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Strengthening the digital fortress: A Pythagorean fuzzy Dombi–Archimedean approach to network security decisions

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Abstract

Network-security planning frequently depends on judgments that are imprecise, incomplete, and sometimes internally conflicting. Conventional numerical decision models are effective when assessments are available in exact form; however, they are less expressive when experts must report both supporting and opposing evidence for the same security alternative. This paper develops a Pythagorean fuzzy Dombi–Archimedean aggregation framework for multi-criteria decision making under such conditions. In the proposed model, each expert assessment is represented as a Pythagorean fuzzy number and is aggregated through Dombi-type flexibility functions embedded in an Archimedean operational structure. Within this unified notation, weighted averaging, ordered weighted averaging, hybrid averaging, weighted geometric, ordered weighted geometric, and hybrid geometric operators are introduced. The main algebraic properties of these operators, including closure, idempotency, boundedness, and monotonicity, are established from the operational laws with explicit admissibility requirements. A network-security case study is then presented to rank six candidate security systems with respect to four risk-oriented criteria. The numerical implementation includes tabular computations, a procedural flowchart, sensitivity analysis with respect to the flexibility parameter, and a comparative ranking discussion. The results show that the proposed framework offers a coherent and adjustable mechanism for ranking security alternatives when the available decision information involves hesitation, conflict, and expert uncertainty.

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1. Introduction

Fuzzy set theory, introduced by Zadeh [1], provides a mathematical language for describing vague concepts through degrees of membership rather than through crisp inclusion or ex-

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clusion. Over time, this viewpoint has shaped several areas of fuzzy mathematics, including fuzzy relations, linguistic modeling, clustering, and decision analysis [2–5]. Its importance in multi-criteria decision making (MCDM) is particularly evident because practical evaluations are rarely supplied as exact numerical measurements. Instead, experts often express their judgments in approximate, linguistic, or partially reliable terms [6, 7].

As the need to represent richer uncertainty structures increased, several extensions of the classical fuzzy set model were introduced. Hesitant fuzzy sets allow more than one possible membership degree to be assigned to the same element, which is useful when an expert cannot reasonably select a single value [8]. Intuitionistic fuzzy sets add an explicit non-membership degree, thereby making it possible to describe evidence in favor of an alternative and evidence against it within the same formal structure [9]. This additional representational capacity has made intuitionistic fuzzy modeling valuable in decision making, preference analysis, pattern recognition, and related uncertainty-based applications [10–16].

Pythagorean fuzzy sets (PFSs), introduced by Yager, extend the intuitionistic fuzzy framework by replacing the linear restriction on membership and non-membership degrees with a squared condition [17]. In this setting, the sum of the squared membership and squared non-membership degrees must not exceed one. The resulting feasible region is larger than that of intuitionistic fuzzy sets, which makes the Pythagorean fuzzy model more appropriate when both positive and negative assessments may be relatively strong. Accordingly, a substantial literature has developed ranking measures, aggregation schemes, and decision-making methods for Pythagorean fuzzy information [18–25]. The q-rung orthopair fuzzy model further generalizes this idea by replacing the squared restriction with a qth-power constraint [26–30].

Aggregation operators are central to MCDM because they transform several criterion-wise assessments into a single overall evaluation. Dombi t-norms and t-conorms are useful in this context because their parameter controls the strictness of the aggregation process [31]. Archimedean t-norms and t-conorms, on the other hand, provide a generator-based construction that includes many familiar fuzzy operational families as special cases [32–37]. A Dombi–Archimedean formulation can therefore combine parameter-driven flexibility with the structural generality offered by Archimedean generators.

The present study develops this combined structure for network-security decision making. Selecting an appropriate security system is naturally an MCDM problem because each candidate architecture must be evaluated across several risk dimensions, including resistance to unauthorized access, vulnerability to denial-of-service attacks, exposure to eavesdropping, and resilience against multi-vector threats. Recent cybersecurity studies have applied machine-learning and uncertainty-based approaches to malware classification, anomaly detection, and network-intrusion identification, demonstrating the importance of systematic and data-informed security assessment [38–40]. Related fuzzy decision-making investigations have further shown that aggregation-based models can support the evaluation of security alternatives when expert judgments are imprecise, incomplete, or conflicting [41–44]. These studies motivate the development of a transparent and adjustable Pythagorean fuzzy framework for ranking network-security systems without attributing cybersecurity claims to unrelated mathematical literature.

The conceptual flow in Figure 1 summarizes how the research gap, mathematical construction, operator analysis, and cybersecurity application are connected.

1.1. Critical review of related studies

Recent Pythagorean fuzzy aggregation studies have produced valuable averaging, geometric, Dombi, Aczel–Alsina, and Archimedean families, but they also leave several methodological issues that motivate the present formulation. Classical Pythagorean fuzzy weighted operators provide simple and interpretable aggregation, although their compensation pattern is usually fixed once the operator has been selected [17, 19, 21, 22]. Pythagorean fuzzy Dombi operators introduce a parameter that adjusts aggregation strictness, but they are generally developed directly from Dombi t-norms and t-conorms rather than from an explicitly stated generator-admissibility framework [23–25]. Archimedean and Dombi–Archimedean studies broaden the operational basis, but in many applications the numerical implementation is performed through a special generator without a transparent derivation from the general formulas [30, 34–36]. The mathematical reliability of decision-support models can also benefit from developments in operator theory, numerical approximation, dynamical-system stability, inequality analysis, and generalized uncertainty structures [45–49]. Although these studies are not directly concerned with cybersecurity, they provide broader methodological perspectives on stability, computation, control, and uncertainty representation that are relevant to the theoretical development of aggregation-based decision models. Accordingly, they are cited here as supporting mathematical literature rather than as evidence of cybersecurity standards, intrusion-detection methods, or security-control practices.

Thus, the present paper does not claim that previous formulas are globally invalid. Rather, it addresses three specific limitations relevant to this study: (i) the need to state the generator assumptions before proving closure and monotonicity; (ii) the need to derive the computational Dombi-scale case from a concrete identity-generator choice; and (iii) the need to report enough intermediate calculations, sensitivity checks, and comparison criteria to make the ranking reproducible.

1.2. Motivation

The motivation for this work comes from two related limitations in existing aggregation-based decision models. First, many Pythagorean fuzzy aggregation operators rely on fixed operational forms, which restricts the decision maker's ability to regulate compensation among criteria. Second, some Dombi–Archimedean formulations are affected by inconsistent notation, mixed parameter usage, or expressions that do not

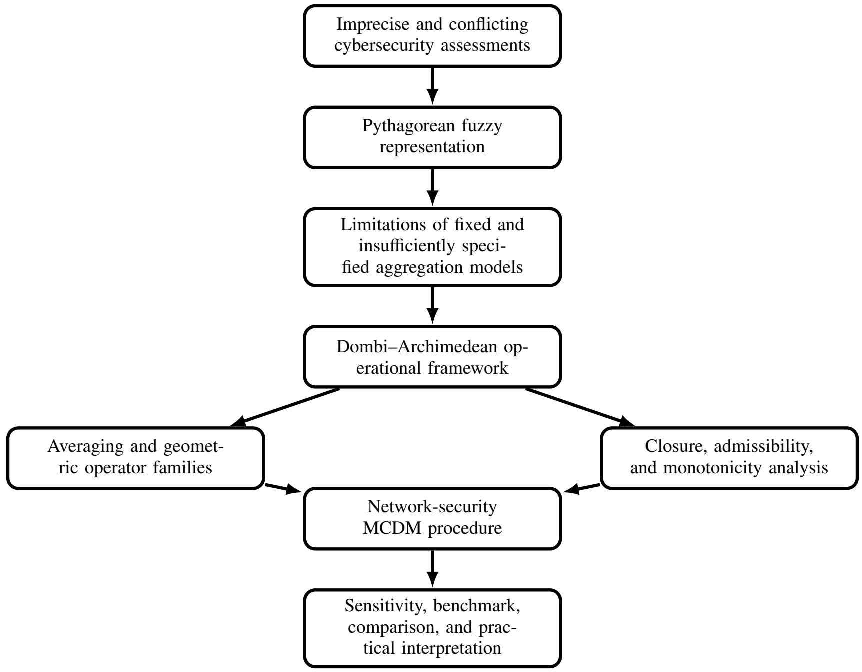

Figure 1: Conceptual flow of the study from uncertain cybersecurity assessments to validation and decision support.

clearly preserve the squared Pythagorean constraint. A formulation is therefore needed so that the aggregation process remains mathematically coherent, reproducible, and faithful to the Pythagorean fuzzy structure.

This issue is especially important in network-security evaluation. A security system may perform strongly against one class of attacks while remaining exposed to another. A useful aggregation model should therefore permit the decision maker to adjust the strictness of the evaluation process without leaving the Pythagorean fuzzy domain. The flexibility parameter in the proposed Dombi–Archimedean framework is designed to provide this adjustment.

1.3. Novelty and contributions

The central contribution of this paper is a corrected and consistently notated Pythagorean fuzzy Dombi–Archimedean aggregation framework for ranking network-security alternatives. The specific contributions are as follows.

The Dombi–Archimedean operational laws are proposed so that membership and non-membership components are aggregated on the squared Pythagorean scale. Weighted, ordered weighted, hybrid, geometric, ordered geometric, and hybrid geometric operators are formulated under a unified symbolic structure. Fundamental properties of the proposed operators are stated and justified, including closure, idempotency, boundedness, and monotonicity. A stepwise MCDM algorithm is developed and illustrated by a network-security system selection problem. Tables, a flowchart, sensitivity analysis, and comparative ranking results are included to make the implementation transparent and reproducible.

Table 1 summarizes the theoretical distinction between the proposed framework and representative related operators.

1.4. Paper organization

The remainder of the paper is structured as follows. Section 2 presents the necessary preliminaries on Pythagorean fuzzy numbers, score functions, Archimedean generators, and Dombi scale functions. Section 3 introduces the corrected Dombi–Archimedean operational laws. Section 4 develops the proposed aggregation operators and proves their basic properties. Section 5 describes the MCDM procedure. Section 6 applies the method to network-security system selection. Section 7 presents the sensitivity and comparative analyses, and Section 8 concludes the study.

2. Preliminaries

This section fixes the notation and recalls the basic definitions used throughout the paper. The symbols are chosen to remove ambiguity from the aggregation formulas and to maintain a consistent mathematical style in the subsequent developments.

The principal notation used throughout the manuscript is summarized in Table 2.

Definition 2.1 (Pythagorean fuzzy set)**.** Following Yager's Pythagorean fuzzy model [17], let X denote a universe of discourse. A Pythagorean fuzzy set $\mathcal{P}$ defined on X is written as

$$\mathcal{P} = \{\langle x, \alpha_{\mathcal{P}}(x), \beta_{\mathcal{P}}(x)\rangle : x \in X\}, \tag{1}$$

where $\alpha_{\mathcal{P}} : X \to [0,1]$ and $\beta_{\mathcal{P}} : X \to [0,1]$ denote the membership and non-membership mappings, respectively, and satisfy

$$0 \le \alpha^2_{\mathcal{P}}(x) + \beta^2_{\mathcal{P}}(x) \le 1. \tag{2}$$

For any fixed element of X, the ordered pair $\widehat{\pi} = (\alpha, \beta)$ is called a Pythagorean fuzzy number (PFN).

Table 1: Theoretical distinction between the proposed framework and representative related operators.

Method family	Main operational basis	Typical limitation for this study	Distinction of the present model
Classical PFWA/PFWG [21, 22]	Algebraic PF aggregation	Fixed compensation pattern	Uses a tunable Dombi flexibility parameter and reports parameter sensitivity
PF Dombi operators [23–25]	Dombi t-norm/t-conorm	Dombi scale is not always connected to a stated Archimedean generator pair	Embeds the Dombi scale in a stated generator framework and derives the computational identity-generator case
PF Aczel–Alsina operators [18]	Aczel–Alsina norms	Uses a different generator family and different compensation behavior	Keeps the Dombi ratio-scale interpretation while allowing generator-based notation
Archimedean fuzzy operators [34–36]	General generator functions	Closure and admissibility may be stated at a high level	Gives explicit admissibility and complement-consistency conditions for Pythagorean fuzzy numbers

Table 2: Notation used in the manuscript.

Symbol	Meaning
$\widehat{\pi} = (\alpha,\beta)$	Pythagorean fuzzy number
α	Membership degree
β	Non-membership degree
$\overline{A}_i$	ith alternative
$\overline{C}_j$	jth criterion
ξ	Dombi flexibility parameter, $\xi > 0$
η	Archimedean generator used in the t-conorm branch
ζ	Archimedean generator used in the t-norm branch
Λ_ξ	Dombi membership-scale function
Γ_ξ	Dombi non-membership-scale function
$\oplus_{\mathcal{AD}}$	Dombi–Archimedean sum
$\otimes_{\mathcal{AD}}$	Dombi–Archimedean product
$\star_{\mathcal{AD}}$	Scalar operation for averaging aggregation
$\diamond_{\mathcal{AD}}$	Scalar operation for geometric aggregation

Definition 2.2 (Score and accuracy)**.** For a PFN $\widehat{\pi} = (\alpha,\beta)$, the score and accuracy functions used for ranking Pythagorean fuzzy information are defined by [18, 19]

$$\mathscr{S}(\widehat{\pi}) = \alpha^2 - \beta^2, \qquad \mathscr{H}(\widehat{\pi}) = \alpha^2 + \beta^2. \tag{3}$$

When two PFNs have unequal score values, the PFN with the larger score is preferred. If the scores are equal, the accuracy value is used as the secondary comparison criterion.

Definition 2.3 (Archimedean generators)**.** In the sense of Archimedean t-norm and t-conorm constructions [32, 33], let η be a continuous, strictly increasing generator associated with the t-conorm branch, and let ζ be a continuous, strictly decreasing generator associated with the t-norm branch. Whenever the relevant inverse mappings exist on the required domains, they are denoted by η^{-1} and ζ^{-1}.

Definition 2.4 (Archimedean scale generators)**.** In the transformed-scale formulation used in this paper, η and ζ are mappings from $[0,\infty)$ to $[0,\infty)$. Both are assumed to be continuous and strictly increasing, satisfy $\eta(0) = \zeta(0) = 0$, and possess strictly increasing inverse mappings η^{-1} and ζ^{-1} on $[0,\infty)$. The t-conorm/t-norm directions are determined by the Dombi scales Λ_ξ and Γ_ξ, respectively; in particular, the decreasing behavior of the non-membership branch is induced by Γ_ξ and the final inverse Dombi map, not by taking ζ to be decreasing.

Definition 2.5 (Admissible Dombi–Archimedean generator pair)**.** A pair (η,ζ) is called admissible for the present PF Dombi–Archimedean construction if the following conditions hold.

(A1) $\eta,\zeta : [0,\infty) \to [0,\infty)$ are continuous and strictly increasing, with $\eta(0) = \zeta(0) = 0$.

(A2) The inverse mappings $\eta^{-1},\zeta^{-1} : [0,\infty) \to [0,\infty)$ exist and are continuous and strictly increasing.

(A3) For all $x, u, y, v \in (0,1)$ satisfying $x+u \leq 1$ and $y+v \leq 1$, the generated kernels satisfy the complement-consistency condition

$$\mathcal{S}_{\mathcal{AD}}(x,y) + \mathcal{T}_{\mathcal{AD}}(u,v) \leq 1. \tag{4}$$

(A4) Boundary values at 0 and 1 are defined by continuous extension whenever an input component is an endpoint.

The numerical study uses the admissible identity pair $\eta(t) = \zeta(t) = t$ on $[0,\infty)$.

Definition 2.6 (Dombi scale functions)**.** Motivated by Dombi-type parametric operations [31], for $y \in (0,1)$ and $\xi > 0$, define

$$\Lambda_\xi(y) = \left(\frac{y}{1-y}\right)^\xi, \qquad \Gamma_\xi(y) = \left(\frac{1-y}{y}\right)^\xi. \tag{5}$$

The scale Λ_ξ is strictly increasing on $(0,1)$, whereas Γ_ξ is strictly decreasing on $(0,1)$. These scale monotonicities determine the directions of the membership and non-membership branches.

3. Dombi–Archimedean operations for PFNs

Let $\widehat{\pi}_1 = (\alpha_1, \beta_1)$ and $\widehat{\pi}_2 = (\alpha_2, \beta_2)$ be two PFNs. For $x, y \in (0,1)$, the following auxiliary Dombi–Archimedean kernels are introduced:

$$\mathcal{S}_{\mathcal{AD}}(x,y) = 1 - \left(1 + \{\eta(\eta^{-1}(\Lambda_\xi(x)) + \eta^{-1}(\Lambda_\xi(y)))\}^{1/\xi}\right)^{-1}, \tag{6}$$

$$\mathcal{T}_{\mathcal{AD}}(x,y) = \left(1 + \{\zeta(\zeta^{-1}(\Gamma_\xi(x)) + \zeta^{-1}(\Gamma_\xi(y)))\}^{1/\xi}\right)^{-1}. \tag{7}$$

The kernel $\mathcal{S}_{\mathcal{AD}}$ represents the t-conorm direction, whereas $\mathcal{T}_{\mathcal{AD}}$ represents the t-norm direction.

Definition 3.1 (Dombi–Archimedean operations)**.** For PFNs $\widehat{\pi}_1 = (\alpha_1, \beta_1)$ and $\widehat{\pi}_2 = (\alpha_2, \beta_2)$, define

$$\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2 = \left(\sqrt{\mathcal{S}_{\mathcal{AD}}(\alpha_1^2, \alpha_2^2)}, \sqrt{\mathcal{T}_{\mathcal{AD}}(\beta_1^2, \beta_2^2)}\right), \tag{8}$$

$$\widehat{\pi}_1 \otimes_{\mathcal{AD}} \widehat{\pi}_2 = \left(\sqrt{\mathcal{T}_{\mathcal{AD}}(\alpha_1^2, \alpha_2^2)}, \sqrt{\mathcal{S}_{\mathcal{AD}}(\beta_1^2, \beta_2^2)}\right). \tag{9}$$

For $\lambda > 0$, the scalar averaging and scalar geometric operations are

$$\begin{aligned} \lambda \star_{\mathcal{AD}} \widehat{\pi}_1 = \Big(&\sqrt{1 - (1 + \{\eta[\lambda\eta^{-1}(\Lambda_\xi(\alpha_1^2))]\}^{1/\xi})^{-1}}, \\ &\sqrt{(1 + \{\zeta[\lambda\zeta^{-1}(\Gamma_\xi(\beta_1^2))]\}^{1/\xi})^{-1}} \Big), \end{aligned} \tag{10}$$

$$\begin{aligned} \lambda \diamond_{\mathcal{AD}} \widehat{\pi}_1 = \Big(&\sqrt{(1 + \{\zeta[\lambda\zeta^{-1}(\Gamma_\xi(\alpha_1^2))]\}^{1/\xi})^{-1}}, \\ &\sqrt{1 - (1 + \{\eta[\lambda\eta^{-1}(\Lambda_\xi(\beta_1^2))]\}^{1/\xi})^{-1}} \Big). \end{aligned} \tag{11}$$

Theorem 3.1 (Closure)**.** Let $\widehat{\pi}_1 = (\alpha_1, \beta_1)$ and $\widehat{\pi}_2 = (\alpha_2, \beta_2)$ be PFNs. If (η, ζ) is an admissible Dombi–Archimedean generator pair in the sense of Definition 5, then $\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2$ and $\widehat{\pi}_1 \otimes_{\mathcal{AD}} \widehat{\pi}_2$ are PFNs. The same conclusion holds for $\lambda \star_{\mathcal{AD}} \widehat{\pi}_1$ and $\lambda \diamond_{\mathcal{AD}} \widehat{\pi}_1$ whenever $\lambda > 0$ and the corresponding generated kernels satisfy the same complement-consistency condition.

Proof. The components obtained from Equations (8)–(11) belong to $[0,1]$ because the kernels take values in $[0,1]$ and the final mapping is a square root. It remains to verify the Pythagorean bound. Put

$$M = \mathcal{S}_{\mathcal{AD}}(\alpha_1^2, \alpha_2^2), \qquad N = \mathcal{T}_{\mathcal{AD}}(\beta_1^2, \beta_2^2).$$

Since $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are PFNs, $\alpha_1^2 + \beta_1^2 \leq 1$ and $\alpha_2^2 + \beta_2^2 \leq 1$. By the complement-consistency condition in Equation (4),

$$M + N = \mathcal{S}_{\mathcal{AD}}(\alpha_1^2, \alpha_2^2) + \mathcal{T}_{\mathcal{AD}}(\beta_1^2, \beta_2^2) \leq 1.$$

The squared membership and non-membership values of $\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2$ are exactly M and N, respectively. Hence the resulting PFN satisfies $M + N \leq 1$. The product operation is obtained by interchanging the t-conorm and t-norm branches, and the same argument applies.

For the identity-generator computational specialization, the inequality can be checked directly. Let

$$A = \left(\frac{x}{1-x}\right)^\xi, \quad B = \left(\frac{y}{1-y}\right)^\xi,$$
$$C = \left(\frac{1-u}{u}\right)^\xi, \quad D = \left(\frac{1-v}{v}\right)^\xi.$$

If $x + u \leq 1$ and $y + v \leq 1$, then $C \geq A$ and $D \geq B$. Therefore $(C+D)^{1/\xi} \geq (A+B)^{1/\xi}$, and

$$\mathcal{T}^*_{\mathcal{AD}}(u,v) \leq \frac{1}{1 + (A+B)^{1/\xi}} = 1 - \mathcal{S}^*_{\mathcal{AD}}(x,y).$$

Thus $\mathcal{S}^*_{\mathcal{AD}}(x,y) + \mathcal{T}^*_{\mathcal{AD}}(u,v) \leq 1$, proving closure for the specialization used in the computations. □

Property 1 (Commutativity)**.** For all PFNs $\widehat{\pi}_1$ and $\widehat{\pi}_2$,

$$\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2 = \widehat{\pi}_2 \oplus_{\mathcal{AD}} \widehat{\pi}_1, \qquad \widehat{\pi}_1 \otimes_{\mathcal{AD}} \widehat{\pi}_2 = \widehat{\pi}_2 \otimes_{\mathcal{AD}} \widehat{\pi}_1. \tag{12}$$

The assertion follows immediately from the symmetry of the sums appearing in the Dombi–Archimedean kernels.

Example 1 (Identity-generator computational specialization)**.** The numerical part of this study uses the identity-generator special case

$$\eta(t) = t, \qquad \zeta(t) = t, \qquad \eta^{-1}(t) = t, \qquad \zeta^{-1}(t) = t, \tag{13}$$

for $t \geq 0$. Substituting Equation (13) into Equations (6) and (7) gives

$$\mathcal{S}^*_{\mathcal{AD}}(x_1, \ldots, x_r) = 1 - \left(1 + \left\{\sum_{j=1}^r \omega_j \left(\frac{x_j}{1-x_j}\right)^\xi\right\}^{1/\xi}\right)^{-1}, \tag{14}$$

$$\mathcal{T}^*_{\mathcal{AD}}(y_1, \ldots, y_r) = \left(1 + \left\{\sum_{j=1}^r \omega_j \left(\frac{1-y_j}{y_j}\right)^\xi\right\}^{1/\xi}\right)^{-1}. \tag{15}$$

Thus, the case study is not a different method; it is the identity-generator Dombi-scale member of the general Dombi–Archimedean family. This explicit derivation removes the earlier ambiguity between the general generator model and the computational implementation.

4. Pythagorean fuzzy Dombi–Archimedean aggregation operators

Let $\widehat{\pi}_j = (\alpha_j, \beta_j)$, $j = 1, 2, \ldots, r$, be a collection of PFNs. The criterion-weight vector is denoted by $\omega = (\omega_1, \ldots, \omega_r)$, where $\omega_j \geq 0$ and $\sum_{j=1}^{r} \omega_j = 1$. A distinct ordered-position weight vector is denoted by $\vartheta = (\vartheta_1, \ldots, \vartheta_r)$, where $\vartheta_j \geq 0$ and $\sum_{j=1}^{r} \vartheta_j = 1$. Criterion weights remain attached to criteria, whereas ordered-position weights are attached to rank positions after score ordering.

Definition 4.1 (Weighted averaging operator)**.** The Pythagorean fuzzy Dombi–Archimedean weighted averaging operator is defined by

$$\mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigoplus_{j=1}^{r} (\omega_j \star_{\mathcal{AD}} \widehat{\pi}_j). \tag{16}$$

The closed form of this operator is established in Theorem 4.2.

Definition 4.2 (Ordered weighted averaging operator)**.** Let $\widehat{\pi}_{\sigma(1)}, \ldots, \widehat{\pi}_{\sigma(r)}$ be a permutation of $\widehat{\pi}_1, \ldots, \widehat{\pi}_r$ arranged in descending order according to the score function. The ordered weighted averaging operator is

$$\mathrm{PFADOWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigoplus_{j=1}^{r} (\omega_j \star_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}). \tag{17}$$

Definition 4.3 (Hybrid averaging operator)**.** Let $\omega = (\omega_1, \ldots, \omega_r)$ be the criterion-weight vector and let $\vartheta = (\vartheta_1, \ldots, \vartheta_r)$ be the ordered-position vector. The hybrid averaging operator first forms $r\omega_j \star_{\mathcal{AD}} \widehat{\pi}_j$, orders the resulting PFNs by score, and then aggregates them as

$$\mathrm{PFADHA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigoplus_{j=1}^{r} (\vartheta_j \star_{\mathcal{AD}} \widehat{\pi}^{*}_{\sigma(j)}), \tag{18}$$

where $\widehat{\pi}^{*}_{\sigma(j)}$ denotes the jth ordered transformed PFN.

Definition 4.4 (Weighted geometric operator)**.** The Pythagorean fuzzy Dombi–Archimedean weighted geometric operator is

$$\mathrm{PFADWG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigotimes_{j=1}^{r} (\omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_j). \tag{19}$$

The corresponding closed form is derived in Theorem 4.5.

Definition 4.5 (Ordered and hybrid geometric operators)**.** The ordered weighted geometric operator and hybrid geometric operator are respectively given by

$$\mathrm{PFADOWG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigotimes_{j=1}^{r} (\omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}), \tag{20}$$

$$\mathrm{PFADHG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigotimes_{j=1}^{r} (\omega_j \diamond_{\mathcal{AD}} \widehat{\pi}^{*}_{\sigma(j)}). \tag{21}$$

4.1. Operator theorems and direct proofs

Each theorem in this subsection is followed immediately by its proof so that the algebraic validity of the proposed operators can be verified within the same logical development.

Theorem 4.1 (Algebraic consistency of the Dombi–Archimedean operations)**.** Let $\widehat{\pi}_1 = (\alpha_1, \beta_1)$, $\widehat{\pi}_2 = (\alpha_2, \beta_2)$, and $\widehat{\pi}_3 = (\alpha_3, \beta_3)$ be PFNs, and let $\lambda, \lambda_1, \lambda_2 > 0$. Under the admissibility assumptions of the generators η and ζ, the following relations hold:

$$\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2 = \widehat{\pi}_2 \oplus_{\mathcal{AD}} \widehat{\pi}_1, \tag{22}$$

$$\widehat{\pi}_1 \otimes_{\mathcal{AD}} \widehat{\pi}_2 = \widehat{\pi}_2 \otimes_{\mathcal{AD}} \widehat{\pi}_1, \tag{23}$$

$$\lambda \star_{\mathcal{AD}} (\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2) = (\lambda \star_{\mathcal{AD}} \widehat{\pi}_1) \oplus_{\mathcal{AD}} (\lambda \star_{\mathcal{AD}} \widehat{\pi}_2), \tag{24}$$

$$\lambda \diamond_{\mathcal{AD}} (\widehat{\pi}_1 \otimes_{\mathcal{AD}} \widehat{\pi}_2) = (\lambda \diamond_{\mathcal{AD}} \widehat{\pi}_1) \otimes_{\mathcal{AD}} (\lambda \diamond_{\mathcal{AD}} \widehat{\pi}_2), \tag{25}$$

$$(\lambda_1 + \lambda_2) \star_{\mathcal{AD}} \widehat{\pi}_1 = (\lambda_1 \star_{\mathcal{AD}} \widehat{\pi}_1) \oplus_{\mathcal{AD}} (\lambda_2 \star_{\mathcal{AD}} \widehat{\pi}_1), \tag{26}$$

$$(\lambda_1 + \lambda_2) \diamond_{\mathcal{AD}} \widehat{\pi}_1 = (\lambda_1 \diamond_{\mathcal{AD}} \widehat{\pi}_1) \otimes_{\mathcal{AD}} (\lambda_2 \diamond_{\mathcal{AD}} \widehat{\pi}_1). \tag{27}$$

Proof. For (22), the membership part of $\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2$ is governed by

$$\eta^{-1}\{\Lambda_{\xi}(\alpha_1^2)\} + \eta^{-1}\{\Lambda_{\xi}(\alpha_2^2)\},$$

and the non-membership part is governed by

$$\zeta^{-1}\{\Gamma_{\xi}(\beta_1^2)\} + \zeta^{-1}\{\Gamma_{\xi}(\beta_2^2)\}.$$

Both sums are unchanged when the subscripts 1 and 2 are interchanged; hence $\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2 = \widehat{\pi}_2 \oplus_{\mathcal{AD}} \widehat{\pi}_1$. The same calculation for

$$\zeta^{-1}\{\Gamma_{\xi}(\alpha_1^2)\} + \zeta^{-1}\{\Gamma_{\xi}(\alpha_2^2)\}, \qquad \eta^{-1}\{\Lambda_{\xi}(\beta_1^2)\} + \eta^{-1}\{\Lambda_{\xi}(\beta_2^2)\}$$

gives (23).

For (24), the membership scale of $\lambda \star_{\mathcal{AD}} (\widehat{\pi}_1 \oplus_{\mathcal{AD}} \widehat{\pi}_2)$ is

$$\lambda\eta^{-1}\{\Lambda_{\xi}(\alpha_1^2)\} + \lambda\eta^{-1}\{\Lambda_{\xi}(\alpha_2^2)\}.$$

The membership scale of $(\lambda \star_{\mathcal{AD}} \widehat{\pi}_1) \oplus_{\mathcal{AD}} (\lambda \star_{\mathcal{AD}} \widehat{\pi}_2)$ is the identical sum. Its non-membership scale is

$$\lambda\zeta^{-1}\{\Gamma_{\xi}(\beta_1^2)\} + \lambda\zeta^{-1}\{\Gamma_{\xi}(\beta_2^2)\},$$

which is also identical on both sides. Therefore (24) holds.

For (25), the two sides have membership scale

$$\lambda\zeta^{-1}\{\Gamma_{\xi}(\alpha_1^2)\} + \lambda\zeta^{-1}\{\Gamma_{\xi}(\alpha_2^2)\}$$

and non-membership scale

$$\lambda\eta^{-1}\{\Lambda_{\xi}(\beta_1^2)\} + \lambda\eta^{-1}\{\Lambda_{\xi}(\beta_2^2)\}.$$

Thus (25) follows.

For (26), the left-hand membership scale is $(\lambda_1 + \lambda_2)\eta^{-1}\{\Lambda_{\xi}(\alpha_1^2)\}$, while the right-hand side produces

$$\lambda_1\eta^{-1}\{\Lambda_{\xi}(\alpha_1^2)\} + \lambda_2\eta^{-1}\{\Lambda_{\xi}(\alpha_1^2)\},$$

which is equal to the left-hand scale. The non-membership scale gives the same equality with $\zeta^{-1}\{\Gamma_{\xi}(\beta_1^2)\}$. Equation (27) is obtained by the corresponding computation with the geometric scales $\zeta^{-1}\{\Gamma_{\xi}(\alpha_1^2)\}$ and $\eta^{-1}\{\Lambda_{\xi}(\beta_1^2)\}$. Hence all six relations are proved. □

Theorem 4.2 (Closed form of the PFADAWA operator)**.** Let $\widehat{\pi}_j = (\alpha_j, \beta_j)$, $j = 1, 2, \ldots, r$, be PFNs and let $\omega_j \geq 0$ with $\sum_{j=1}^{r} \omega_j = 1$. Then the operator in Equation (16) has the closed form

$$\mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_A, \mathfrak{n}_A), \tag{28}$$

where

$$\mathfrak{m}_A = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(\alpha_j^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{29}$$

$$\mathfrak{n}_A = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(\beta_j^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{30}$$

Proof. The proof is by induction on the number of aggregated PFNs. For $r = 1$, the weight condition gives $\omega_1 = 1$, and Equation (28) reduces to $\widehat{\pi}_1$. For $r = 2$, Definition (16) gives

$$\mathrm{PFADAWA}(\widehat{\pi}_1, \widehat{\pi}_2) = (\omega_1 \star_{\mathcal{AD}} \widehat{\pi}_1) \oplus_{\mathcal{AD}} (\omega_2 \star_{\mathcal{AD}} \widehat{\pi}_2).$$

Using the scalar operation in Equation (10) and the addition operation in Equation (8), the transformed membership scale becomes

$$\omega_1 \eta^{-1}\{\Lambda_\xi(\alpha_1^2)\} + \omega_2 \eta^{-1}\{\Lambda_\xi(\alpha_2^2)\},$$

and the transformed non-membership scale becomes

$$\omega_1 \zeta^{-1}\{\Gamma_\xi(\beta_1^2)\} + \omega_2 \zeta^{-1}\{\Gamma_\xi(\beta_2^2)\}.$$

Substituting these two sums into the inverse Dombi–Archimedean maps gives Equations (29) and (30) for $r = 2$.

Assume the result is true for $r = s$. Then

$$\mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_s) = (\mathfrak{m}_{A,s}, \mathfrak{n}_{A,s}),$$

where the two components are obtained from the sums over $j = 1, \ldots, s$. Aggregating one more term gives

$$\begin{aligned}\mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_s, \widehat{\pi}_{s+1}) = {} & \mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_s) \\ & \oplus_{\mathcal{AD}} (\omega_{s+1} \star_{\mathcal{AD}} \widehat{\pi}_{s+1}).\end{aligned}$$

The first factor contributes the transformed sums up to s, while the second contributes

$$\omega_{s+1} \eta^{-1}\{\Lambda_\xi(\alpha_{s+1}^2)\}, \qquad \omega_{s+1} \zeta^{-1}\{\Gamma_\xi(\beta_{s+1}^2)\}.$$

The new transformed sums are therefore exactly the sums from $j = 1$ to $s + 1$. This proves the formula for $s + 1$, and the induction is complete. □

Theorem 4.3 (Closed form of the PFADOWA operator)**.** Let σ be a permutation such that

$$\mathscr{S}(\widehat{\pi}_{\sigma(1)}) \geq \mathscr{S}(\widehat{\pi}_{\sigma(2)}) \geq \cdots \geq \mathscr{S}(\widehat{\pi}_{\sigma(r)}).$$

Then

$$\mathrm{PFADOWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_{OA}, \mathfrak{n}_{OA}), \tag{31}$$

where

$$\mathfrak{m}_{OA} = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(\alpha_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{32}$$

$$\mathfrak{n}_{OA} = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(\beta_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{33}$$

Proof. After the score-ordering permutation has been fixed, the ordered input sequence is

$$(\widehat{\pi}_{\sigma(1)}, \widehat{\pi}_{\sigma(2)}, \ldots, \widehat{\pi}_{\sigma(r)}).$$

By Definition (17), the operator is

$$\mathrm{PFADOWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigoplus_{j=1}^{r} (\omega_j \star_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}).$$

The jth scalar term contributes the transformed membership value

$$\omega_j \eta^{-1}\{\Lambda_\xi(\alpha_{\sigma(j)}^2)\}$$

and the transformed non-membership value

$$\omega_j \zeta^{-1}\{\Gamma_\xi(\beta_{\sigma(j)}^2)\}.$$

Adding these contributions over $j = 1, 2, \ldots, r$ gives the two sums appearing in $\mathfrak{m}_{OA}$ and $\mathfrak{n}_{OA}$. Substitution into the Dombi–Archimedean inverse forms therefore yields Equation (31). □

Theorem 4.4 (Closed form of the PFADHA operator)**.** Let $\Omega = (\Omega_1, \ldots, \Omega_r)$ be a criterion-weight vector and let $\omega = (\omega_1, \ldots, \omega_r)$ be an ordered-position vector. Define

$$\widehat{\pi}_j^{\circ} = r\Omega_j \star_{\mathcal{AD}} \widehat{\pi}_j = (u_j, v_j), \qquad j = 1, 2, \ldots, r, \tag{34}$$

and let $\widehat{\pi}_{\sigma(j)}^{\circ} = (u_{\sigma(j)}, v_{\sigma(j)})$ be the descending score order of the transformed PFNs. Then

$$\mathrm{PFADHA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_{HA}, \mathfrak{n}_{HA}), \tag{35}$$

where

$$\mathfrak{m}_{HA} = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(u_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{36}$$

$$\mathfrak{n}_{HA} = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(v_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{37}$$

Proof. The first stage of PFADHA transforms each original PFN into $\widehat{\pi}_j^{\circ} = r\Omega_j \star_{\mathcal{AD}} \widehat{\pi}_j = (u_j, v_j)$. After score ordering, the aggregation definition becomes

$$\mathrm{PFADHA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigoplus_{j=1}^{r} (\omega_j \star_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}^{\circ}).$$

The scalar contribution of $\widehat{\pi}_{\sigma(j)}^{\circ}$ to the membership branch is

$$\omega_j \eta^{-1}\{\Lambda_\xi(u_{\sigma(j)}^2)\},$$

and its contribution to the non-membership branch is

$$\omega_j \zeta^{-1}\{\Gamma_\xi(v_{\sigma(j)}^2)\}.$$

Summing these expressions over all ordered positions and substituting the two sums into the inverse Dombi–Archimedean averaging maps gives (36) and (37). Hence Equation (35) is proved directly from the definition. □

Theorem 4.5 (Closed form of the PFADWG operator)**.** For PFNs $\widehat{\pi}_j = (\alpha_j, \beta_j)$ and weights $\omega_j \geq 0$ with $\sum_{j=1}^{r} \omega_j = 1$, the weighted geometric operator has the closed form

$$\mathrm{PFADWG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_G, \mathfrak{n}_G), \tag{38}$$

where

$$\mathfrak{m}_G = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(\alpha_j^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{39}$$

$$\mathfrak{n}_G = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(\beta_j^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{40}$$

Proof. For $r = 1$, $\omega_1 = 1$ and Equation (38) returns $\widehat{\pi}_1$. For $r = 2$, Definition (19) gives

$$\mathrm{PFADWG}(\widehat{\pi}_1, \widehat{\pi}_2) = (\omega_1 \diamond_{\mathcal{AD}} \widehat{\pi}_1) \otimes_{\mathcal{AD}} (\omega_2 \diamond_{\mathcal{AD}} \widehat{\pi}_2).$$

The membership branch of the first scalar term contributes $\omega_1 \zeta^{-1}\{\Gamma_\xi(\alpha_1^2)\}$ and the second contributes $\omega_2 \zeta^{-1}\{\Gamma_\xi(\alpha_2^2)\}$. Thus the combined membership scale is

$$\omega_1 \zeta^{-1}\{\Gamma_\xi(\alpha_1^2)\} + \omega_2 \zeta^{-1}\{\Gamma_\xi(\alpha_2^2)\}.$$

The non-membership branch contributes

$$\omega_1 \eta^{-1}\{\Lambda_\xi(\beta_1^2)\} + \omega_2 \eta^{-1}\{\Lambda_\xi(\beta_2^2)\}.$$

Mapping these two sums back through the geometric Dombi–Archimedean forms gives (39) and (40) for two inputs.

Assume the formula holds for $r = s$. Then the transformed membership sum is

$$\sum_{j=1}^{s} \omega_j \zeta^{-1}\{\Gamma_\xi(\alpha_j^2)\},$$

and the transformed non-membership sum is

$$\sum_{j=1}^{s} \omega_j \eta^{-1}\{\Lambda_\xi(\beta_j^2)\}.$$

Adding $\omega_{s+1} \diamond_{\mathcal{AD}} \widehat{\pi}_{s+1}$ appends the terms

$$\omega_{s+1} \zeta^{-1}\{\Gamma_\xi(\alpha_{s+1}^2)\}, \qquad \omega_{s+1} \eta^{-1}\{\Lambda_\xi(\beta_{s+1}^2)\}.$$

The two sums therefore extend from $j = 1, \ldots, s$ to $j = 1, \ldots, s+1$. The induction is complete, so the closed form holds for every r. □

Theorem 4.6 (Closed form of the PFADOWG operator)**.** Let σ arrange the PFNs in descending score order. Then

$$\mathrm{PFADOWG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_{OG}, \mathfrak{n}_{OG}), \tag{41}$$

where

$$\mathfrak{m}_{OG} = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(\alpha_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{42}$$

$$\mathfrak{n}_{OG} = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(\beta_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{43}$$

Proof. By Definition (20),

$$\mathrm{PFADOWG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigotimes_{j=1}^{r} (\omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}).$$

For the ordered term $\widehat{\pi}_{\sigma(j)}$, the membership contribution under $\diamond_{\mathcal{AD}}$ is

$$\omega_j \zeta^{-1}\{\Gamma_\xi(\alpha_{\sigma(j)}^2)\},$$

and the non-membership contribution is

$$\omega_j \eta^{-1}\{\Lambda_\xi(\beta_{\sigma(j)}^2)\}.$$

The product aggregation adds these transformed contributions over all ordered positions. Substituting the resulting two sums into the geometric inverse forms gives (42) and (43). Thus (41) follows directly. □

Theorem 4.7 (Closed form of the PFADHG operator)**.** Let $\Omega = (\Omega_1, \ldots, \Omega_r)$ and define

$$\widehat{\pi}_j^{\diamond} = r\Omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_j = (s_j, t_j), \qquad j = 1, 2, \ldots, r. \tag{44}$$

Let $\widehat{\pi}_{\sigma(j)}^{\diamond} = (s_{\sigma(j)}, t_{\sigma(j)})$ denote the descending score order of the transformed PFNs. Then

$$\mathrm{PFADHG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = (\mathfrak{m}_{HG}, \mathfrak{n}_{HG}), \tag{45}$$

where

$$\mathfrak{m}_{HG} = \sqrt{\left(1 + \left\{\zeta\left[\sum_{j=1}^{r} \omega_j \zeta^{-1}\{\Gamma_\xi(s_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}, \tag{46}$$

$$\mathfrak{n}_{HG} = \sqrt{1 - \left(1 + \left\{\eta\left[\sum_{j=1}^{r} \omega_j \eta^{-1}\{\Lambda_\xi(t_{\sigma(j)}^2)\}\right]\right\}^{1/\xi}\right)^{-1}}. \tag{47}$$

Proof. The hybrid geometric construction first produces $\widehat{\pi}_j^{\diamond} = r\Omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_j = (s_j, t_j)$. After score ordering, the definition becomes

$$\mathrm{PFADHG}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \bigotimes_{j=1}^{r} (\omega_j \diamond_{\mathcal{AD}} \widehat{\pi}_{\sigma(j)}^{\diamond}).$$

The ordered transformed input $\widehat{\pi}_{\sigma(j)}^{\diamond}$ contributes

$$\omega_j \zeta^{-1}\{\Gamma_\xi(s_{\sigma(j)}^2)\}$$

to the membership scale and

$$\omega_j \eta^{-1}\{\Lambda_\xi(t_{\sigma(j)}^2)\}$$

to the non-membership scale. Summing these terms over $j = 1, 2, \ldots, r$ and applying the geometric inverse transformations proves (46) and (47). Hence the stated PFADHG closed form is established. □

Theorem 4.8 (Idempotency of the non-hybrid operators)**.** If $\widehat{\pi}_1 = \widehat{\pi}_2 = \cdots = \widehat{\pi}_r = \widehat{\pi}$, $\sum_{j=1}^{r} \omega_j = 1$, and $\sum_{j=1}^{r} \vartheta_j = 1$, then

$$\mathrm{PFADAWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \mathrm{PFADOWA}(\widehat{\pi}_1, \ldots, \widehat{\pi}_r) = \widehat{\pi}$$

and

$$\mathrm{PFADWG}(\widehat{\pi}_1,\ldots,\widehat{\pi}_r)=\mathrm{PFADOWG}(\widehat{\pi}_1,\ldots,\widehat{\pi}_r)=\widehat{\pi}.$$

No idempotency claim is made here for PFADHA or PFADHG, because their criterion-dependent pre-weighting generally transforms identical inputs into non-identical PFNs.

Proof. For PFADAWA, the membership scale is

$$\sum_{j=1}^{r}\omega_j\eta^{-1}\{\Lambda_\xi(\alpha^2)\}=\eta^{-1}\{\Lambda_\xi(\alpha^2)\},$$

and the non-membership scale is

$$\sum_{j=1}^{r}\omega_j\zeta^{-1}\{\Gamma_\xi(\beta^2)\}=\zeta^{-1}\{\Gamma_\xi(\beta^2)\}.$$

The inverse scale transformations return α^2 and β^2, and the square root returns $\widehat{\pi}$. Score ordering does not change identical inputs; replacing ω by ϑ proves the PFADOWA case. The same reasoning on the dual geometric scales proves the PFADWG and PFADOWG cases. □

Theorem 4.9 (Componentwise boundedness of the proposed operators)**.** Define the componentwise lower and upper PFNs by

$$\widehat{\pi}_c^-=\left(\min_j\alpha_j,\max_j\beta_j\right),\qquad \widehat{\pi}_c^+=\left(\max_j\alpha_j,\min_j\beta_j\right),\tag{48}$$

and define $\widehat{\pi}_a\preceq_{\mathrm{c}}\widehat{\pi}_b$ if $\alpha_a\le\alpha_b$ and $\beta_a\ge\beta_b$. Then every non-hybrid proposed averaging or geometric operator $\mathcal{A}$ satisfies

$$\widehat{\pi}_c^-\preceq_{\mathrm{c}}\mathcal{A}(\widehat{\pi}_1,\ldots,\widehat{\pi}_r)\preceq_{\mathrm{c}}\widehat{\pi}_c^+.\tag{49}$$

The same conclusion holds for hybrid operators when the normalized hybrid transformation maps all transformed PFNs into the same componentwise interval.

Proof. The previous score-order statement has been replaced by a componentwise statement because score order alone is not a complete componentwise order. Let $a^-=\min_j\alpha_j$ and $a^+=\max_j\alpha_j$. Since $\Lambda_\xi(x)$ is increasing in $x\in(0,1)$ and the admissible generator scale is monotone, every transformed membership value lies between the transformed values of $(a^-)^2$ and $(a^+)^2$. A convex weighted sum of values lying in this interval also lies in the same interval. Applying the inverse generator, the inverse Dombi map, and the square root preserves the same membership bound.

Similarly, let $b^-=\min_j\beta_j$ and $b^+=\max_j\beta_j$. The non-membership aggregation is governed by the transformed Dombi scale Γ_ξ together with the decreasing inverse PF map; therefore the aggregated non-membership component lies between b^- and b^+. Hence the aggregated PFN is bounded between $\widehat{\pi}_c^-$ and $\widehat{\pi}_c^+$ under $\preceq_{\mathrm{c}}$. The hybrid case follows when the normalized hybrid pre-weighted PFNs remain in the componentwise interval before the final ordered aggregation. □

Theorem 4.10 (Monotonicity of the weighted non-ordered operators)**.** Let $\widehat{\pi}_j=(\alpha_j,\beta_j)$ and $\widehat{\pi}'_j=(\alpha'_j,\beta'_j)$ satisfy $\alpha_j\le\alpha'_j$ and $\beta_j\ge\beta'_j$ for every j. Under the identity-generator specialization, and under any admissible general generator pair whose composite branch mappings have the same monotonic directions,

$$\mathrm{PFADAWA}(\widehat{\pi}_1,\ldots,\widehat{\pi}_r)\preceq_{\mathrm{c}}\mathrm{PFADAWA}(\widehat{\pi}'_1,\ldots,\widehat{\pi}'_r),\tag{50}$$

$$\mathrm{PFADWG}(\widehat{\pi}_1,\ldots,\widehat{\pi}_r)\preceq_{\mathrm{c}}\mathrm{PFADWG}(\widehat{\pi}'_1,\ldots,\widehat{\pi}'_r).\tag{51}$$

Proof. For PFADAWA under the identity-generator specialization, define

$$F(z)=\sqrt{1-\frac{1}{1+z^{1/\xi}}},\qquad z_\alpha=\sum_{j=1}^{r}\omega_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^{\xi}.$$

Both $\alpha\mapsto(\alpha^2/(1-\alpha^2))^{\xi}$ and F are increasing; hence the aggregated membership cannot decrease when every α_j increases. For the non-membership component, let

$$G(z)=\sqrt{\frac{1}{1+z^{1/\xi}}},\qquad z_\beta=\sum_{j=1}^{r}\omega_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^{\xi}.$$

The inner map is decreasing in β, and G is decreasing in its argument. Therefore, when every β_j decreases, the aggregated non-membership component cannot increase. This proves PFADAWA monotonicity under $\preceq_{\mathrm{c}}$. PFADWG follows by interchanging the two dual branches. The same argument applies to the general generator form when the stated composite branch mappings preserve these monotonic directions. □

Remark 1 (Ordered and hybrid operators)**.** Global componentwise monotonicity is not asserted for PFADOWA, PFADOWG, PFADHA, or PFADHG under score-based ordering, because modifying inputs may change the score permutation and score order is not equivalent to componentwise order. Conditional monotonicity holds when the ordering permutation remains fixed, but that restricted statement is not used in the numerical application.

5. MCDM procedure under Pythagorean fuzzy information

Consider the alternative set $\{\overline{A}_1,\overline{A}_2,\ldots,\overline{A}_m\}$ and the criterion set $\{\overline{C}_1,\overline{C}_2,\ldots,\overline{C}_n\}$. The expert evaluations are arranged in the Pythagorean fuzzy decision matrix

$$\mathcal{R}=[\widehat{\pi}_{ij}]_{m\times n}=[(\alpha_{ij},\beta_{ij})]_{m\times n}.\tag{52}$$

For benefit-type criteria, the original entries are retained. For cost-type criteria, normalization is carried out by interchanging the membership and non-membership degrees:

$$\widehat{\pi}_{ij}^N=\begin{cases}(\alpha_{ij},\beta_{ij}), & \overline{C}_j \text{ is a benefit criterion,}\\ (\beta_{ij},\alpha_{ij}), & \overline{C}_j \text{ is a cost criterion.}\end{cases}\tag{53}$$

The overall PFN associated with alternative $\overline{A}_i$ is then computed by

$$\widehat{\pi}_i=\mathrm{PFADAWA}(\widehat{\pi}_{i1}^N,\widehat{\pi}_{i2}^N,\ldots,\widehat{\pi}_{in}^N),\tag{54}$$

and the final ordering is obtained by arranging the score values $\mathscr{S}(\widehat{\pi}_i)$ in descending order.

Algorithm 1 Pythagorean fuzzy Dombi–Archimedean MCDM procedure.

1: Input the PF decision matrix $\mathcal{R}$, the weight vector ω, and the flexibility parameter ξ.
2: Classify each criterion as benefit or cost.
3: Normalize the matrix using Equation (53).
4: Aggregate each alternative by PFADAWA, PFADOWA, PFADWG, or PFADOWG.
5: Compute the score and accuracy values using Equation (3).
6: Rank the alternatives in descending order of score; use accuracy only for ties.
7: Output the best alternative and the complete ranking order.

5.1. Computational complexity

For an MCDM problem with m alternatives and n criteria, normalization requires $O(mn)$ operations. Aggregation of all alternatives by PFADAWA or PFADWG also requires $O(mn)$ scale evaluations and weighted additions. If ordered operators are used, an additional sorting cost of $O(mn \log n)$ is required because the n criterion-wise PFNs are ordered for each alternative. The final score computation and ranking require $O(m)$ and $O(m \log m)$ operations, respectively. Therefore, the total complexity is

$$O(mn + m \log m) \quad \text{for weighted operators,} \tag{55}$$

$$O(mn \log n + m \log m) \quad \text{for ordered and hybrid operators.} \tag{56}$$

The memory requirement is $O(mn)$ when the full normalized matrix is stored and $O(n)$ when alternatives are processed sequentially.

The workflow in Figure 2 summarizes the computational sequence used in the proposed MCDM procedure.

6. Case study: network-security system selection

A technology-oriented organization seeks to select a network-security system capable of reducing unauthorized access, denial-of-service disruption, eavesdropping, and multi-vector attack exposure. Six candidate systems, denoted by $\overline{A}_1, \ldots, \overline{A}_6$, are assessed with respect to the following four criteria:

$\overline{C}_1$: risk of back-door exploitation;

$\overline{C}_2$: resistance to denial-of-service attacks;

$\overline{C}_3$: resistance to eavesdropping and passive monitoring;

$\overline{C}_4$: risk of multi-vector compromise.

The criteria $\overline{C}_1$ and $\overline{C}_4$ are now explicitly defined as cost-oriented risk criteria, while $\overline{C}_2$ and $\overline{C}_3$ remain benefit-oriented protection criteria. Thus, normalization by interchanging membership and non-membership degrees is applied only to risk-type criteria. The PF decision matrix is an illustrative expert-style benchmark constructed to demonstrate the proposed method; it is not presented as a confidential real-world audit dataset. All values needed to reproduce the computations are reported in Tables 3–5. The criterion-weight vector is

$$\omega = (0.3471, 0.2316, 0.2308, 0.1905), \qquad \sum_{j=1}^{4} \omega_j = 1. \tag{57}$$

For the ordered operators, the ordered-position weight vector is

$$\vartheta = (0.4, 0.3, 0.2, 0.1), \qquad \sum_{j=1}^{4} \vartheta_j = 1. \tag{58}$$

The criterion weights ω in Equation (57) and ordered-position weights ϑ in Equation (58) have distinct roles. The flexibility parameter is set to $\xi = 2$ unless stated otherwise.

As illustrated in Figure 3, $\overline{A}_2$ and $\overline{A}_6$ exhibit the strongest average membership profiles, while $\overline{A}_6$ has the lowest average non-membership degree.

To compute Tables 5 and 6, we use the identity generators $\eta(t) = \zeta(t) = t$ with $\eta^{-1}(t) = \zeta^{-1}(t) = t$ for $t \geq 0$, the flexibility parameter $\xi = 2$, the criterion weights in Equation (57), and, where ordering is required, the ordered-position weights in Equation (58). Substitution into Equations (14)–(54) yields the reported aggregated PFNs and score values.

Figure 4 shows that PFADAWA selects $\overline{A}_2$ as the leading alternative, whereas the stricter geometric aggregation favors $\overline{A}_6$. This difference reflects the distinct compensation behavior of the two operator families rather than a change in the underlying data.

To extend the numerical validation beyond a single weighting vector, three benchmark cybersecurity priority profiles are considered in Table 7. These profiles represent intrusion-prevention priority, communication-protection priority, and resilience priority. The leading alternative remains $\overline{A}_2$ in all benchmark profiles, while $\overline{A}_6$ remains second, showing that the principal decision is stable under cybersecurity-relevant changes in criterion emphasis.

Using the score function, the ranking orders produced by the four operators are obtained as follows. For the weighted averaging operator,

$$\text{PFADAWA}: \quad \overline{A}_2 \succ \overline{A}_6 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_4 \succ \overline{A}_1. \tag{59}$$

For the ordered weighted averaging operator,

$$\text{PFADOWA}: \quad \overline{A}_2 \succ \overline{A}_6 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_4 \succ \overline{A}_1. \tag{60}$$

For the weighted geometric operator,

$$\text{PFADWG}: \quad \overline{A}_6 \succ \overline{A}_2 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_1 \succ \overline{A}_4. \tag{61}$$

For the ordered weighted geometric operator,

$$\text{PFADOWG}: \quad \overline{A}_6 \succ \overline{A}_2 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_1 \succ \overline{A}_4. \tag{62}$$

The two averaging-type operators select $\overline{A}_2$ as the best network-security system, whereas the two geometric-type operators select $\overline{A}_6$ as the best alternative. This difference is mathematically

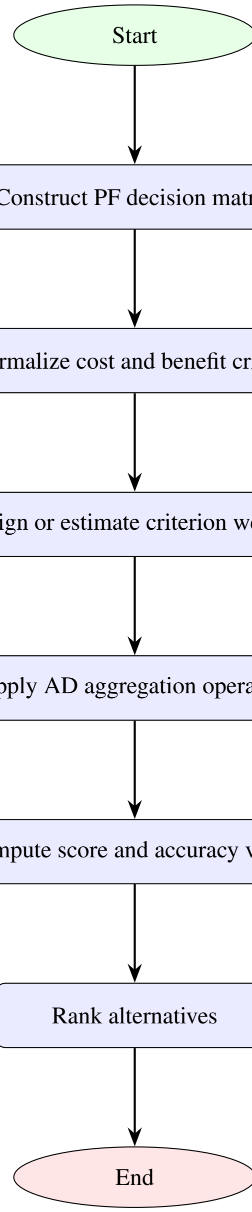

Figure 2: Flowchart of the Pythagorean fuzzy Dombi–Archimedean MCDM procedure.

Table 3: Expert PF decision matrix for network-security alternatives.

Alternative	$\overline{C}_1$	$\overline{C}_2$	$\overline{C}_3$	$\overline{C}_4$
$\overline{A}_1$	$\langle 0.7, 0.4\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.4, 0.8\rangle$	$\langle 0.5, 0.3\rangle$
$\overline{A}_2$	$\langle 0.7, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.3, 0.3\rangle$
$\overline{A}_3$	$\langle 0.6, 0.3\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.7, 0.4\rangle$
$\overline{A}_4$	$\langle 0.9, 0.4\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.7, 0.2\rangle$
$\overline{A}_5$	$\langle 0.7, 0.4\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.7, 0.2\rangle$
$\overline{A}_6$	$\langle 0.4, 0.6\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.3, 0.5\rangle$

meaningful because the averaging operators allow stronger compensation among criteria, while the geometric operators penalize weak criterion-wise evaluations more severely. Hence, the ordered and non-ordered versions confirm that the main competition is between $\overline{A}_2$ and $\overline{A}_6$, and the final preference depends on whether the decision maker adopts a compensatory or a stricter non-compensatory aggregation attitude.

Table 4: Normalized PF decision matrix.

Alternative	$\overline{C}_1$	$\overline{C}_2$	$\overline{C}_3$	$\overline{C}_4$
$\overline{A}_1$	$\langle 0.4, 0.7\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.4, 0.8\rangle$	$\langle 0.3, 0.5\rangle$
$\overline{A}_2$	$\langle 0.5, 0.7\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.3, 0.3\rangle$
$\overline{A}_3$	$\langle 0.3, 0.6\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.4, 0.7\rangle$
$\overline{A}_4$	$\langle 0.4, 0.9\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.2, 0.7\rangle$
$\overline{A}_5$	$\langle 0.4, 0.7\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.2, 0.7\rangle$
$\overline{A}_6$	$\langle 0.6, 0.4\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.5, 0.3\rangle$

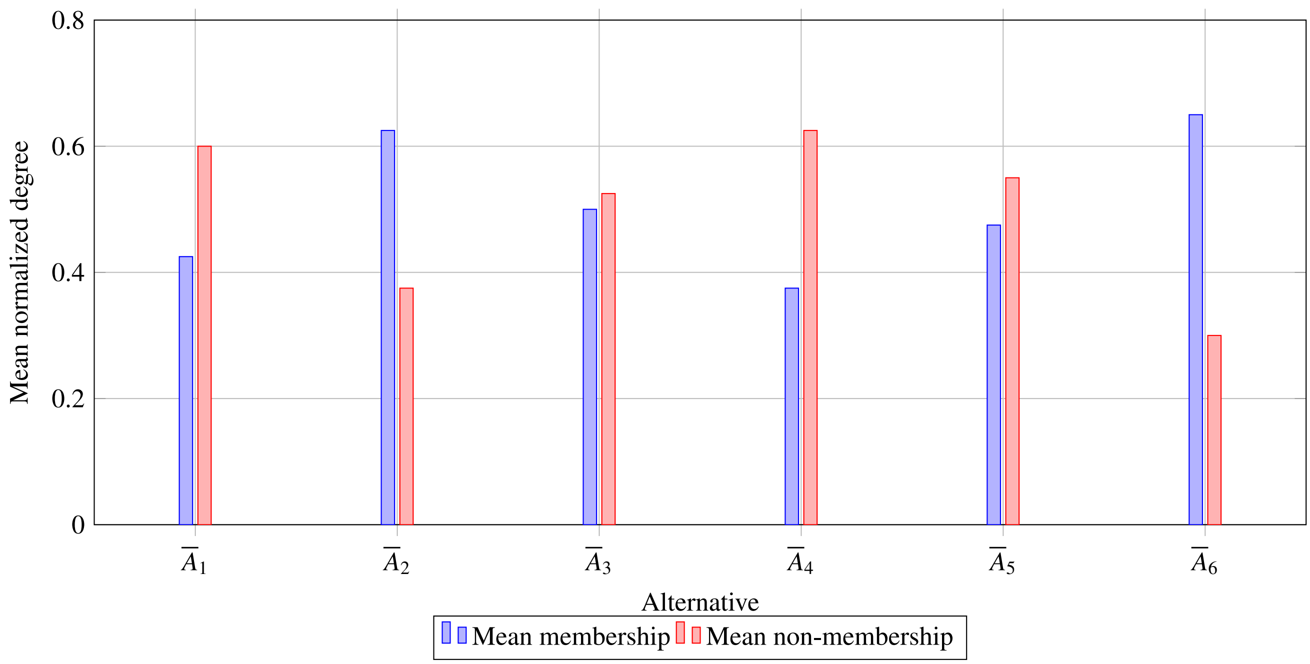

Figure 3: Mean membership and non-membership degrees obtained from the normalized PF decision matrix in Table 4. The figure provides a compact visual check of the alternative-wise normalized profiles before aggregation.

Table 5: Aggregated PFNs and scores for the operators at $\xi = 2$.

Alternative	PFADAWA	S	H	PFADWG	S	H
$\overline{A}_1$	$\langle 0.4865, 0.5048\rangle$	-0.0182	0.4915	$\langle 0.3777, 0.7138\rangle$	-0.3669	0.6522
$\overline{A}_2$	$\langle 0.8308, 0.2645\rangle$	0.6203	0.7602	$\langle 0.4166, 0.6026\rangle$	-0.1895	0.5366
$\overline{A}_3$	$\langle 0.5940, 0.4016\rangle$	0.1915	0.5140	$\langle 0.3662, 0.5991\rangle$	-0.2249	0.4930
$\overline{A}_4$	$\langle 0.4222, 0.4095\rangle$	0.0105	0.3459	$\langle 0.2851, 0.8481\rangle$	-0.6379	0.8005
$\overline{A}_5$	$\langle 0.5948, 0.4041\rangle$	0.1905	0.5171	$\langle 0.2891, 0.6481\rangle$	-0.3364	0.5035
$\overline{A}_6$	$\langle 0.7136, 0.2614\rangle$	0.4408	0.5775	$\langle 0.6008, 0.3401\rangle$	0.2452	0.4766

7. Sensitivity and comparative analysis

The flexibility parameter ξ controls the extent to which the Dombi scale affects the aggregation process. Changing ξ modifies the rate at which strong and weak criterion evaluations influence the final score. Table 8 reports the score values obtained by varying ξ from 1 to 5.

Figure 5 shows the alternative scores across the tested flexibility-parameter values.

The sensitivity results show that the strongest alternatives remain stable for most values of ξ. This supports the robustness of the leading decision while also indicating that middle-ranked alternatives may exchange positions when the aggregation flexibility is modified.

The sensitivity analysis was also extended to criterion weights. Table 9 reports the ranking obtained when each criterion weight is increased by 10% and the full weight vector is renormalized. The leading pair $\overline{A}_2 \succ \overline{A}_6$ is preserved in every

Table 6: Ordered-position calculations for PFADOWA and PFADOWG at $\xi = 2$ and $\vartheta = (0.4, 0.3, 0.2, 0.1)$.

Alternative	Descending score permutation	PFADOWA	S	PFADOWG	S
$\overline{A}_1$	$(\overline{C}_2, \overline{C}_4, \overline{C}_1, \overline{C}_3)$	$\langle 0.5221, 0.4594\rangle$	0.0615	$\langle 0.3691, 0.6523\rangle$	−0.2893
$\overline{A}_2$	$(\overline{C}_3, \overline{C}_2, \overline{C}_4, \overline{C}_1)$	$\langle 0.8611, 0.2476\rangle$	0.6802	$\langle 0.4213, 0.4889\rangle$	−0.0615
$\overline{A}_3$	$(\overline{C}_2, \overline{C}_3, \overline{C}_1, \overline{C}_4)$	$\langle 0.6377, 0.3614\rangle$	0.2761	$\langle 0.4097, 0.5524\rangle$	−0.1373
$\overline{A}_4$	$(\overline{C}_2, \overline{C}_3, \overline{C}_4, \overline{C}_1)$	$\langle 0.4425, 0.3655\rangle$	0.0622	$\langle 0.2840, 0.7692\rangle$	−0.5111
$\overline{A}_5$	$(\overline{C}_3, \overline{C}_2, \overline{C}_1, \overline{C}_4)$	$\langle 0.6250, 0.3621\rangle$	0.2595	$\langle 0.3325, 0.5993\rangle$	−0.2486
$\overline{A}_6$	$(\overline{C}_2, \overline{C}_3, \overline{C}_1, \overline{C}_4)$	$\langle 0.7481, 0.2381\rangle$	0.5030	$\langle 0.6444, 0.3134\rangle$	0.3170

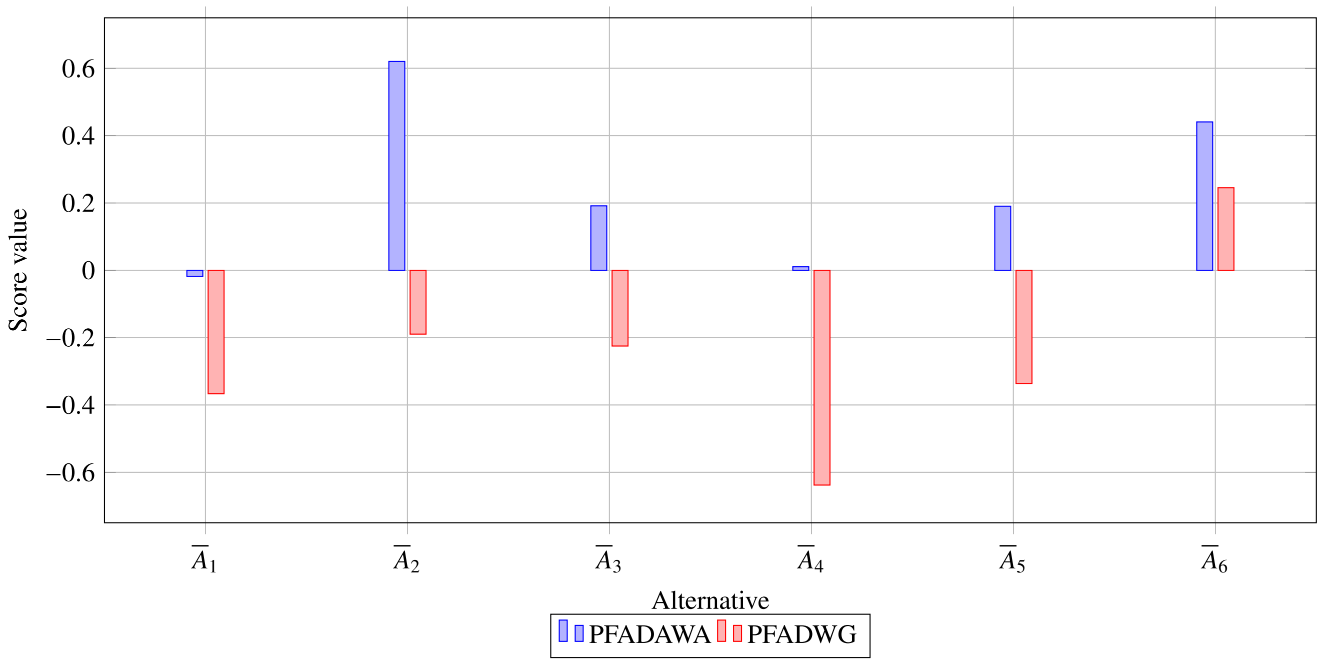

Figure 4: Alternative-wise score comparison for the PFADAWA and PFADWG results reported in Table 5.

Table 7: Benchmark profiles for additional robustness checking.

Benchmark profile	Weight vector $(\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4)$	Score vector $(\overline{A}_1, \ldots, \overline{A}_6)$	Ranking order
Intrusion-prevention	(0.4000, 0.2500, 0.2000, 0.1500)	(−0.0104, 0.6096, 0.1981, 0.0195, 0.1856, 0.4464)	$\overline{A}_2 \succ \overline{A}_6 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_4 \succ \overline{A}_1$
Communication-protection	(0.2500, 0.2000, 0.3500, 0.2000)	(−0.0372, 0.6551, 0.1880, −0.0025, 0.2162, 0.4372)	$\overline{A}_2 \succ \overline{A}_6 \succ \overline{A}_5 \succ \overline{A}_3 \succ \overline{A}_4 \succ \overline{A}_1$
Resilience	(0.2500, 0.2000, 0.2000, 0.3500)	(−0.0263, 0.6037, 0.1677, −0.0123, 0.1638, 0.4219)	$\overline{A}_2 \succ \overline{A}_6 \succ \overline{A}_3 \succ \overline{A}_5 \succ \overline{A}_4 \succ \overline{A}_1$

Table 8: Sensitivity of PFADAWA scores under different flexibility levels.

ξ	$S(\overline{A}_1)$	$S(\overline{A}_2)$	$S(\overline{A}_3)$	$S(\overline{A}_4)$	$S(\overline{A}_5)$	$S(\overline{A}_6)$
1	−0.0992	0.5153	0.0910	−0.0847	0.0810	0.3924
2	−0.0182	0.6203	0.1915	0.0105	0.1905	0.4408
3	0.0353	0.6670	0.2486	0.0503	0.2484	0.4770
4	0.0706	0.6926	0.2822	0.0722	0.2821	0.5023
5	0.0944	0.7086	0.3039	0.0868	0.3038	0.5200

perturbation scenario. Only the middle positions of $\overline{A}_3$ and $\overline{A}_5$ change when the confidentiality-related criterion $\overline{C}_3$ is emphasized.

Before Table 10, the comparison operators and parameter settings are specified explicitly. The classical PFWA/PFWG definitions follow the standard Pythagorean aggregation literature [21, 22], while the Pythagorean fuzzy Aczel–Alsina weighted averaging (PFAAWA) operator follows the Aczel–

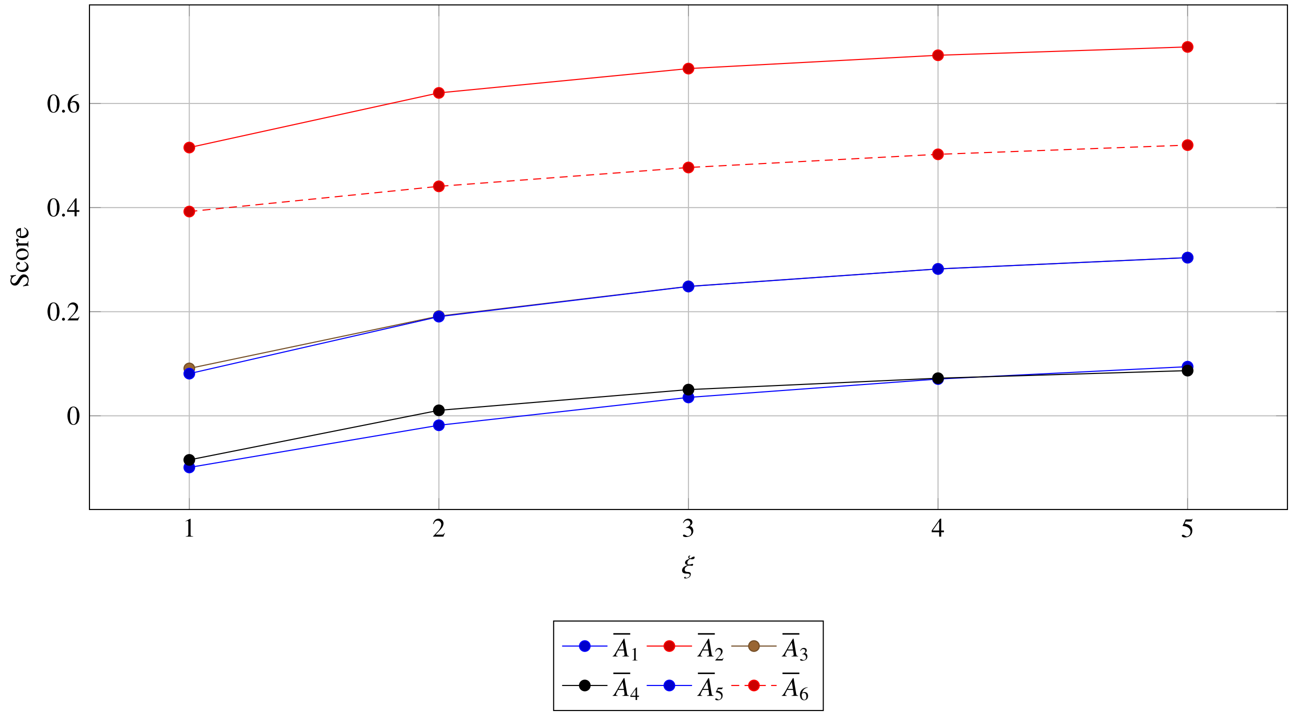

Figure 5: Score variation of the alternatives under different values of ξ.

Table 9: Criterion-weight sensitivity of PFADAWA scores at $\xi = 2$.

Scenario	Renormalized weight vector	Score vector $(\overline{A}_1, \ldots, \overline{A}_6)$	Ranking order
Baseline	$(0.3471, 0.2316, 0.2308, 0.1905)$	$(-0.0182, 0.6203, 0.1915, 0.0105, 0.1905, 0.4408)$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_3 > \overline{A}_5 > \overline{A}_4 > \overline{A}_1$
$+10\%\ \overline{C}_1$	$(0.3690, 0.2238, 0.2231, 0.1841)$	$(-0.0230, 0.6157, 0.1857, 0.0076, 0.1849, 0.4369)$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_3 > \overline{A}_5 > \overline{A}_4 > \overline{A}_1$
$+10\%\ \overline{C}_2$	$(0.3392, 0.2490, 0.2256, 0.1862)$	$(-0.0090, 0.6211, 0.2010, 0.0175, 0.1948, 0.4477)$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_3 > \overline{A}_5 > \overline{A}_4 > \overline{A}_1$
$+10\%\ \overline{C}_3$	$(0.3393, 0.2264, 0.2482, 0.1862)$	$(-0.0216, 0.6262, 0.1907, 0.0088, 0.1949, 0.4401)$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_5 > \overline{A}_3 > \overline{A}_4 > \overline{A}_1$
$+10\%\ \overline{C}_4$	$(0.3406, 0.2273, 0.2265, 0.2056)$	$(-0.0197, 0.6181, 0.1884, 0.0079, 0.1871, 0.4385)$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_3 > \overline{A}_5 > \overline{A}_4 > \overline{A}_1$

Alsina construction in [18]. For the Pythagorean fuzzy Aczel–Alsina weighted averaging operator, the parameter is set to $p = 2$. All comparison methods use the normalized matrix in Table 4, the criterion weights in Equation (57), and the score function in Equation (3). The values in Table 10 compare the proposed method with the existing operators.

Figure 6 presents the rank-correlation comparison with PFADAWA.

The comparative results show that the proposed averaging operator creates a clear separation between the leading system and the remaining alternatives. Existing PF aggregation operators may produce different middle-rank positions because their compensation behavior is fixed. By contrast, the Dombi–Archimedean operators introduce an adjustable mechanism through ξ, enabling decision makers to examine whether the ranking remains stable under stricter or more permissive aggregation settings.

7.1. Practical implications for cybersecurity decision makers

The proposed framework supports cybersecurity planning in three ways. First, it separates risk-type criteria from benefit-type criteria, which helps analysts avoid mixing exposure indicators with protection indicators. Second, the flexibility parameter ξ allows a security committee to examine whether a preferred system remains attractive when weak criterion values are penalized more strongly. Third, the weight-sensitivity and benchmark-profile analyses show which alternatives are stable under different operational priorities. In practice, the method may be used as a transparent screening tool before detailed penetration testing, cost assessment, and compliance evaluation under cybersecurity-management frameworks.

8. Conclusion

This study developed a Pythagorean fuzzy Dombi–Archimedean aggregation framework for network-security decision making. The revision clarifies the relationship between the general generator model and the identity-generator Dombi-scale computation, states admissibility assumptions explicitly, strengthens closure and monotonicity arguments, replaces score-order boundedness with componentwise boundedness, and corrects the cost/benefit interpretation of the cy-

Table 10: Comparative ranking with representative PF aggregation operators.

Operator	Score vector $(\overline{A}_1, \ldots, \overline{A}_6)$	Ranking order
PFADAWA	$-0.0182, 0.6203, 0.1915, 0.0105, 0.1905, 0.4408$	$\overline{A}_2 > \overline{A}_6 > \overline{A}_3 > \overline{A}_5 > \overline{A}_4 > \overline{A}_1$
PFADWG	$-0.3669, -0.1895, -0.2249, -0.6379, -0.3364, 0.2452$	$\overline{A}_6 > \overline{A}_2 > \overline{A}_3 > \overline{A}_5 > \overline{A}_1 > \overline{A}_4$
PFWA	$0.2983, 0.3196, 0.1871, 0.3444, 0.3721, 0.3675$	$\overline{A}_5 > \overline{A}_6 > \overline{A}_4 > \overline{A}_2 > \overline{A}_1 > \overline{A}_3$
PFWG	$0.2549, 0.2951, 0.2227, 0.3183, 0.3333, 0.3213$	$\overline{A}_5 > \overline{A}_6 > \overline{A}_4 > \overline{A}_2 > \overline{A}_1 > \overline{A}_3$
PFAAWA	$0.2773, 0.2842, 0.2185, 0.3017, 0.3233, 0.3121$	$\overline{A}_5 > \overline{A}_6 > \overline{A}_4 > \overline{A}_2 > \overline{A}_1 > \overline{A}_3$

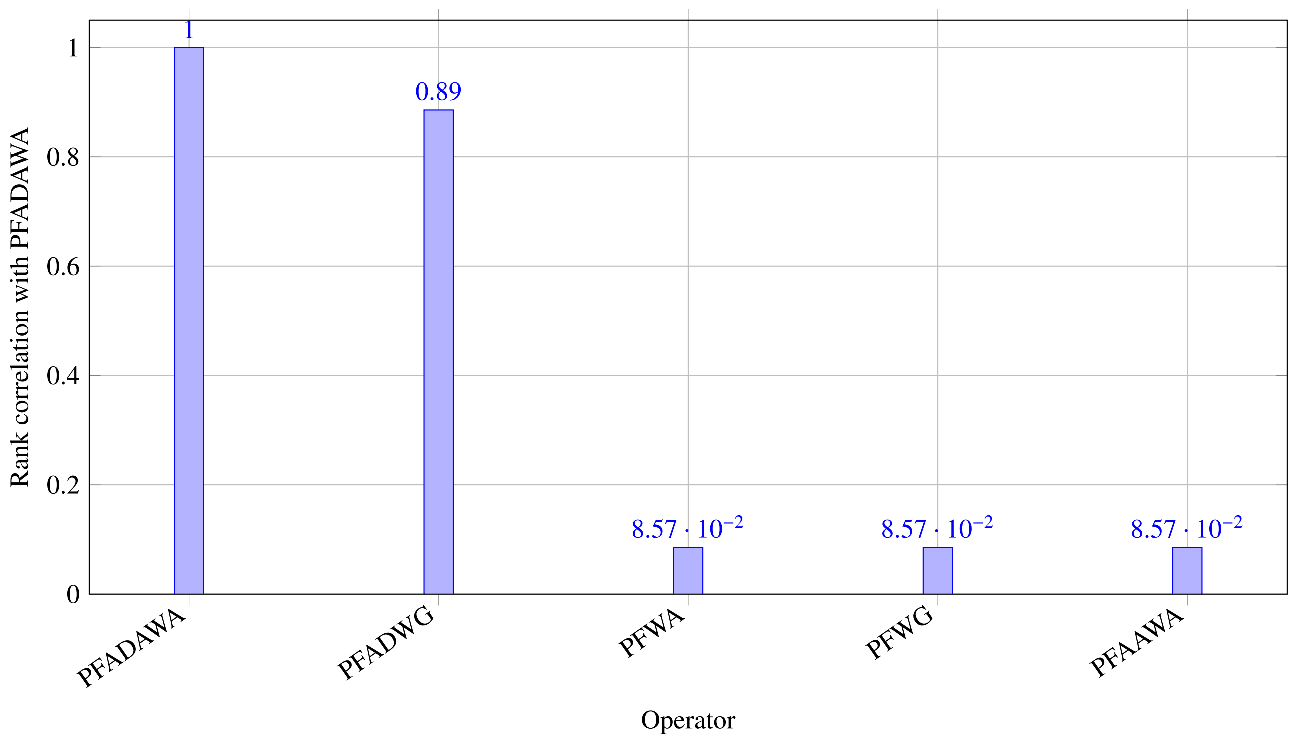

Figure 6: Rank-correlation comparison with PFADAWA. The figure avoids interpreting raw best-score magnitudes across operators as direct evidence of superiority.

bersecurity criteria. The proposed weighted averaging, ordered weighted averaging, hybrid averaging, weighted geometric, ordered weighted geometric, and hybrid geometric operators provide a flexible mechanism for combining uncertain expert assessments. The case study, sample computation, benchmark profiles, computational-complexity analysis, parameter sensitivity, and criterion-weight sensitivity demonstrate that the method is reproducible and practically interpretable.

Future research may extend the proposed framework in several important directions. First, the model can be generalized to group decision-making environments in which expert weights are partially known, dynamically changing, or completely unknown. Such an extension would make the framework more suitable for real cybersecurity panels, where the credibility, expertise, and reliability of different evaluators may vary across technical domains. Second, the proposed operators may be reformulated under interval-valued Pythagorean fuzzy information, q-rung orthopair fuzzy information, spherical fuzzy information, and complex-valued fuzzy environments to capture deeper forms of uncertainty, hesitation, and phase-based assessment. Third, future studies should validate the proposed decision-making structure using public cybersecurity datasets, real audit logs, vulnerability-assessment records, penetration-testing results, and incident-response reports. This would allow the ranking outcomes to be compared with empirical security performance rather than relying only on expert-constructed evaluations.

The proposed Pythagorean fuzzy Dombi–Archimedean framework provides a systematic mechanism for aggregating uncertain cybersecurity evaluations and ranking competing security alternatives. The numerical analysis demonstrates that the flexibility parameter, criterion weights, and aggregation structure can influence the final ranking; therefore, sensitivity analysis and transparent reporting are essential for reliable decision support. Future research may extend the framework through interval-valued assessments, group decision-making environments, objective weight-determination techniques, and validation using real cybersecurity performance data.

Future research may investigate topology-aware and propagation-sensitive extensions of the proposed fuzzy cybersecurity decision framework. In particular, the mathematical treatment of protected topological states and wave-localization phenomena in complex physical structures [50, 51] may provide abstract methodological inspiration for modelling structural robustness, disturbance propagation, and localized vulnerability patterns in interconnected cyber networks. Such an extension could combine Pythagorean fuzzy aggregation with graph topology, network-interaction structures, and dynamic attack-propagation mechanisms.

Further extensions of the proposed model may incorporate the aggregation of intuitionistic fuzzy (IF) functions with graph-theoretic topology and network interaction relations. Dynamic mechanisms could also be employed to model the propagation of cyber threats over time. In addition, the fixed-point and fractional approaches discussed in [50–58] may provide a foundation for developing fractional differential models of cyber-threat propagation under uncertainty. Such models could facilitate the analysis of threat stability, memory effects, temporal dynamics, and propagation behaviour.

NOMENCLATURE

$\widehat{\pi} = (\alpha, \beta)$	Pythagorean fuzzy number with membership and non-membership degrees
ξ	Dombi–Archimedean flexibility parameter
ω_i	Weight of the ith criterion or argument
$\mathscr{S}(\widehat{\pi})$	Score function used for ranking PFNs
PFADAWA	Pythagorean fuzzy Dombi–Archimedean weighted averaging operator
PFADWG	Pythagorean fuzzy Dombi–Archimedean weighted geometric operator

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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