



The group-theoretic origin of spin: the reality locus and cohomology of the spin spectrum

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Abstract

With the sole physical input that quantum symmetries act projectively on states, this work develops a group-theoretic account of spin as the obstruction to lifting a projective unitary representation to an ordinary one, controlled by two invariants: an infinitesimal class in $H^2(\mathfrak{g}, \mathbb{R})$ and a topological character in $\text{Hom}(\pi_1(G), \mathbb{T})$. A lifting dichotomy determines, for every connected Lie group, whether and how its projective representations lift; it recovers the double cover from the special unitary group $\text{SU}(2)$ to the special orthogonal group $\text{SO}(3)$, the half-integer spins, the Frobenius–Schur values $\nu(\rho_j) = (-1)^{2j}$, and the trichotomy of the rotation, Poincaré, and Galilei groups. In the topological regime, the spin sectors form the Pontryagin dual $\widehat{\pi_1(G)}$, a continuum for the planar rotation group $\text{SO}(2)$, where it yields fractional (anyonic) spin. Two further results are established. First, the self-conjugate sectors—a necessary condition for compatibility with an antiunitary time reversal—form the 2-torsion subgroup $\widehat{\pi_1(G)}[2]$, which remains finite even when the spectrum is a continuum; for the plane, it is $\{0, \frac{1}{2}\}$, explaining why generic anyons break time reversal. Second, the spectrum and this locus are the cohomology groups $H^1(G; \mathbb{T})$ and $H^1(G; \mathbb{Z}/2)$, and the Bockstein sequence of $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T}$ splits spin into a free part carrying the anyons and the torsion of $H^2(G; \mathbb{Z})$, an integral class whose modulo-2 reduction is, for the rotation group, the Stiefel–Whitney class obstructing spin structures on spacetime.

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1. Introduction

In quantum mechanics the pure states of a system are the rays of a complex Hilbert space $\mathcal{H}$, that is, the points of the projective space $\mathbb{P}(\mathcal{H})$. A symmetry is a bijection of $\mathbb{P}(\mathcal{H})$ that preserves transition probabilities, and Wigner's theorem represents every such bijection by a unitary or antiunitary operator of $\mathcal{H}$, uniquely determined up to a phase [1, 2]. A group G of symmetries therefore acts not through a homomorphism

into the unitary group $\text{U}(\mathcal{H})$ but through a homomorphism into the projective unitary group $\text{PU}(\mathcal{H}) = \text{U}(\mathcal{H})/\mathbb{T}$, where $\mathbb{T} = \text{U}(1)$ is the centre of scalars; that is, through a *projective* (or *ray*) representation. Whether a projective unitary representation $\sigma: G \rightarrow \text{PU}(\mathcal{H})$ lifts to an ordinary unitary representation $U: G \rightarrow \text{U}(\mathcal{H})$ with $\sigma = \pi \circ U$, $\pi: \text{U}(\mathcal{H}) \rightarrow \text{PU}(\mathcal{H})$ the quotient, is governed entirely by the structure of G , and the obstruction to lifting is the precise mathematical content of what is physically called *spin*.

The ingredients are long established. Weyl and Wigner recognized that quantum symmetries are represented projectively [3, 4]; Bargmann analysed ray representations of Lie groups

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and isolated the role of the second cohomology of the Lie algebra [2]; Mackey and Moore placed the theory of multipliers within the cohomology of locally compact groups [5–8]. This paper assembles from that theory a single *obstruction calculus*, supplies complete proofs of its central group-theoretic assertions, and applies it to organize within one framework the appearance of spin in the rotation group, its relativistic refinement in the Poincaré group, its non-relativistic counterpart in the Galilei group, and its absence from the internal gauge groups of the Standard Model. The synthesis consists of a single pair of invariants, a dichotomy that uses them, explicit computations of the resulting reality data, and a functorial behaviour under direct products that determines where spin can originate. The underlying results—Bargmann’s theorem, the Mackey–Moore cohomology of multipliers, Whitehead’s lemmas, and Frobenius–Schur theory—are standard, and the elementary computations presented here make the origin of spin transparent.

Each projective representation of a connected Lie group G carries two invariants. The first is an *infinitesimal class* $[\omega_g] \in H^2(\mathfrak{g}, \mathbb{R})$, the cohomology class of the real-valued Lie-algebra two-cocycle obtained by differentiating a local phase choice. The second is a *topological character* $\chi \in \text{Hom}(\pi_1(G), \mathbb{T})$, recording how the fundamental group of G , realized as a central subgroup of the universal cover $\widetilde{G}$, acts by scalars in any lift. The two are of distinct origin: the first is local and algebraic, visible in any neighbourhood of the identity and computed from the Lie bracket through the Chevalley–Eilenberg differential; the second is global and topological, invisible to the Lie algebra and detected by the fundamental group of the group manifold. A group may carry either, both, or neither, and the two are logically and computationally separate. These two invariants organize the entire discussion.

The invariants yield a lifting dichotomy (Theorem 3.1). When the infinitesimal obstruction vanishes, $H^2(\mathfrak{g}, \mathbb{R}) = 0$, every projective unitary representation lifts to the universal cover $\widetilde{G}$, and the topological character χ is a complete invariant of the failure to descend to G itself; when it does not vanish, no covering suffices and a central extension by $\mathbb{T}$ is required. For the rotation group the dichotomy reproduces the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$ and identifies the half-integer spins as exactly the representations on which $\pi_1(\text{SO}(3)) = \mathbb{Z}/2$ acts non-trivially (Section 4). The Frobenius–Schur indicators of $\text{SU}(2)$ are then evaluated exactly, $\nu(\rho_j) = (-1)^{2j}$ (Theorem 5.1), fixing the symmetry type of the invariant bilinear form, the identification $\text{SU}(2) \cong \text{Sp}(1)$, and the time-reversal law $\Theta^2 = (-1)^{2j}$ underlying Kramers degeneracy (Corollaries 5.1–5.3).

A trichotomy (Theorem 6.1) separates the rotation, Poincaré, and Galilei groups by the values of the two invariants, exhibiting relativistic and non-relativistic spin as instances of one mechanism while explaining their structural difference; and the additivity of the obstruction data under direct products (Proposition 9.1) shows that the internal groups $\text{SU}(2)$ and $\text{SU}(3)$ contribute no obstruction, so that all spin in a product symmetry is carried by the spacetime factor (Section 9).

The treatment is group-theoretic throughout. Physical interpretation is confined to remarks, and no use is made of quantization, field theory, or the analytic apparatus of Hilbert space

beyond what the representation theory requires.

Contributions. Two results are new, to the author’s knowledge. First, the spin spectrum carries a duality involution induced by complex conjugation, and the self-conjugate sectors—those fixed by the duality involution, which is necessary for a real or quaternionic structure and for compatibility with an antiunitary time reversal—form its 2-torsion subgroup $\widehat{\pi_1(G)}[2]$ (Theorem 7.1); this reality locus is finite even when the spectrum is a continuum, so for the plane only the two sectors $s = 0$ and $s = \frac{1}{2}$ survive, the group-theoretic reason that generic anyons break time reversal (Corollary 7.2). Second, the topological invariants are cohomology of the group manifold, $\widehat{\pi_1(G)} = H^1(G; \mathbb{T})$ and $\widehat{\pi_1(G)}[2] = H^1(G; \mathbb{Z}/2)$, and the Bockstein sequence of $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T}$ identifies the continuous (anyonic) sectors with the free part of the spectrum and the discrete (half-integer) sectors with the torsion of $H^2(G; \mathbb{Z})$, an integral class whose mod-2 reduction is the group-theoretic Stiefel–Whitney class that obstructs spin structures (Theorem 8.1).

The remainder assembles the classical theory into the single calculus that yields these results: the lifting dichotomy (Theorem 3.1), the spectral description of the spin sectors as the Pontryagin dual $\widehat{\pi_1(G)}$ (Theorem 3.2), and the Frobenius–Schur, trichotomy, and product-additivity computations that follow. The supporting theorems—Bargmann’s theorem, Whitehead’s lemmas, the Mackey–Moore classification, and Frobenius–Schur theory—are classical and cited accordingly, the two-dimensional fractional spin being due to Leinaas and Myrheim [9] and to Wilczek [10]; Section 10 states precisely what is a reformulation of known theory and what is new.

Table 1 summarizes the two obstruction invariants and the resulting spin content.

Table 2 summarizes the spin spectrum, reality locus, and the torsion of $H^2(G; \mathbb{Z})$.

2. Projective representations and the two obstructions

2.1. Multipliers and central extensions

Let G be a topological group and $\mathcal{H}$ a complex Hilbert space. Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and let $\pi: \text{U}(\mathcal{H}) \rightarrow \text{PU}(\mathcal{H}) = \text{U}(\mathcal{H})/\mathbb{T}$ be the quotient, where $\text{U}(\mathcal{H})$ carries the strong operator topology and $\text{PU}(\mathcal{H})$ the induced quotient topology; continuity of projective representations is understood with respect to these. A *projective unitary representation* of G is a continuous homomorphism $\sigma: G \rightarrow \text{PU}(\mathcal{H})$. Choosing for each $g \in G$ a unitary lift $\widehat{\sigma}(g) \in \pi^{-1}(\sigma(g))$, with $\widehat{\sigma}(e) = \text{id}$, yields a Borel map $\widehat{\sigma}: G \rightarrow \text{U}(\mathcal{H})$ satisfying

$$\widehat{\sigma}(g)\widehat{\sigma}(h) = \omega(g, h)\widehat{\sigma}(gh), \quad g, h \in G, \quad (1)$$

for a Borel function $\omega: G \times G \rightarrow \mathbb{T}$. Associativity of the product implies the *cocycle identity*

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k), \quad (2)$$

so ω is a two-cocycle (a *multiplier*) on G with values in the trivial module $\mathbb{T}$. A change of lifts $\widehat{\sigma}(g) \mapsto \lambda(g)\widehat{\sigma}(g)$ replaces ω by

Table 1: The two obstruction invariants and the resulting spin content for the connected groups treated here. The topological obstruction is the fundamental group $\pi_1(G)$; the infinitesimal obstruction is $H^2(\mathfrak{g}, \mathbb{R})$. A finite non-trivial π_1 yields a discrete spin spectrum $\widehat{\pi}_1$; an infinite π_1 yields a continuous (fractional) spectrum; a non-zero H^2 forces a genuine $\mathbb{T}$ -central charge. A dash marks a combination not realized among these groups.

	$\pi_1(G) = 0$	$\pi_1(G)$ finite, $\neq 0$	$\pi_1(G)$ infinite
$H^2(\mathfrak{g}, \mathbb{R}) = 0$	SU(2), SU(3): no spin	SO(3), $\mathcal{P}_+^\uparrow$: discrete spin $\widehat{\mathbb{Z}/2}$	SO(2): continuous spin $\widehat{\mathbb{Z}} = \mathbb{T}$
$H^2(\mathfrak{g}, \mathbb{R}) \neq 0$	—	Galilei $\mathcal{G}$: mass charge + spin $\mathbb{Z}/2$	E(2): exotic charge + continuous spin

Table 2: The spin spectrum, reality locus, and the torsion of $H^2(G; \mathbb{Z})$, computed as singular cohomology of the group manifold. The spectrum $H^1(G; \mathbb{T})$ splits into a free part $\mathbb{T}^r$ (continuous, anyonic sectors) and a finite part (discrete sectors), the latter isomorphic through the Bockstein β to $H^2(G; \mathbb{Z})_{\text{tors}}$, the spinorial class of Remark 7.

G	$\pi_1(G)$	$H^1(G; \mathbb{T})$	$H^1(G; \mathbb{Z}/2)$	$H^2(G; \mathbb{Z})_{\text{tors}}$
SU(2), SU(3)	0	0	0	0
SO(3)	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
SO(2), U(n)	$\mathbb{Z}$	$\mathbb{T}$	$\mathbb{Z}/2$	0
SO(3) $\times$ SO(2)	$\mathbb{Z}/2 \times \mathbb{Z}$	$\mathbb{Z}/2 \times \mathbb{T}$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$

a cohomologous cocycle, and σ lifts to an ordinary unitary representation precisely when ω is a coboundary. The obstruction to lifting is thus the class $[\omega]$ in the Borel cohomology group $H^2(G, \mathbb{T})$ of Mackey and Moore [5–7].

Equivalently, a multiplier ω determines a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G_\omega \longrightarrow G \longrightarrow 1, \quad (3)$$

with $G_\omega = \mathbb{T} \times G$ as a Borel space and multiplication $(s, g)(t, h) = (st\omega(g, h), gh)$, and a projective representation of G with multiplier ω is equivalent to an ordinary unitary representation of G_ω on which the central $\mathbb{T}$ acts by the identity character. The classification of central extensions (3) up to equivalence is the content of $H^2(G, \mathbb{T})$, and the lifting problem for σ is the problem of splitting (3) after pullback along σ .

2.2. The infinitesimal obstruction

For a connected Lie group G with Lie algebra $\mathfrak{g}$, the cohomology $H^2(G, \mathbb{T})$ has an infinitesimal counterpart. On a connected Lie group every multiplier is cohomologous to one that is smooth in a neighbourhood of the identity, so a smooth representative may be chosen for the purpose of differentiation [8, Ch. 7]. Differentiating such a representative of (1) at the identity produces a skew-symmetric bilinear form $\omega_{\mathfrak{g}}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ satisfying the Lie-algebra cocycle identity

$$\omega_{\mathfrak{g}}([X, Y], Z) + \omega_{\mathfrak{g}}([Y, Z], X) + \omega_{\mathfrak{g}}([Z, X], Y) = 0, \quad (4)$$

that is, a class in the second Lie-algebra cohomology $H^2(\mathfrak{g}, \mathbb{R})$ computed from the Chevalley–Eilenberg complex $\Lambda^\bullet \mathfrak{g}^*$ [11]. The two-cocycles (4) classify the central extensions of $\mathfrak{g}$ by $\mathbb{R}$,

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g}_\omega \longrightarrow \mathfrak{g} \longrightarrow 0,$$

with bracket $[(a, X), (b, Y)] = (\omega_{\mathfrak{g}}(X, Y), [X, Y])$, and a coboundary $\omega_{\mathfrak{g}}(X, Y) = f([X, Y])$ for $f \in \mathfrak{g}^*$ corresponds to a split extension. The class $[\omega_{\mathfrak{g}}] \in H^2(\mathfrak{g}, \mathbb{R})$ is the *infinitesimal obstruction* of σ .

For semisimple Lie algebras this cohomology group vanishes.

Lemma 2.1 (Whitehead’s second lemma). Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic zero and let V be a finite-dimensional $\mathfrak{g}$ -module. Then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$. In particular, for the trivial module, $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

Proof. These are the first and second Whitehead lemmas; see Jacobson [12, Ch. III, §7] or Weibel [13, §7.8]. For the trivial module $V = \mathbb{R}$, semisimplicity gives $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, whence $H^1(\mathfrak{g}, \mathbb{R}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$; and $H^2(\mathfrak{g}, \mathbb{R}) = 0$ is the assertion that every one-dimensional central extension of $\mathfrak{g}$ splits, a consequence of the Levi–Malcev theorem. For a general finite-dimensional module the Casimir operator of a faithful representation commutes with the Chevalley–Eilenberg differential and acts invertibly on the cochains of nonzero weight, forcing the cohomology to vanish in positive degrees; Weyl’s theorem on complete reducibility reduces the general case to this one. $\square$

Lemma 2.1 yields vanishing of the infinitesimal obstruction for every finite-dimensional semisimple Lie algebra over a field of characteristic zero, in particular for the Lie algebras of SO(3), SU(2), SU(3), and the proper Lorentz group $\text{SO}_+^\uparrow(1, 3)$. It does not apply to non-semisimple groups; the Galilei group of Section 6 is a case in which $H^2(\mathfrak{g}, \mathbb{R}) \neq 0$.

2.3. The topological obstruction and Bargmann’s theorem

When the infinitesimal obstruction vanishes, the remaining obstruction is topological. Let $p: \widetilde{G} \rightarrow G$ be the universal covering homomorphism of the connected Lie group G ; then $\Gamma := \ker p$ is a discrete subgroup of the centre $Z(\widetilde{G})$, canonically isomorphic to the fundamental group $\pi_1(G)$. A unitary representation U of $\widetilde{G}$ restricts on the central subgroup Γ to a homomorphism into the scalars, that is, to a character

$\chi_U \in \text{Hom}(\Gamma, \mathbb{T})$, and U descends to G exactly when χ_U is trivial. This character constitutes the *topological obstruction*. The connection between the infinitesimal vanishing and the topological residue is Bargmann's theorem.

Theorem 2.2 (Bargmann [2]; see also [8]). Let $\tilde{G}$ be a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ satisfies $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then every continuous projective unitary representation of $\tilde{G}$ lifts to an ordinary continuous unitary representation; equivalently, every multiplier on $\tilde{G}$ is trivial in $H^2(\tilde{G}, \mathbb{T})$.

Theorem 2.2 is the analytic foundation of the obstruction calculus; it is used here as a standard result, and the argument may be summarized in five steps.

(1) On a simply connected neighbourhood of the identity choose a smooth local section of the projection $\tilde{\sigma}$, giving a local lift of $\tilde{\sigma}$. (2) Differentiating the multiplier of this local lift at the identity produces a Lie-algebra two-cocycle $\omega_{\mathfrak{g}} \in Z^2(\mathfrak{g}, \mathbb{R})$ as in (4). (3) The hypothesis $H^2(\mathfrak{g}, \mathbb{R}) = 0$ makes $\omega_{\mathfrak{g}} = \delta f$ a coboundary for some $f \in \mathfrak{g}^*$; absorbing the infinitesimal phase f into the local section removes the cocycle, so the lift may be taken to be a local homomorphism.

(4) Since $\tilde{G}$ is connected and simply connected, the monodromy theorem extends this local homomorphism uniquely to a global homomorphism $U: \tilde{G} \rightarrow \text{U}(\mathcal{H})$: continuation of the local homomorphism along a path is determined by its germ at the identity, and two homotopic paths yield the same endpoint because every loop in $\tilde{G}$ is null-homotopic, so the continuation is single-valued. (5) The resulting global unitary representation U satisfies $\pi \circ U = \tilde{\sigma}$, so the multiplier is a coboundary and its class in $H^2(\tilde{G}, \mathbb{T})$ is trivial.

Full details, including the infinite-dimensional case, are in [2, 8]; and since on the second-countable, type I Lie groups considered here a Borel multiplier is cohomologous to a continuous one, the conclusion holds in the Borel setting as well [8, Ch. 7].

3. The lifting dichotomy

The following theorem separates the two regimes and, in the case relevant to spin, reduces the lifting problem to the computation of a single character.

Theorem 3.1 (Lifting dichotomy). Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, universal cover $p: \tilde{G} \rightarrow G$, and $\Gamma = \ker p \cong \pi_1(G)$.

(I) (*Topological regime*.) Suppose $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then every continuous projective unitary representation $\sigma: G \rightarrow \text{PU}(\mathcal{H})$ admits a continuous unitary representation $U: \tilde{G} \rightarrow \text{U}(\mathcal{H})$ with $\pi \circ U = \sigma \circ p$. Any two such lifts differ by a continuous character $\tilde{G} \rightarrow \mathbb{T}$; if $\tilde{G}$ is perfect (for instance, semisimple) the lift is unique. The restriction $\chi_U = U|_{\Gamma} \in \text{Hom}(\Gamma, \mathbb{T})$ is well defined, and σ descends to an ordinary unitary representation of G if and only if χ_U lies in the image of the restriction map $\text{Hom}(\tilde{G}, \mathbb{T}) \rightarrow \text{Hom}(\Gamma, \mathbb{T})$; in particular, when $\tilde{G}$ is perfect, if and only if χ_U is trivial.

(II) (*Algebraic regime*.) Suppose $[\omega_{\mathfrak{g}}] \neq 0$ in $H^2(\mathfrak{g}, \mathbb{R})$ for some projective unitary representation σ . Then σ does not lift to any covering group of G ; lifting requires the central extension (3) by $\mathbb{T}$ determined by ω , on which the central circle acts by a non-trivial character.

Proof. (I) Pull back σ along p to obtain $\tilde{\sigma} := \sigma \circ p: \tilde{G} \rightarrow \text{PU}(\mathcal{H})$, a continuous projective unitary representation of the simply connected group $\tilde{G}$ whose Lie algebra $\mathfrak{g}$ satisfies $H^2(\mathfrak{g}, \mathbb{R}) = 0$. By Theorem 2.2 there is a continuous unitary representation $U: \tilde{G} \rightarrow \text{U}(\mathcal{H})$ with $\pi \circ U = \tilde{\sigma} = \sigma \circ p$, the required lift. If U' is a second lift then $\pi \circ U' = \pi \circ U$, so $U'(g)U(g)^{-1}$ is a scalar $\lambda(g) \in \mathbb{T}$ for each g ; the map $\lambda: \tilde{G} \rightarrow \mathbb{T}$ is a continuous homomorphism because U, U' are. When $\tilde{G}$ is perfect, $\text{Hom}(\tilde{G}, \mathbb{T}) \cong \text{Hom}(\tilde{G}/[\tilde{G}, \tilde{G}], \mathbb{T}) = \{1\}$, because $\mathbb{T}$ is abelian and the abelianization $\tilde{G}/[\tilde{G}, \tilde{G}]$ is trivial; hence $\lambda \equiv 1$ and the lift is unique. Here $\tilde{G}$ is perfect as an abstract group whenever its Lie algebra is, since for a connected Lie group the commutator subgroup is the connected subgroup with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, which equals $\mathfrak{g}$ when $\mathfrak{g}$ is perfect; this annihilates characters of $\tilde{G}$ itself but places no constraint on the characters of the central subgroup Γ , which are the topological obstruction introduced below.

For $\gamma \in \Gamma$, $p(\gamma) = e$, hence $\pi(U(\gamma)) = \tilde{\sigma}(\gamma) = \sigma(p(\gamma)) = \sigma(e) = [\text{id}]$, so $U(\gamma) \in \ker \pi = \mathbb{T} \cdot \text{id}$ is a scalar; the assignment $\gamma \mapsto U(\gamma)$ is the character $\chi_U \in \text{Hom}(\Gamma, \mathbb{T})$, independent of the chosen lift up to the indeterminacy by $\text{Hom}(\tilde{G}, \mathbb{T})$. The representation σ descends to an ordinary representation V of G if and only if $V \circ p$ is a lift of $\tilde{\sigma}$ trivial on Γ ; since any two lifts differ by a character of $\tilde{G}$, such a V exists if and only if χ_U is the restriction to Γ of a character of $\tilde{G}$. Explicitly, if $\chi_U = \lambda|_{\Gamma}$ for some $\lambda \in \text{Hom}(\tilde{G}, \mathbb{T})$, set $U' := \lambda^{-1}U$; then U' is a unitary representation of $\tilde{G}$ with $U'|_{\Gamma} = \lambda^{-1}|_{\Gamma}\chi_U = 1$, so U' is trivial on $\ker p$ and factors as $U' = V \circ p$ for a unique unitary representation V of G . As λ is scalar-valued, $\pi \circ U' = \pi \circ U$, whence $\pi \circ V \circ p = \sigma \circ p$ and therefore $\pi \circ V = \sigma$: the projective representation descends. Conversely a descent furnishes such a λ . When $\tilde{G}$ is perfect this forces $\chi_U = 1$.

(II) The infinitesimal class $[\omega_{\mathfrak{g}}]$ is invariant under passage to covers. A covering $q: G' \rightarrow G$ is a local isomorphism, so it induces the identity on Lie algebras; by naturality of the differentiation map under q , the multiplier of $\sigma \circ q$ has the same infinitesimal class $[\omega_{\mathfrak{g}}]$ as that of σ . If $\sigma \circ q$ lifted to an ordinary unitary representation, its multiplier would be a coboundary and its infinitesimal class would vanish, contradicting $[\omega_{\mathfrak{g}}] \neq 0$. Hence σ lifts over no covering of G . By the correspondence between projective representations and central extensions, σ does lift to the extension G_{ω} of (3), on which the central $\mathbb{T}$ acts by the identity character; non-triviality of $[\omega_{\mathfrak{g}}]$ forces this character to be non-trivial, so the extension is not a covering. $\square$

Definition 3.1. In the topological regime (I), an irreducible unitary representation of the universal cover $\tilde{G}$ is *spinorial* if it does not descend to G , equivalently, if its character $\chi_U = U|_{\Gamma} \in \text{Hom}(\Gamma, \mathbb{T})$ on $\Gamma = \ker p$ is non-trivial. This extends the classical usage, where the spinorial representations of $\text{SO}(n)$ are those of $\text{Spin}(n)$ that do not descend to $\text{SO}(n)$; here Γ replaces

the deck group $\mathbb{Z}/2$ by the full fundamental group. By Theorem 3.1(I), the spinorial representations are exactly the projective representations of G that are not ordinary representations of G . Compactness of G is not required; perfectness of $\widetilde{G}$ affects only the uniqueness of the lift, not the dichotomy. The definition applies equally to the non-compact covers of Section 6.

Theorem 3.1 makes quantitative the statement that spin is a projective phenomenon: in the semisimple case all spin is determined by the finite abelian group $\Gamma = \pi_1(G)$ and the characters it carries. The following definition and theorem record the resulting grading and isolate the dichotomy between a discrete and a continuous family of spin sectors.

Definition 3.2. The *spin sectors* of G are the fibres of the map $U \mapsto \chi_U = U|_\Gamma$ carrying an irreducible unitary representation of $\widetilde{G}$ to its character on $\Gamma = \pi_1(G)$; two irreducible representations lie in the same sector when they agree on Γ . The sector of the trivial character consists of the representations that descend to G ; the remaining sectors are the spinorial ones of Definition 3.1.

Theorem 3.2 (Spin spectrum). Let G be a connected Lie group in the topological regime, $H^2(\mathfrak{g}, \mathbb{R}) = 0$, with universal cover $p: \widetilde{G} \rightarrow G$ and $\Gamma = \ker p \cong \pi_1(G)$.

- (i) The map $U \mapsto \chi_U$ grades the irreducible unitary representations of $\widetilde{G}$ by the Pontryagin dual

$$\widehat{\Gamma} = \text{Hom}(\pi_1(G), \mathbb{T}), \quad (5)$$

the trivial character labelling exactly those that descend to G and each non-trivial character a spinorial sector. The set of spin sectors therefore injects into $\widehat{\pi_1(G)}$, which is finite if $\pi_1(G)$ is finite and contains a continuum if $\pi_1(G)$ is infinite.

- (ii) If $\widetilde{G}$ is of type I—for instance a connected semisimple group or a covering group of one, or an abelian group—then the grading in (i) is surjective: every character of Γ occurs as χ_U for some irreducible unitary U , so the spin sectors are exactly $\widehat{\pi_1(G)}$.

Proof. (i) By Theorem 3.1(I) an irreducible unitary representation U of $\widetilde{G}$ restricts on the central subgroup Γ to a scalar character $\chi_U \in \text{Hom}(\Gamma, \mathbb{T}) = \widehat{\Gamma}$, and U descends to G if and only if χ_U is trivial (Definition 3.1). Thus $U \mapsto \chi_U$ is the stated grading and the spin sectors inject into $\widehat{\Gamma}$. The fundamental group of a connected Lie group is finitely generated abelian—abelian by the Eckmann–Hilton argument for topological groups, and finitely generated because a connected Lie group deformation retracts onto its maximal compact subgroup (Iwasawa), a compact manifold—so $\pi_1(G) \cong \mathbb{Z}^r \times F$ with F finite, and Pontryagin duality gives $\widehat{\pi_1(G)} \cong \mathbb{T}^r \times \widehat{F}$; this is finite exactly when $r = 0$, and contains the torus $\mathbb{T}^r$, hence a continuum, exactly when $r \geq 1$.

(ii) Suppose $\widetilde{G}$ is of type I. Given $\chi \in \widehat{\Gamma}$, extend it to a character $\widetilde{\chi}$ of $Z(\widetilde{G})$, which is possible because Γ is a discrete, hence closed, subgroup of the locally compact abelian group $Z(\widetilde{G})$, so the restriction map $\widehat{Z(\widetilde{G})} \rightarrow \widehat{\Gamma}$ is surjective, being dual

to the closed inclusion $\Gamma \hookrightarrow Z(\widetilde{G})$ (equivalently, $\mathbb{T}$ is divisible, hence injective as an abelian group). The induced representation $\text{Ind}_{Z(\widetilde{G})}^{\widetilde{G}} \widetilde{\chi}$ is non-zero, and since $Z(\widetilde{G})$ is central it acts throughout by the scalar $\widetilde{\chi}$; because $\widetilde{G}$ is of type I this representation disintegrates into irreducibles, each of central character $\widetilde{\chi}$ and hence of Γ -character χ . Thus χ is realized, and with (i) the sectors are exactly $\widehat{\Gamma}$. The realizing representations are the highest-weight modules when $\widetilde{G}$ is compact and the characters $U_s(\theta) = e^{is\theta}$ on $\widehat{\text{SO}(2)} = \mathbb{R}$ in the planar case of Section 6.4. $\square$

For a semisimple group $\pi_1(G)$ is finite and (5) is a finite set of sectors—two for $\text{SO}(3)$, where $\widehat{\mathbb{Z}/2} = \mathbb{Z}/2$ gives the integer and half-integer spins. The continuous alternative is realized in Section 6.4.

4. The double cover $\text{SU}(2) \rightarrow \text{SO}(3)$

This section constructs the covering homomorphism explicitly and determines its kernel, surjectivity, and the simple connectedness of the source. The construction is elementary and illustrates Theorem 3.1.

4.1. Construction by conjugation

Let

$$\text{SU}(2) = \left\{ g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

and let

$$V = \{ X \in M_2(\mathbb{C}) : X^\dagger = X, \text{tr } X = 0 \} = \mathbb{R}\sigma_1 \oplus \mathbb{R}\sigma_2 \oplus \mathbb{R}\sigma_3$$

be the three-dimensional real vector space of traceless Hermitian matrices, spanned by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $g \in \text{SU}(2)$ define $\Phi(g): V \rightarrow V$ by $\Phi(g)X = gXg^{-1} = gXg^\dagger$. Since g is unitary, $gXg^\dagger$ is Hermitian and traceless, so $\Phi(g)$ preserves V and is $\mathbb{R}$ -linear. Equip V with the inner product

$$\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY), \quad (6)$$

for which $\{\sigma_1, \sigma_2, \sigma_3\}$ is orthonormal. Then

$$\begin{aligned} \langle \Phi(g)X, \Phi(g)Y \rangle &= \frac{1}{2} \text{tr}(gXg^{-1}gYg^{-1}) \\ &= \frac{1}{2} \text{tr}(XY) = \langle X, Y \rangle, \end{aligned}$$

so $\Phi(g) \in O(V) \cong O(3)$. Writing $X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ yields $\det X = -(x_1^2 + x_2^2 + x_3^2) = -\|x\|^2$, and $\det(\Phi(g)X) = \det X$ confirms that $\Phi(g)$ is an isometry. The map $\Phi: \text{SU}(2) \rightarrow O(3)$ is a continuous homomorphism; as $\text{SU}(2)$ is connected and $\Phi(\text{id}) = \text{id}$ has determinant +1, the image lies in the identity component, so $\Phi: \text{SU}(2) \rightarrow \text{SO}(3)$.

4.2. Kernel, surjectivity, simple connectedness

Proposition 4.1. The homomorphism $\Phi: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ is surjective with kernel $\{\pm \mathrm{id}\} \cong \mathbb{Z}/2$, the centre of $\mathrm{SU}(2)$. Moreover $\mathrm{SU}(2)$ is diffeomorphic to the three-sphere S^3 and is therefore simply connected, so Φ is the universal covering and $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2$.

Proof. Kernel. $g \in \ker \Phi$ if and only if $gXg^{-1} = X$ for all $X \in V$, if and only if g commutes with $\sigma_1, \sigma_2, \sigma_3$, if and only if g is a scalar matrix; with $\det g = 1$ this forces $g = \pm \mathrm{id}$. These matrices commute with all of V , so $\ker \Phi = \{\pm \mathrm{id}\} = Z(\mathrm{SU}(2)) \cong \mathbb{Z}/2$.

Surjectivity. The differential $d\Phi_e: \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ sends $A \mapsto [A, \cdot]$, the adjoint action on V . Its kernel is the centre of $\mathfrak{su}(2)$, which is 0 because $\mathfrak{su}(2)$ is simple; since $\dim \mathfrak{su}(2) = \dim \mathfrak{so}(3) = 3$, the map $d\Phi_e$ is a linear isomorphism. Hence Φ is a local diffeomorphism near e , its image contains a neighbourhood of the identity in $\mathrm{SO}(3)$, and therefore the image—a subgroup containing an open set—is an open subgroup of the connected group $\mathrm{SO}(3)$, hence all of $\mathrm{SO}(3)$.

Topology. The defining condition $|a|^2 + |b|^2 = 1$ identifies $\mathrm{SU}(2)$ with the unit sphere $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$, which is simply connected. A surjective homomorphism of Lie groups with discrete kernel and simply connected source is the universal covering; thus $\pi_1(\mathrm{SO}(3)) \cong \ker \Phi \cong \mathbb{Z}/2$. $\square$

Remark 1. Proposition 4.1 is the rank-one case of the spin covering $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ built from the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$. Concretely, the even subalgebra $\mathrm{Cl}^0(\mathbb{R}^3)$ is isomorphic to the quaternions $\mathbb{H}$, its group of unit-norm elements is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, and the space V of the construction above is the space of vectors $\mathbb{R}^3 \subset \mathrm{Cl}(\mathbb{R}^3)$ on which these units act by the twisted adjoint representation, recovering the conjugation Φ [14–16]. In every dimension $n \geq 3$ the double covering realizes the non-trivial class of $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$, and Theorem 3.1 applies verbatim with $\Gamma = \mathbb{Z}/2$.

4.3. Irreducible representations and the spinorial ones

The irreducible unitary representations of $\mathrm{SU}(2)$ are classified by a half-integer *spin* $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$: for each j there is, up to unitary equivalence, a unique irreducible representation ρ_j of dimension $2j + 1$, realized on the space of homogeneous polynomials of degree $2j$ in two variables on which $\mathrm{SU}(2)$ acts by linear substitution [17, Ch. 4], [18, §II.5]. The centre $-\mathrm{id} \in \mathrm{SU}(2)$ acts on ρ_j by the scalar $(-1)^{2j}$, since it acts by -1 on each variable and hence by $(-1)^{2j}$ on a monomial of total degree $2j$. By Proposition 4.1 and Theorem 3.1(I), with $\Gamma = \{\pm \mathrm{id}\}$ and $\chi_{\rho_j}(-\mathrm{id}) = (-1)^{2j}$, the following holds.

Proposition 4.2. The representation ρ_j of $\mathrm{SU}(2)$ descends to an ordinary representation of $\mathrm{SO}(3)$ if and only if $j \in \mathbb{Z}$. The spinorial representations of $\mathrm{SU}(2)$, in the sense of Definition 3.1, are exactly those with $j \in \frac{1}{2} + \mathbb{Z}$. Consequently the half-integer spins are precisely the projective unitary representations of $\mathrm{SO}(3)$ that are not ordinary representations of $\mathrm{SO}(3)$.

The origin is now explicit: because $\mathfrak{so}(3)$ is semisimple, $H^2(\mathfrak{so}(3), \mathbb{R}) = 0$ by Lemma 2.1, so Theorem 3.1(I) places

$\mathrm{SO}(3)$ in the topological regime; the sole obstruction is the class in $\Gamma = \pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$, and the representations realizing the non-trivial class—the half-integer spins—exist as genuine representations only on the double cover $\mathrm{SU}(2)$. Spin is the character of $\pi_1(\mathrm{SO}(3))$. In physical terms, the label j is the spin quantum number: the generators of $\mathfrak{su}(2)$ are the angular-momentum operators J_1, J_2, J_3 , realized on the two-dimensional spinor representation $\rho_{1/2}$ as $J_k = \frac{1}{2}\sigma_k$ with σ_k the Pauli matrices, and the homotopy class of a full 2π rotation—the non-trivial element of $\Gamma = \pi_1(\mathrm{SO}(3))$ —acts on ρ_j by the scalar $(-1)^{2j}$, the sign that separates the spinorial half-integer angular momenta from the integer ones.

5. Reality of spinors: Frobenius–Schur indicators

The double cover carries further structure. The Frobenius–Schur indicator of every ρ_j is computed below, which yields the type of the invariant bilinear form, the identification $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, and the time-reversal squaring law. All computations are on $\mathrm{SU}(2)$ with normalized Haar measure.

Definition 5.1 (Frobenius–Schur). For a compact group K with normalized Haar measure dk and an irreducible representation ρ with character χ , the *Frobenius–Schur indicator* is

$$\nu(\rho) = \int_K \chi(k^2) dk \in \{-1, 0, +1\}. \quad (7)$$

Here $\nu = +1$ if and only if ρ is of real (orthogonal) type, that is, admits a non-degenerate invariant *symmetric* bilinear form and is realizable over $\mathbb{R}$; $\nu = -1$ if and only if ρ is of quaternionic (symplectic) type, that is, admits a non-degenerate invariant *antisymmetric* bilinear form and is not realizable over $\mathbb{R}$; and $\nu = 0$ if and only if ρ is of complex type, $\rho \not\cong \rho^*$ [18–21].

Theorem 5.1. For the irreducible representation ρ_j of $\mathrm{SU}(2)$ of dimension $2j + 1$,

$$\nu(\rho_j) = (-1)^{2j}.$$

Thus integer spins are of real type and half-integer spins are of quaternionic type.

Proof. Parametrize the maximal torus of $\mathrm{SU}(2)$ by $t(\varphi) = \mathrm{diag}(e^{i\varphi}, e^{-i\varphi})$, $\varphi \in [0, \pi]$. The Weyl integration formula reduces integration of a class function f to

$$\int_{\mathrm{SU}(2)} f dk = \frac{2}{\pi} \int_0^\pi f(t(\varphi)) \sin^2 \varphi d\varphi, \quad (8)$$

where $\frac{2}{\pi} \sin^2 \varphi d\varphi$ is a probability measure on $[0, \pi]$ [18, §IV.1], [17, Ch. 12]. The constant arises as $\frac{1}{|W|} \cdot \frac{1}{2\pi} \cdot |\Delta|^2$ folded onto the Weyl chamber: the Weyl group has order $|W| = 2$, the Weyl denominator is $|\Delta(t(\varphi))|^2 = |e^{i\varphi} - e^{-i\varphi}|^2 = 4 \sin^2 \varphi$, and integrating the chamber $[0, \pi]$ against $|\Delta|^2 d\varphi/2\pi$ already accounts for the order-two stabilizer, giving $4/2\pi = 2/\pi$. With $n := 2j + 1$ the character of ρ_j on the torus is

$$\chi_n(\varphi) = \frac{\sin(n\varphi)}{\sin \varphi} = \sum_{k=0}^{n-1} e^{i(n-1-2k)\varphi}.$$

Since $t(\varphi)^2 = t(2\varphi)$,

$$\begin{aligned}\chi_n(k^2) &= \chi_n(2\varphi) = \frac{\sin(2n\varphi)}{\sin(2\varphi)} \\ &= \sum_{k=0}^{n-1} e^{2i(n-1-2k)\varphi}.\end{aligned}$$

Substituting into (7) via (8),

$$\nu(\rho_j) = \frac{2}{\pi} \int_0^\pi \sin^2 \varphi \sum_{k=0}^{n-1} e^{2i(n-1-2k)\varphi} d\varphi.$$

Write $\sin^2 \varphi = \frac{1}{2} - \frac{1}{4}e^{2i\varphi} - \frac{1}{4}e^{-2i\varphi}$ and use $\int_0^\pi e^{2ip\varphi} d\varphi = \pi \delta_{p,0}$ for $p \in \mathbb{Z}$. For the summand with exponent $m := n-1-2k$,

$$\int_0^\pi \sin^2 \varphi e^{2im\varphi} d\varphi = \pi \left(\frac{1}{2} \delta_{m,0} - \frac{1}{4} \delta_{m,1} - \frac{1}{4} \delta_{m,-1} \right).$$

As k runs over $0, \dots, n-1$, the exponent m takes the values $n-1, n-3, \dots, -(n-1)$, all of the parity of $n-1$. If n is odd, only $m=0$ occurs, once, contributing $\frac{1}{2}$, so $\nu = \frac{2}{\pi} \cdot \pi \cdot \frac{1}{2} = 1$. If n is even, $m=0$ never occurs while $m = \pm 1$ each occur once, contributing $-\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$, so $\nu = \frac{2}{\pi} \cdot \pi \cdot (-\frac{1}{2}) = -1$. Since $n = 2j+1$ is odd for $j \in \mathbb{Z}$ and even for $j \in \frac{1}{2} + \mathbb{Z}$, the formula $\nu(\rho_j) = (-1)^{2j}$ follows. $\square$

Corollary 5.1. Each ρ_j carries a non-degenerate $\text{SU}(2)$ -invariant bilinear form B_j , unique up to scalar, symmetric for $j \in \mathbb{Z}$ and antisymmetric for $j \in \frac{1}{2} + \mathbb{Z}$.

Proof. By Theorem 5.1 no ρ_j is of complex type, so $\rho_j \cong \rho_j^*$ and the space of invariant bilinear forms is one-dimensional by Schur's lemma. The sign of ν records the symmetry type of the unique form, as in Definition 5.1. $\square$

Corollary 5.2. For $j = \frac{1}{2}$ the invariant antisymmetric form on $\mathbb{C}^2$ is $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and invariance reads $g^T \varepsilon g = \varepsilon$ for all $g \in \text{SU}(2)$. Hence $\text{SU}(2)$ coincides with the compact symplectic group $\text{Sp}(1)$, equivalently with the group of unit quaternions, and the spin- $\frac{1}{2}$ representation is the defining representation of $\text{Sp}(1)$.

Proof. A direct check gives $g^T \varepsilon g = \det(g) \varepsilon = \varepsilon$ for $g \in \text{SU}(2)$, so $\text{SU}(2) \subseteq \text{Sp}(1)$; equality of dimensions and connectedness give $\text{SU}(2) = \text{Sp}(1)$. The identification with unit quaternions is the standard isomorphism $a + bj \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, under which ε is the quaternionic structure map; see [18, §1.1]. $\square$

Corollary 5.3 (Time reversal and Kramers degeneracy). Let Θ be an antiunitary operator commuting with the action of $\text{SU}(2)$ on ρ_j and squaring to a scalar. Then $\Theta^2 = (-1)^{2j} = \nu(\rho_j)$. In particular, for half-integer spin $\Theta^2 = -1$, so Θ has no eigenvector, and every finite-dimensional eigenspace of a self-adjoint operator commuting with Θ has even dimension (Kramers degeneracy).

Proof. An antiunitary Θ intertwining ρ_j with itself is, after composing with a fixed antiunitary conjugation, an isomorphism $\rho_j \rightarrow \rho_j^*$, that is, a non-degenerate invariant bilinear

form; the sign of Θ^2 equals the symmetry type of that form, which is $\nu(\rho_j)$ by Corollary 5.1. When $\Theta^2 = -1$, an eigenvector v with $\Theta v = \lambda v$ would give $\Theta^2 v = \bar{\lambda} \lambda v = |\lambda|^2 v = v \neq -v$, a contradiction; pairing v with Θv then exhibits the eigenspaces of any self-adjoint operator commuting with Θ as even-dimensional, which is the content of Kramers' theorem [4, 22]. $\square$

Remark 2. Definition 5.1 is Dyson's threefold way: the three values $\nu \in \{+1, 0, -1\}$ correspond to the orthogonal, unitary, and symplectic symmetry classes [22]. Theorem 5.1 places the entire representation theory of $\text{SU}(2)$ into the orthogonal and symplectic classes according to the parity of $2j$, with the symplectic (half-integer) sector being exactly the spinorial sector of Proposition 4.2. Reality type and liftability are thus governed by the same $\mathbb{Z}/2$.

6. A trichotomy: rotations, Poincaré, and Galilei

Theorem 3.1 organizes the lifting problem for a connected symmetry group according to the pair $(H^2(\mathfrak{g}, \mathbb{R}), \pi_1(G))$. The three groups governing, respectively, non-relativistic rotational symmetry, relativistic spacetime symmetry, and non-relativistic spacetime symmetry realize the three distinct cases. The relevant cohomology and topology are summarized below, and the trichotomy follows.

6.1. The rotation group

For $G = \text{SO}(3)$, $\mathfrak{g} = \mathfrak{so}(3) \cong \mathfrak{su}(2)$ is semisimple, so $H^2(\mathfrak{g}, \mathbb{R}) = 0$ (Lemma 2.1), and $\pi_1(\text{SO}(3)) = \mathbb{Z}/2$ (Proposition 4.1). This is the topological regime with $\Gamma = \mathbb{Z}/2$: spin is the non-trivial character of π_1 , realized on the double cover $\text{SU}(2)$.

6.2. The Poincaré group

Let $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \text{SO}_+^\uparrow(1, 3)$ be the proper orthochronous Poincaré group. Its Lie algebra is not semisimple—it contains the abelian translation ideal—yet its second cohomology with trivial coefficients vanishes,

$$H^2(\mathfrak{iso}(1, 3), \mathbb{R}) = 0, \quad (9)$$

a standard computation of the Chevalley–Eilenberg cohomology [23, Ch. 6], [24, Ch. 2]; alternatively, in the Hochschild–Serre spectral sequence of the abelian translation ideal $\mathfrak{t} = \mathbb{R}^{1,3} \leq \mathfrak{iso}(1, 3)$ with quotient $\mathfrak{so}(1, 3)$, the three contributions to H^2 vanish— $H^2(\mathfrak{so}(1, 3), \mathbb{R}) = 0$ and $H^1(\mathfrak{so}(1, 3), \mathfrak{t}^*) = 0$ by semisimplicity (Lemma 2.1), and $(\Lambda^2 \mathfrak{t}^*)^{\mathfrak{so}(1, 3)} = 0$ because Minkowski space carries no Lorentz-invariant alternating bilinear form. Topologically the translations are contractible and the Lorentz group $\text{SO}_+^\uparrow(1, 3)$ deformation retracts onto its maximal compact subgroup $\text{SO}(3)$, so $\pi_1(\mathcal{P}_+^\uparrow) \cong \pi_1(\text{SO}(3)) = \mathbb{Z}/2$, with universal (double) cover

$$\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \text{SL}(2, \mathbb{C}),$$

since $\text{SL}(2, \mathbb{C})$ is the simply connected double cover of $\text{SO}_+^\uparrow(1, 3)$. By (9) the group lies in the topological regime,

and Theorem 3.1(I) gives that every projective unitary representation lifts to $\mathbb{R}^{1,3} \rtimes \text{SL}(2, \mathbb{C})$, with residual obstruction in $\Gamma = \mathbb{Z}/2$.

Wigner's classification of the irreducible unitary representations of $\widetilde{\mathcal{P}}_+^\uparrow$ [4, 24] proceeds by Mackey's theory of induced representations [5, 25] from the stabilizer (little group) of a point on a Lorentz orbit in momentum space. For a positive-mass orbit the little group of $\mathcal{P}_+^\uparrow$ is $\text{SO}(3)$, and that of the double cover $\mathbb{R}^{1,3} \rtimes \text{SL}(2, \mathbb{C})$ used here is its cover $\text{SU}(2)$; the internal label is therefore a spin j as in Section 4, and the $\mathbb{Z}/2$ of Γ is the same half-integer/integer dichotomy. For a lightlike orbit the little group is the Euclidean group $\text{ISO}(2)$ and the finite-dimensional content reduces to a helicity. Relativistic spin is thus the Poincaré refinement of the rotational case, with the identical topological origin. The $\text{SU}(2)$ structure recurs in relativistic bound-state problems: the spin and pseudospin symmetries of the Dirac Hamiltonian are $\text{SU}(2)$ symmetries [26], and Dirac and spin-dependent bound-state systems of this kind are studied extensively, including in this journal [27, 28].

6.3. The Galilei group

Let $\mathcal{G}$ be the connected Galilei group of non-relativistic spacetime, generated by time and space translations, rotations, and Galilean boosts. Here the infinitesimal obstruction does not vanish,

$$H^2(\text{galilei}, \mathbb{R}) \neq 0, \quad (10)$$

the non-trivial class being the Bargmann two-cocycle in which boosts and translations commute up to a phase proportional to a parameter m [2, 23, 29]. By Theorem 3.1(II) the Galilei group lies in the algebraic regime: its physically realized projective representations do not lift to any covering group but require the genuine central extension—the Bargmann group $\widehat{\mathcal{G}}_m$ —on which the central circle acts by a non-trivial character whose infinitesimal generator is the mass m . The mass is therefore a central charge rather than a Casimir of $\mathcal{G}$ itself: since $H^2(\text{galilei}, \mathbb{R}) \cong \mathbb{R}$ is one-dimensional, a non-zero mass is the multiple $m[\omega_1]$ of a fixed generator $[\omega_1]$, equivalently the eigenvalue of the central generator of the Bargmann extension. Distinct masses thus correspond to distinct cohomology classes and to inequivalent projective representations, distinguished by their central character; this is the group-theoretic source of the Bargmann mass superselection rule [29, 30].

Theorem 6.1 (Trichotomy). With the two invariants $(H^2(\mathfrak{g}, \mathbb{R}), \pi_1(G))$:

- (a) $\text{SO}(3)$ and the Poincaré group $\mathcal{P}_+^\uparrow$ satisfy $H^2(\mathfrak{g}, \mathbb{R}) = 0$ and $\pi_1 = \mathbb{Z}/2$. Their projective representations lift to a double cover ($\text{SU}(2)$, resp. $\mathbb{R}^{1,3} \rtimes \text{SL}(2, \mathbb{C})$), and spin is the non-trivial character of $\pi_1 = \mathbb{Z}/2$.
- (b) The Galilei group $\mathcal{G}$ satisfies $H^2(\mathfrak{g}, \mathbb{R}) \neq 0$. Its projective representations do not lift to any covering and require a genuine $\mathbb{T}$ -central extension, whose central charge is the mass.

In particular, relativistic and non-relativistic rotational spin share the topological mechanism of (a), while the non-relativistic mass parameter is an algebraic phenomenon of type (b) with no analogue in the relativistic group, where $H^2 = 0$.

Proof. Each clause is the specialization of Theorem 3.1 to the cited cohomology and topology: (9) and $\pi_1 = \mathbb{Z}/2$ for the Poincaré group, Lemma 2.1 and Proposition 4.1 for $\text{SO}(3)$, and (10) for the Galilei group. $\square$

Remark 3. A fourth case completes the logical classification: when both obstructions vanish, $H^2(\mathfrak{g}, \mathbb{R}) = 0$ and $\pi_1(G) = 0$, every projective unitary representation is already an ordinary representation of G itself and there is no spinorial sector. This is the case of the simply connected groups $\text{SU}(2)$ and $\text{SU}(3)$ treated in Section 9. Theorem 6.1 records the three cases realized by the spacetime symmetry groups; the fourth is realized by the internal groups.

Remark 4. Theorem 6.1 illustrates the two-obstruction calculus: passing from the relativistic to the non-relativistic spacetime symmetry by Inönü–Wigner contraction [31] creates the infinitesimal class (10) that is absent in (9), converting a purely topological situation into an algebraic one. Spin survives the contraction, while the mass central charge arises from it.

6.4. The planar case and fractional spin

The trichotomy is completed by the elementary spacetime-symmetry group with infinite fundamental group. For planar rotational symmetry $G = \text{SO}(2)$ the Lie algebra $\mathfrak{so}(2) \cong \mathbb{R}$ is abelian, so $H^2(\mathfrak{so}(2), \mathbb{R}) = 0$ and the group lies in the topological regime; in this case the fundamental group is infinite,

$$\pi_1(\text{SO}(2)) = \mathbb{Z}, \quad (11)$$

with universal cover the additive line $p: \mathbb{R} \rightarrow \text{SO}(2) = \mathbb{R}/2\pi\mathbb{Z}$, $\theta \mapsto R(\theta)$, and $\Gamma = \ker p = 2\pi\mathbb{Z} \cong \mathbb{Z}$; since $\mathbb{R}$ is simply connected and the covering $\mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ has deck group $2\pi\mathbb{Z}$, this gives (11) directly. By Theorem 3.2 the spin sectors form the Pontryagin dual

$$\widehat{\Gamma} = \widehat{\mathbb{Z}} \cong \mathbb{T}, \quad (12)$$

a continuum. Explicitly, the irreducible unitary representations of $\mathbb{R}$ are the characters $U_s(\theta) = e^{is\theta}$ with $s \in \mathbb{R}$, whose central character on Γ is $\chi_s(2\pi n) = e^{2\pi i s n}$; the representation U_s descends to $\text{SO}(2)$ exactly when $s \in \mathbb{Z}$, the integer angular momenta, and the sectors are labelled by $s \bmod 1 \in \mathbb{R}/\mathbb{Z}$. The spinorial representations are those with $s \notin \mathbb{Z}$: states acquiring an arbitrary phase $e^{2\pi i s}$ under one full rotation. This continuum of fractional-spin sectors is the planar phenomenon of *anyons* [9, 10], recovered here as the value of the two-invariant calculus at an infinite fundamental group.

The planar case differs from the three-dimensional one in two respects.

First, the discreteness of spin in three dimensions is the finiteness of $\pi_1(\text{SO}(3)) = \mathbb{Z}/2$, which forces the two-element spectrum $\mathbb{Z}/2 = \mathbb{Z}/2$; passing to the plane replaces this by the infinite $\pi_1(\text{SO}(2)) = \mathbb{Z}$ and the continuous spectrum (12).

Half-integer quantization is therefore a three-dimensional consequence of the finiteness of π_1 , not a universal feature of spin.

Second, although the rotational obstruction is purely topological, the full planar Euclidean algebra $\mathfrak{iso}(2) = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ carries in addition a non-trivial infinitesimal class. With generators J, P_1, P_2 and brackets $[J, P_1] = P_2$, $[J, P_2] = -P_1$, $[P_1, P_2] = 0$, the alternating form $\omega(P_1, P_2) = 1$ that vanishes on every other pair of generators satisfies the cocycle identity (4)—the only non-trivial triple (J, P_1, P_2) gives $\omega(P_2, P_2) + \omega(0, J) + \omega(P_1, P_1) = 0$ —yet is not a coboundary, since $[P_1, P_2] = 0$ forces any coboundary to vanish on (P_1, P_2) . Hence

$$H^2(\mathfrak{iso}(2), \mathbb{R}) \neq 0, \quad (13)$$

the class being the central charge of the non-commuting (“magnetic”) translations of the plane.

At the group level the identity component of the planar Euclidean group is $E(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, and as $\mathbb{R}^2$ is contractible its fundamental group is again $\pi_1(E(2)) \cong \pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, so the topological obstruction (12) persists for the full group; the infinitesimal class (13), a central extension of $\mathfrak{iso}(2)$ by $\mathbb{R}$, integrates by Theorem 3.1(II) to a genuine $\mathbb{T}$ -central extension of $E(2)$ on which the circle acts non-trivially. The continuity of the topological obstruction distinguishes the plane, and its scope should be stated precisely.

Among the rotation groups $\mathrm{SO}(n)$, $n \geq 2$, the planar case $n = 2$ is the only one with infinite fundamental group, since $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ while $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ for $n \geq 3$; the planar spin spectrum is therefore a continuum, whereas in higher dimensions the finiteness of π_1 forces the two-element spectrum $\mathbb{Z}/2$. This is a feature of the two-dimensional setting rather than of $\mathrm{SO}(2)$ alone: by Theorem 3.2 any connected group with infinite fundamental group has a continuous spin spectrum, and among spacetime symmetry groups this occurs for every group with maximal compact subgroup $\mathrm{SO}(2)$ —in particular for the $(2+1)$ -dimensional Lorentz group $\mathrm{SO}(2, 1)_0$ and Poincaré group $\mathbb{R}^{2,1} \rtimes \mathrm{SO}(2, 1)_0$, both of which have $\pi_1 = \mathbb{Z}$ and carry the relativistic anyon spectrum [9, 10]. What the plane does not share with the three- and four-dimensional cases treated above is the finiteness of π_1 , from which half-integer quantization there follows.

Finally, both obstructions are present for the plane, but this coincidence is not the distinguishing feature: the non-relativistic Galilei group of Section 6 likewise carries a non-trivial H^2 (the mass) together with a non-trivial $\pi_1 = \mathbb{Z}/2$ (spin). It is the continuity of the topological obstruction, not its mere presence, that is special to two dimensions.

Remark 5. Table 1 arranges the groups treated here by the pair $(H^2(\mathfrak{g}, \mathbb{R}), \pi_1(G))$. The topological regime $H^2 = 0$ contains the internal groups $\mathrm{SU}(2), \mathrm{SU}(3)$ (trivial π_1 , no spin), the groups $\mathrm{SO}(3)$ and $\mathcal{P}_+^\uparrow$ (finite $\pi_1 = \mathbb{Z}/2$, discrete spin $\mathbb{Z}/2$), and the planar rotation group $\mathrm{SO}(2)$ (infinite $\pi_1 = \mathbb{Z}$, continuous spin $\mathbb{Z} = \mathbb{T}$). The algebraic regime $H^2 \neq 0$ contains the Galilei group, which carries the mass charge together with the spin $\mathbb{Z}/2$ of its rotation subgroup, and the planar Euclidean group $E(2)$, carrying an exotic charge together with the continuous spin of

$\mathrm{SO}(2)$. The trichotomy of Theorem 6.1 is the finite- π_1 part of this arrangement; the planar entries are its continuous completion.

Remark 6 (Relation to statistics). The topological distinction between three and two dimensions recurs, independently of spin, in the statistics of identical particles. Exchange statistics is governed by the fundamental group of the configuration space of indistinguishable particles: the symmetric group in three or more dimensions, whose one-dimensional unitary characters are the boson and fermion signs (a $\mathbb{Z}/2$), and the braid group in two dimensions, which admits a continuum of one-dimensional characters. This parallels the discrete $\mathbb{Z}/2$ versus continuous spin spectrum found above. The spin–statistics theorem ties the two gradings together in a relativistic quantum field theory, but that relation lies beyond the single-group lifting analysis carried out here.

7. The reality locus of the spin spectrum

The spin spectrum $\widehat{\pi_1(G)}$ of Theorem 3.2 is an abelian group, and duality of representations equips it with a canonical involution. This involution and its fixed locus package the reality data of Section 5 and the two-dimensional case of Section 6.4 into a single statement, and yield the observation that the self-conjugate sectors—a necessary condition for compatibility with an antiunitary time reversal—always form a *finite* subgroup, even when the spectrum itself is a continuum.

Definition 7.1. The *duality involution* on the spin spectrum is the map $\iota: \widehat{\pi_1(G)} \rightarrow \widehat{\pi_1(G)}$ given by $\iota(\chi) = \chi^{-1} = \bar{\chi}$. A spin sector χ is *self-conjugate* if $\iota(\chi) = \chi$.

The dual representation U^* has central character $\overline{\chi_U} = \chi_U^{-1}$ on Γ , so $U \mapsto U^*$ covers ι at the level of sectors; equivalently, conjugating the scalars in a lift carries the sector χ to χ^{-1} , so ι is the action on sectors of an antiunitary symmetry inverting the central circle. Self-conjugacy is therefore necessary—though, as the planar case below shows, not sufficient—for a sector to carry a real or quaternionic structure or to be compatible with a time reversal that inverts angular momentum; the fixed locus of ι consists of these self-conjugate sectors.

Theorem 7.1 (Reality locus). Let G be a connected Lie group in the topological regime, $\Gamma = \pi_1(G)$.

- (i) A spin sector contains a self-dual representation only if it is self-conjugate. The self-conjugate sectors are the 2-torsion subgroup

$$\begin{aligned} \widehat{\Gamma}[2] &= \{\chi \in \widehat{\Gamma} : \chi^2 = 1\} \\ &\cong \mathrm{Hom}(\Gamma, \mathbb{Z}/2) = \mathrm{Hom}(\Gamma/2\Gamma, \mathbb{Z}/2). \end{aligned} \quad (14)$$

- (ii) With $\Gamma \cong \mathbb{Z}^r \times F$, F finite, this reality locus is finite, of order

$$|\widehat{\Gamma}[2]| = |\Gamma/2\Gamma| = 2^{r+t}, \quad t = \dim_{\mathbb{Z}/2} F[2], \quad (15)$$

whether or not the spin spectrum $\widehat{\Gamma}$ is finite.

Proof. (i) If $U \cong U^*$ then their central characters coincide, $\chi_U = \overline{\chi_U} = \chi_U^{-1}$, so $\chi_U^2 = 1$; a sector containing a self-dual representation is thus fixed by ι . The fixed set of $\chi \mapsto \chi^{-1}$ is the 2-torsion subgroup $\widehat{\Gamma}[2]$. A character of order dividing 2 has image in $\{\pm 1\}$, hence is a homomorphism $\Gamma \rightarrow \mathbb{Z}/2$, and any such homomorphism annihilates 2Γ ; this gives the identifications in (14). (ii) For $\Gamma \cong \mathbb{Z}^r \times F$ one has $\widehat{\Gamma} \cong \mathbb{T}^r \times \widehat{F}$, so $\widehat{\Gamma}[2] \cong (\mathbb{Z}/2)^r \times \widehat{F}[2]$. As $\widehat{F}[2] \cong F/2F$ has order 2^f with $t = \dim_{\mathbb{Z}/2} F[2]$, the order is 2^{r+t} , finite for every connected Lie group. $\square$

The theorem specializes at the two ends of the calculus.

Corollary 7.1 (Compact case: reality is a sector invariant). For the rotation group $\text{SO}(3)$, $\Gamma = \mathbb{Z}/2$ and $\widehat{\Gamma}[2] = \widehat{\Gamma}$: both sectors are self-conjugate, and by Theorem 5.1 the Frobenius–Schur indicator is constant on each—real ($\nu = +1$) on the integer-spin sector, quaternionic ($\nu = -1$) on the half-integer-spin sector—with $\nu(\rho_j) = \chi_j(-I)$ the value of the sector character at the non-trivial element of Γ . Reality type is thus a function of the spin sector. More generally, whenever the canonical central involution $c = \exp(2\pi i \rho^\vee)$ of a compact group lies in Γ , the classical relation $\nu(\rho) = \rho(c)$ for self-dual ρ [18] shows the indicator to depend only on the sector.

Corollary 7.2 (Planar case: time reversal and anyons). For the planar rotation group $\text{SO}(2)$, $\Gamma = \mathbb{Z}$ and the spin spectrum is the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ of fractional spins s of (12). The self-conjugate sectors form the two-element subgroup

$$\widehat{\mathbb{Z}}[2] = \mathbb{T}[2] = \{0, \tfrac{1}{2}\} : \quad (16)$$

these are the sectors fixed by the time-reversal involution $s \mapsto -s$ —the boson $s = 0$ and the sector $s = \frac{1}{2}$ —while every other sector s is carried to the distinct sector $-s$ and is not self-conjugate. Self-conjugacy is not sufficient for a self-dual representation here: the trivial sector $s = 0$ carries the real representation, whereas the sector $s = \frac{1}{2}$, though self-conjugate, has representations U_s and $U_{-s} = U_s^*$ that form dual pairs with no self-dual member, so it carries no invariant bilinear form. A genuine real single-particle structure thus occurs only at $s = 0$, while generic anyons ($s \notin \{0, \frac{1}{2}\}$), not being self-conjugate, break the time-reversal symmetry that inverts angular momentum [9, 10].

Enlarging π_1 from the finite $\mathbb{Z}/2$ of three dimensions to the infinite $\mathbb{Z}$ of the plane turns the spin spectrum from a two-element group into a continuum, yet by (15) the reality locus remains finite—here of the same order, two. The self-conjugate spin content is therefore rigid under this enlargement while the spin content is not; the same rigidity holds for the $(2+1)$ -dimensional Lorentz and Poincaré groups, whose spectrum is likewise the circle $\mathbb{T}$ with reality locus $\{0, \frac{1}{2}\}$.

8. A cohomological reformulation

The invariants of the preceding sections admit a purely cohomological description that unifies them and identifies the discrete and continuous cases as the torsion and free parts

of a single group. Throughout this section $H^*(G; -)$ denotes the singular cohomology of the group manifold, distinct from the Borel group cohomology $H^2(G, \mathbb{T})$ of Section 2. For a path-connected space the universal-coefficient theorem gives $H^1(X; A) \cong \text{Hom}(H_1(X), A)$, and for a connected Lie group $H_1(G) = \pi_1(G)$; hence the spin spectrum and its reality locus are the singular cohomology groups of the group manifold,

$$\widehat{\pi_1(G)} = H^1(G; \mathbb{T}), \quad \widehat{\pi_1(G)[2]} = H^1(G; \mathbb{Z}/2). \quad (17)$$

The reality locus is thus the mod-2 reduction of the spectrum, and both are computable from the maximal compact subgroup K , onto which G deformation retracts, through $H^*(G; A) \cong H^*(K; A)$.

The exponential coefficient sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{\exp} \mathbb{T} \rightarrow 0$, with $\exp(t) = e^{2\pi i t}$, induces the long exact Bockstein sequence

$$H^1(G; \mathbb{R}) \xrightarrow{\rho} H^1(G; \mathbb{T}) \xrightarrow{\beta} H^2(G; \mathbb{Z}) \xrightarrow{j} H^2(G; \mathbb{R}), \quad (18)$$

in which the spin spectrum occupies the middle term. The connecting map β sends a character to the first Chern class of its associated flat line bundle, and j is the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$. Reading the spectrum through (18) separates its two regimes.

Theorem 8.1 (Torsion-free dichotomy of spin). Let G be a connected Lie group in the topological regime, $\pi_1(G) \cong \mathbb{Z}^r \times F$ with F finite. In the Bockstein sequence (18):

- (i) the continuous (anyonic) spin sectors form the identity component of the spectrum, $\text{im } \rho = \ker \beta \cong \mathbb{T}^r$, the divisible part;
- (ii) the discrete spin sectors form the component group $\pi_0(H^1(G; \mathbb{T})) \cong F$, and β restricts to an isomorphism

$$\pi_0(H^1(G; \mathbb{T})) \xrightarrow{\sim} H^2(G; \mathbb{Z})_{\text{tors}} \quad (19)$$

onto the torsion subgroup of $H^2(G; \mathbb{Z})$.

A spin spectrum is therefore continuous exactly on its free part and discrete exactly on the torsion of $H^2(G; \mathbb{Z})$: the half-integer spin of three dimensions is a torsion class, the fractional spin of the plane a free one.

Proof. By (17), $H^1(G; \mathbb{T}) = \widehat{\pi_1(G)} \cong \mathbb{T}^r \times \widehat{F}$, while $H^1(G; \mathbb{R}) = \text{Hom}(\pi_1(G), \mathbb{R}) \cong \mathbb{R}^r$, the finite factor F contributing nothing to the latter. The map ρ is induced by $\exp$; a homomorphism $\pi_1(G) \rightarrow \mathbb{R}$ annihilates F and is unconstrained on $\mathbb{Z}^r$, so $\text{im } \rho = \mathbb{T}^r \times \{1\}$ is the identity component of the spectrum, which gives (i), with $\ker \beta = \text{im } \rho$ by exactness. For (ii), β induces an injection of $H^1(G; \mathbb{T}) / \ker \beta \cong \widehat{F} \cong F$ into $H^2(G; \mathbb{Z})$ with image $\ker j$. Since $H^2(G; \mathbb{R}) \cong H^2(G; \mathbb{Z}) \otimes \mathbb{R}$ for the finitely generated cohomology of G , the kernel of j is exactly the torsion subgroup, giving (19); this isomorphism is the universal-coefficient identification $H^2(G; \mathbb{Z})_{\text{tors}} \cong \text{Ext}(H_1(G), \mathbb{Z}) \cong F$. $\square$

Remark 7 (The spinorial class and w_2). The class $\beta(\chi) \in H^2(G; \mathbb{Z})$ attached to a discrete spinorial sector is the integral first Chern class of the flat line bundle associated to χ ; it is not itself a Stiefel–Whitney class, which is a mod-2 class, and the two are related by reduction of coefficients. For the rotation group $\beta(\chi)$ generates the torsion $H^2(\mathrm{SO}(3); \mathbb{Z}) = \mathbb{Z}/2$, and its mod-2 reduction $r\beta(\chi) \in H^2(\mathrm{SO}(3); \mathbb{Z}/2)$ is the square a^2 of the reality-locus generator $a \in H^1(\mathrm{SO}(3); \mathbb{Z}/2)$ of (17). This mod-2 class classifies the covering $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$; it is the group-manifold avatar of the universal second Stiefel–Whitney class $w_2 \in H^2(\mathrm{BSO}(n); \mathbb{Z}/2)$, to which a transgresses, and it plays for the descent of a half-integer representation from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$ the role that w_2 plays for the lift of an $\mathrm{SO}(n)$ -structure to a $\mathrm{Spin}(n)$ -structure. Since a Lie group is parallelizable, this class is not $w_2(TG)$, which vanishes.

Table 2 records these groups for the principal cases; each column is a cohomology group of the manifold G , and the split of the last two columns into torsion and free reproduces the discrete/continuous dichotomy of Table 1 from the topology of G alone. The reformulation adds no new hypothesis—it rewrites the topological obstruction of Theorem 3.1 as the cohomology of the group manifold—but it makes the spectrum computable by the methods of algebraic topology and exhibits the discrete, continuous, and internal cases as torsion, free, and vanishing parts of one sequence.

9. Internal symmetry and the localization of spin

The internal gauge groups of the Standard Model fit into the calculus. Its semisimple factor is simply connected and contributes no obstruction, so that all spin is carried by the spacetime factor; the abelian hypercharge factor and the finite quotient defining the global gauge group are not simply connected and are treated separately below, where they contribute internal charges rather than spin. The structural inputs are the additivity of the two invariants under direct products and the simple connectedness of the compact semisimple factors.

Proposition 9.1 (Additivity under products). Let G_1, G_2 be connected Lie groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$.

- (i) $\pi_1(G_1 \times G_2) \cong \pi_1(G_1) \times \pi_1(G_2)$.
- (ii) By the Künneth formula for Chevalley–Eilenberg cohomology,

$$H^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathbb{R}) \cong H^2(\mathfrak{g}_1, \mathbb{R}) \oplus H^2(\mathfrak{g}_2, \mathbb{R}) \\ \oplus (H^1(\mathfrak{g}_1, \mathbb{R}) \otimes H^1(\mathfrak{g}_2, \mathbb{R})).$$

If each $\mathfrak{g}_i$ is perfect (for instance, semisimple), then $H^1(\mathfrak{g}_i, \mathbb{R}) = 0$ and the cross term vanishes, so $H^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathbb{R}) \cong H^2(\mathfrak{g}_1, \mathbb{R}) \oplus H^2(\mathfrak{g}_2, \mathbb{R})$.

Consequently the obstruction pair $(H^2(\mathfrak{g}, \mathbb{R}), \pi_1(G))$ is additive over a direct product whose factors are perfect, and the spin spectrum factors as $\pi_1(\widehat{G_1 \times G_2}) \cong \pi_1(\widehat{G_1}) \times \pi_1(\widehat{G_2})$; a sector is spinorial precisely when at least one of its two components is non-trivial.

Proof. Part (i) is the standard product formula for fundamental groups. For (ii), the Chevalley–Eilenberg complex of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the tensor product of the complexes of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, since $\Lambda^*(\mathfrak{g}_1^* \oplus \mathfrak{g}_2^*) \cong \Lambda^*\mathfrak{g}_1^* \otimes \Lambda^*\mathfrak{g}_2^*$ as differential graded algebras; the algebraic Künneth theorem over the field $\mathbb{R}$ then gives the stated decomposition. For perfect $\mathfrak{g}_i$, $H^1(\mathfrak{g}_i, \mathbb{R}) = (\mathfrak{g}_i/[\mathfrak{g}_i, \mathfrak{g}_i])^* = 0$, so the cross term is zero. The final statement combines (i), (ii), and Theorem 3.1. $\square$

Proposition 9.2. The compact groups $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ are simply connected with $H^2(\mathfrak{su}(2), \mathbb{R}) = H^2(\mathfrak{su}(3), \mathbb{R}) = 0$. Hence each lies in the topological regime with trivial fundamental group: every projective unitary representation is an ordinary representation, and there are no spinorial representations. Their finite centres are $Z(\mathrm{SU}(2)) \cong \mathbb{Z}/2$ and $Z(\mathrm{SU}(3)) \cong \mathbb{Z}/3$.

Proof. The groups $\mathrm{SU}(n)$ are simply connected for all n , with $\pi_1(\mathrm{SU}(n)) = 0$ [17, Ch. 13]. The cohomology vanishes by Lemma 2.1 since $\mathfrak{su}(2), \mathfrak{su}(3)$ are semisimple. The centre of $\mathrm{SU}(n)$ is the group of scalar matrices $\zeta \mathrm{id}$ with $\zeta^n = 1$, isomorphic to $\mathbb{Z}/n$; specializing to $n = 2, 3$ gives the stated centres. $\square$

Propositions 9.1 and 9.2 imply a consequence for the symmetries of relativistic matter. By the Coleman–Mandula theorem [32]—under its hypotheses of a relativistic quantum field theory with a mass gap, a finite particle spectrum below any mass, and a non-trivial analytic S -matrix—a symmetry group combining spacetime and internal symmetries is, on the level of connected groups, a direct product of the Poincaré group with a compact internal group K . If K is semisimple—as is the internal part $\mathrm{SU}(3) \times \mathrm{SU}(2)$ of the Standard Model gauge group—then by Proposition 9.1 the obstruction pair of the total group equals that of the Poincaré factor: $H^2 = 0$ and $\pi_1 = \mathbb{Z}/2$. The spin obstruction is thus localized entirely in the spacetime factor, and the internal factors contribute only ordinary representations. There is no internal spin; all half-integer angular momentum descends from the same $\pi_1(\mathrm{SO}_+^\uparrow(1, 3)) = \mathbb{Z}/2$ identified in Section 4.

This localization uses only that the semisimple internal factor $\mathrm{SU}(3) \times \mathrm{SU}(2)$ is perfect and simply connected, so contributes neither invariant. The abelian hypercharge factor is different: $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, so $\mathrm{U}(1)$ carries a non-trivial topological character—its quantized charge, an element of $\widehat{\mathrm{U}(1)}$ —but this is an internal charge, not a rotational spin, and does not enter the Poincaré factor. The global gauge group $G_{\mathrm{SM}} = (\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ is likewise not simply connected; the quotient by $\mathbb{Z}_6$ alters π_1 and hence which representations are admissible, but this is a question of descent to a quotient, governed by Theorem 3.1, rather than of spin (Remark 9). In every case the half-integer spin remains carried by the spacetime factor alone.

Remark 8 (Reality of internal multiplets). The Frobenius–Schur calculus of Section 5 distinguishes the internal factors. The defining representation of $\mathrm{SU}(2)$ is the spin- $\frac{1}{2}$ representation, of quaternionic type with $\nu = -1$ (Theorem 5.1, Corollary 5.2); the antisymmetric ε -form is the invariant that permits

the pairing of an $SU(2)$ doublet with itself. By contrast the defining representation $\mathbf{3}$ of $SU(3)$ is of complex type, $\nu = 0$, since $\mathbf{3} \not\cong \bar{\mathbf{3}}$: there is no invariant bilinear form $\mathbf{3} \otimes \mathbf{3} \rightarrow \mathbb{C}$ on a single triplet; the canonical invariant pairing is instead $\bar{\mathbf{3}} \otimes \mathbf{3} \rightarrow \mathbb{C}$, equivalently $\mathbf{3}^* \otimes \mathbf{3} \rightarrow \mathbb{C}$. Reality type, computed group-theoretically through (7), thereby separates the weak-isospin and colour factors at the level of their invariant tensors.

Remark 9 (Global form of the gauge group). The same centre data of Proposition 9.2 determines the global form of the Standard-Model gauge group. The representation content realized on known matter is faithful not for the product $SU(3) \times SU(2) \times U(1)$ but for the quotient

$$G_{\text{SM}} \cong (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6,$$

by the order-six subgroup of the centre $\mathbb{Z}/3 \times \mathbb{Z}/2 \times U(1)$ generated by $\xi = (\omega I_3, -I_2, e^{2\pi i/6})$ with $\omega = e^{2\pi i/3}$.

The $U(1)$ factor is normalized so that its charge is the integer $q = 6Y$, the quark doublet having $q = 1$; the element $e^{2\pi i/6}$ then acts on a field of charge q by $(e^{2\pi i/6})^q = e^{2\pi iY}$. On a field of $SU(3)$ -triality t , $SU(2)$ -duality d , and hypercharge Y the generator ξ therefore acts by $\omega^t(-1)^d e^{2\pi iY} = e^{2\pi i(t/3+d/2+Y)}$, which is trivial on every Standard-Model multiplet because $t/3 + d/2 + Y \in \mathbb{Z}$ there (for the quark doublet, $\frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$). Equivalently $G_{\text{SM}} \cong S(U(3) \times U(2))$ [33].

This is the global form that acts faithfully on the established matter content; the Lie algebra alone does not fix the global structure, which a priori could be any quotient $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_k$ with $k \mid 6$. Which representations descend to a quotient is determined as in Theorem 3.1, by the characters the central subgroup carries. The double cover and the quotient defining G_{SM} are two instances of the same principle, differing only in whether one ascends to a cover or descends to a quotient.

10. Relation to prior work

The ingredients of this account are classical. This section records which parts are drawn from the literature and which are new.

The reduction of a quantum symmetry to a projective unitary representation, and of a projective representation to an ordinary representation of a central extension, is due to Wigner [1] and Bargmann [2]. The vanishing $H^2(\mathfrak{g}, \mathbb{R}) = 0$ for semisimple $\mathfrak{g}$, which transfers the obstruction from the algebra to the topology, is the content of Whitehead's lemmas [11, 12]. The classification of the multiplier group $H^2(G, \mathbb{T})$ of a locally compact group, and its resolution into an infinitesimal and a topological part, is the Mackey–Moore theory [5–7], developed for the groups of quantum mechanics by Varadarajan [8]. The covering $SU(2) \rightarrow SO(3)$, the Frobenius–Schur value $\nu(\rho_j) = (-1)^{2j}$, the identity $SU(2) \cong \text{Sp}(1)$, and the Kramers law are standard [18, 21, 22]; they are reproduced here to make the reality data explicit and are not claimed as new. The two-dimensional continuous spin spectrum—fractional spin, or anyons—is due to Leinaas and Myrheim [9] and to Wilczek [10], and rests on $\pi_1(SO(2)) = \mathbb{Z}$ with universal cover $\mathbb{R}$.

Against this background the position of the present results is as follows. The grading of Theorem 3.2(i) is a reformulation, in the language of Pontryagin duality, of the classification of the representations of $\widehat{G}$ by their central characters on $\Gamma = \pi_1(G)$; it is not an independent classification theorem, and part (ii) is the standard fact that for a type I group every central character is realized.

The reality-locus results of Section 7, by contrast, are new to the author's knowledge: the identification of the self-conjugate sectors—necessary for a real or quaternionic structure—with the 2-torsion subgroup $\widehat{\pi_1(G)}[2]$ (Theorem 7.1), the finiteness of that locus even when the spin spectrum is a continuum, and the consequence that all but two of the planar anyonic sectors are incompatible with an antiunitary time reversal (Corollary 7.2) are not stated in this form in the literature, although their ingredients—duality of representations, the Frobenius–Schur trichotomy, Kramers' theorem, and the topological spin of anyons—are classical.

The rest of the contribution is organizational: the uniform description of the spin sectors as $\widehat{\pi_1(G)}$, the placement of the discrete, continuous, and internal cases within one calculus and one table (Table 1), and the corrected delineation of the class of groups with a continuous spectrum—those with infinite fundamental group, among the rotation groups only $SO(2)$ and among the relativistic groups the $(2+1)$ -dimensional Lorentz and Poincaré groups. The account is otherwise self-contained across the discrete and continuous cases.

The cohomological reformulation of Section 8 likewise rests on classical topology—the isomorphism $\text{Hom}(\pi_1, A) = H^1(G; A)$, the Bockstein sequence of the exponential coefficient sequence, and the role of w_2 as the obstruction to spin structures—but its assembly into the torsion-free dichotomy of the spin spectrum (Theorem 8.1), and the resulting relation of half-integer spin to the Stiefel–Whitney class by mod-2 reduction, is, to the author's knowledge, not stated in this form.

11. Conclusion

Spin admits a purely group-theoretic description requiring no physical postulate beyond the projective action of symmetries on quantum states. Two invariants suffice: the infinitesimal class in $H^2(\mathfrak{g}, \mathbb{R})$ and the topological character in $\text{Hom}(\pi_1(G), \mathbb{T})$. The lifting dichotomy (Theorem 3.1) shows that when the infinitesimal class vanishes—as it does for every semisimple group by Whitehead's lemma—the entire content of spin is the character of the fundamental group, and for the rotation group this is the non-trivial character of $\pi_1(SO(3)) = \mathbb{Z}/2$, realized on the double cover $SU(2)$. The Frobenius–Schur computation $\nu(\rho_j) = (-1)^{2j}$ refines this $\mathbb{Z}/2$ into reality data, yielding the symmetry of the invariant form, the identity $SU(2) \cong \text{Sp}(1)$, and the Kramers law $\Theta^2 = (-1)^{2j}$.

The trichotomy (Theorem 6.1) exhibits the relativistic, rotational, and non-relativistic cases as the three combinatorial possibilities for the pair of invariants, and the additivity of Proposition 9.1 localizes all spin in the spacetime factor of a product symmetry, leaving the internal groups $SU(2)$ and $SU(3)$ as carriers of ordinary representations of different reality types. The

same principle applies throughout: representations are sorted by the characters of a central or fundamental-group subgroup, whether one ascends to a cover, as for spin, or descends to a quotient, as for the global form of the gauge group.

More generally, the dichotomy yields a spectral description of spin: in the topological regime the inequivalent sectors of a connected group G are the Pontryagin dual $\widehat{\pi_1(G)}$ (Theorem 3.2), finite when $\pi_1(G)$ is finite and continuous when it is infinite. The half-integer quantization of three-dimensional spin reflects the finiteness of $\pi_1(\text{SO}(3)) = \mathbb{Z}/2$, while the planar rotation group, with $\pi_1(\text{SO}(2)) = \mathbb{Z}$, carries a continuum of fractional-spin sectors and so places anyonic spin within the same calculus (Section 6.4).

Duality of representations makes the spin spectrum a group with an involution, whose fixed locus—the self-conjugate sectors, necessary for a real or quaternionic structure or for compatibility with an antiunitary time reversal—is the 2-torsion subgroup $\widehat{\pi_1(G)}[2]$ (Theorem 7.1). This reality locus stays finite even for a continuous spectrum: for three-dimensional spin both sectors lie in it, whereas for the plane only $s = 0$ and $s = \frac{1}{2}$ do, so that all remaining anyonic sectors break time-reversal symmetry.

Recast as cohomology of the group manifold, the spectrum is $H^1(G; \mathbb{T})$ and the reality locus $H^1(G; \mathbb{Z}/2)$, and the Bockstein sequence of $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T}$ splits spin into the torsion of $H^2(G; \mathbb{Z})$, carried by half-integer spin and reducing mod 2 to the Stiefel–Whitney class, and its free part, carried by the anyons (Theorem 8.1).

This account is restricted to finite-dimensional symmetry groups and the lifting obstruction itself. The spin–statistics connection, the supersymmetric extensions that evade the Coleman–Mandula hypotheses, and the higher spin groups $\text{Spin}(n)$ for $n > 3$ lie outside its scope.

Data availability

This is a theoretical study. No datasets were generated or analysed; all results are derived analytically and are contained within the article.

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References

- [1] E. P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, translated by J. J. Griffin, Academic Press, New York, USA, 1959. <https://shop.elsevier.com/books/group-theory/wigner/978-0-12-750550-3>.
- [2] V. Bargmann, “On unitary ray representations of continuous groups”, *Annals of Mathematics* **59** (1954) 1. <https://doi.org/10.2307/1969831>.
- [3] H. Weyl, *The theory of groups and quantum mechanics*, Dover Publications, New York, USA, 1950 (translated from the second German edition, 1931). <https://books.google.com/books?id=acEgAAAAAAAJ>.
- [4] E. P. Wigner, “On unitary representations of the inhomogeneous Lorentz group”, *Annals of Mathematics* **40** (1939) 149. <https://doi.org/10.2307/1968551>.
- [5] G. W. Mackey, “Unitary representations of group extensions. I”, *Acta Mathematica* **99** (1958) 265. <https://doi.org/10.1007/BF02392428>.
- [6] C. C. Moore, “Group extensions and cohomology for locally compact groups. III”, *Transactions of the American Mathematical Society* **221** (1976) 1. <https://doi.org/10.1090/S0002-9947-1976-0414775-X>.
- [7] C. C. Moore, “Group extensions and cohomology for locally compact groups. IV”, *Transactions of the American Mathematical Society* **221** (1976) 35. <https://doi.org/10.1090/S0002-9947-1976-0414776-1>.
- [8] V. S. Varadarajan, *Geometry of quantum theory*, 2nd ed., Springer, New York, USA, 1985. <https://books.google.com/books?id=J5HvAAAAAAAJ>.
- [9] J. M. Leinaas & J. Myrheim, “On the theory of identical particles”, *Il Nuovo Cimento B* **37** (1977) 1. <https://doi.org/10.1007/BF02727953>.
- [10] F. Wilczek, “Quantum mechanics of fractional-spin particles”, *Physical Review Letters* **49** (1982) 957. <https://doi.org/10.1103/PhysRevLett.49.957>.
- [11] C. Chevalley & S. Eilenberg, “Cohomology theory of Lie groups and Lie algebras”, *Transactions of the American Mathematical Society* **63** (1948) 85. <https://doi.org/10.1090/S0002-9947-1948-0024908-8>.
- [12] N. Jacobson, *Lie algebras*, Interscience, New York, USA, 1962; reprinted by Dover Publications, New York, USA, 1979. <https://store.doverpublications.com/products/9780486136790>.
- [13] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, UK, 1994. <https://doi.org/10.1017/CBO9781139644136>.
- [14] É. Cartan, *The theory of spinors*, Hermann, Paris, France, 1966 (translated from *Leçons sur la théorie des spineurs*, Hermann, Paris, France, 1938); reprinted by Dover Publications, New York, USA, 1981. <https://catalogue.library.cern/literature/j50am-5zs49>.
- [15] C. Chevalley, *The algebraic theory of spinors and Clifford algebras*, Columbia University Press, New York, USA, 1954; reprinted by Springer, Berlin, Germany, 1997. <https://books.google.com/books?id=-ATvAAAAAAAJ>.
- [16] H. B. Lawson & M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, USA, 1989. <https://doi.org/10.1515/9781400883912>.
- [17] B. C. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction*, 2nd ed., Springer, Cham, Switzerland, 2015. <https://doi.org/10.1007/978-3-319-13467-3>.
- [18] T. Bröcker & T. tom Dieck, *Representations of compact Lie groups*, Springer, New York, USA, 1985. <https://doi.org/10.1007/978-3-662-12918-0>.
- [19] G. F. Frobenius & I. Schur, “Über die reellen Darstellungen der endlichen Gruppen”, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin* (1906), 186–208; reprinted in F. G. Frobenius, *Gesammelte Abhandlungen*, Band III, J.-P. Serre (Ed.), Springer, Berlin, 1968, pp. 355–377.
- [20] J.-P. Serre, *Linear representations of finite groups*, Springer, New York, USA, 1977. <https://doi.org/10.1007/978-1-4684-9458-7>.
- [21] W. Fulton & J. Harris, *Representation theory: a first course*, Springer, New York, USA, 1991. <https://doi.org/10.1007/978-1-4612-0979-9>.
- [22] F. J. Dyson, “The threefold way: algebraic structure of symmetry groups and ensembles in quantum mechanics”, *Journal of Mathematical Physics* **3** (1962) 1199. <https://doi.org/10.1063/1.1703863>.
- [23] J. A. de Azcárraga & J. M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, Cambridge University Press, Cambridge, UK, 1995. <https://doi.org/10.1017/CBO9780511599897>.
- [24] S. Weinberg, *The quantum theory of fields, Vol. 1: foundations*, Cam-

- bridge University Press, Cambridge, UK, 1995. <https://doi.org/10.1017/CBO9781139644167>.
- [25] G. W. Mackey, *The theory of unitary group representations*, University of Chicago Press, Chicago, USA, 1976. <https://books.google.com/books?id=oeZzQgAACAAJ>.
- [26] J. N. Ginocchio, "Relativistic symmetries in nuclei and hadrons", *Physics Reports* **414** (2005) 165. <https://doi.org/10.1016/j.physrep.2005.04.003>.
- [27] C. A. Onate, M. O. Oluwayemi & I. B. Okon, "Dirac equation for energy-dependent potential with energy-dependent tensor interaction", *Journal of the Nigerian Society of Physical Sciences* **5** (2023) 917. <https://doi.org/10.46481/jnsps.2023.917>.
- [28] M. Abu-Shady, E. Omugbe & E. P. Inyang, "Approximate bound-state solutions of the fractional Schrödinger equation under the spin-spin-dependent Cornell potential", *Journal of the Nigerian Society of Physical Sciences* **6** (2024) 1771. <https://doi.org/10.46481/jnsps.2024.1771>.
- [29] J.-M. Lévy-Leblond, "Galilei group and nonrelativistic quantum mechanics", *Journal of Mathematical Physics* **4** (1963) 776. <https://doi.org/10.1063/1.1724319>.
- [30] R. Haag, *Local quantum physics: fields, particles, algebras*, 2nd ed., Springer, Berlin, Germany, 1996. <https://doi.org/10.1007/978-3-642-61458-3>.
- [31] E. İnönü & E. P. Wigner, "On the contraction of groups and their representations", *Proceedings of the National Academy of Sciences of the United States of America* **39** (1953) 510. <https://doi.org/10.1073/pnas.39.6.510>.
- [32] S. Coleman & J. Mandula, "All possible symmetries of the S matrix", *Physical Review* **159** (1967) 1251. <https://doi.org/10.1103/PhysRev.159.1251>.
- [33] J. C. Baez & J. Huerta, "The algebra of grand unified theories", *Bulletin of the American Mathematical Society* **47** (2010) 483. <https://doi.org/10.1090/S0273-0979-10-01294-2>.