicted late-time accelerated expansion and dark energy-dominated behaviour are qualitatively consistent with the general picture of modern cosmology. A detailed statistical comparison with observational datasets, such as Cosmic Chronometer H(z) measurements, the Pantheon+ Type Ia supernova compilation, BAO, and CMB observations, would provide stronger constraints on the model parameters. Such an observational analysis is beyond the scope of the present work and will be considered in future studies.

6. Conclusions

In this work, we have developed a cosmological model by incorporating Tsallis Holographic dark energy (THDE) into a five-dimensional Kaluza–Klein framework within $f(R,T)$ gravity. By adopting a suitable form of the scale factor, exact analytical solutions of the modified field equations were obtained, enabling a detailed investigation of the expansion rate, deceleration parameter, energy densities, pressure, and EoS parameter.

The analysis demonstrates that the proposed model describes an accelerated expansion phase, characterized by a negative deceleration parameter. The pressure and EoS parameter remain negative throughout the considered evolution. The matter density remains very small for the adopted parameter set, while the THDE component remains dominant in the considered late-time regime. The arbitrary constant λ, appearing in the adopted deceleration parameter ansatz, influences the expansion dynamics and the late-time evolution of the model. The statefinder trajectory evolves toward the ΛCDM fixed point and approaches it asymptotically in the future.

The revised analysis of the energy conditions shows that the NEC, WEC, and DEC are satisfied, whereas the SEC is violated. The violation of the SEC is consistent with the accelerated expansion of the Universe. However, the energy conditions are not interpreted as a criterion for classical stability. The stability analysis based on the squared sound speed indicates classical instability at the perturbative level for the considered parameter set. The present stability analysis is restricted to the effective squared sound speed, while a complete cosmological perturbation analysis involving scalar perturbations and structure formation is left for future work.

Overall, the proposed model provides an exact analytical description of the late-time cosmological evolution within five-dimensional $f(R,T)$ gravity with THDE. The results highlight the effects of matter–geometry coupling and the multidimensional nature of gravity on the evolution of the cosmological parameters. The present study is primarily focused on the background dynamics of the late-time Universe. Since no observational parameter estimation has been performed and the negative squared sound speed indicates classical instability for the chosen parameter set, the perturbative and observational viability of the model remains to be established. Extending the analysis to include cosmological perturbations, observational constraints, and a parametrization capable of describing both the early decelerating and present accelerating phases would be valuable directions for future research. The present analysis is also restricted to a non-interacting THDE scenario; possible energy exchange between the dark-energy and matter sectors in $f(R,T)$ gravity will be investigated in future work.

Data availability

No datasets were created, examined, or used during this study.

Declaration of competing interest

The authors declare that the content of this paper is unaffected by any financial or personal conflicts of interest.

Funding

The authors did not receive any specific financial assistance or funding to carry out this study.

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