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The Wiener index and spectrum of the nil clean graph of a finite commutative ring

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Abstract

For a finite commutative ring R, the nil clean graph $G_N(R)$ is the undirected simple graph with the elements of R as its vertex set, where two distinct vertices x and y are joined by an edge exactly when $x+y$ decomposes as a nilpotent element plus an idempotent element – in other words, when the sum $x+y$ is a nil clean element of R. This article builds upon and extends several results established in Ref. [1]. We introduce the Wiener index of the nil clean graph of a finite commutative ring R, and as an application we determine this index for $G_N(\mathbb{Z}_n)$ and for the nil clean graph of finite products of rings of the form $\mathbb{Z}_n$, drawing on the framework of Refs. [1, 2]. The paper concludes with a description of the spectrum of the nil clean graph for several classes of rings.

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1. Introduction

Associating a graph structure with a ring was an idea originally put forward in Ref. [3], and since then, bringing algebraic tools to bear on graph-theoretic questions has become a widely used technique. As a consequence, numerous graph invariants have been linked to algebraic properties of the underlying ring, spurring considerable growth in this area of research. As one instance of this construction, every commutative ring R with unity gives rise to a graph on the vertex set R, where two elements x and y are adjacent whenever $x+y$ can be written as a sum of a nilpotent element and an idempotent element; this is precisely the nil clean graph $G_N(R)$.

The study in Ref. [1] investigated fundamental graph-theoretic features of this construction – completeness, girth, clique number, chromatic number, and connectedness, among others. A related construction, the nil clean divisor graph, was subsequently examined in Ref. [4].

Our focus here is a different invariant altogether: the Wiener index, among the most widely applied graph indices in the literature. It traces back to H. Wiener's 1947 work, with its formal graph-theoretic definition later credited to Hosoya in Ref. [5]; the index has since been computed for a variety of graph families; see Refs. [6–8] and the survey in Ref. [9]. Recall that the Wiener index of a graph sums the pairwise distances over all vertex pairs. Wiener's original observation in Ref. [10] linked this quantity to the boiling points of alkanes via their molecular structure graphs, a discovery that triggered extensive subsequent research connecting graph indices to properties of chem-

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ical compounds.

In parallel, the spectral theory of graphs built from algebraic structures has drawn substantial attention in recent decades, with notable work on Cayley graphs (Ref. [11]), commuting graphs (Ref. [12]), and power graphs (Refs. [13, 14]).

The remainder of the paper is organized as follows. We first collect the notation and background results needed later. Section 2 shows how an equivalence relation on R allows the nil clean graph to be written as a generalized composition of complete and totally disconnected graphs, and uses this to compute distances between arbitrary vertex pairs in such a composition. Section 3 derives the Wiener index for this generalized composition and applies it to products of $\mathbb{Z}_n$-type rings together with their equivalence classes. Section 4 introduces the Wiener matrix and its total trace, and establishes the relationship linking the total trace of the Wiener matrix to the Wiener index itself. Section 5 specializes this to the nil clean graph of products of rings $\mathbb{Z}_p$ for prime p. Finally, Section 6 determines the spectrum of the nil clean graph for several families of rings.

We now review the basic graph-theoretic vocabulary, notions, and facts that will be used later, following Ref. [15]. A graph G consists of a pair (V, E), where $V = V(G)$ denotes the vertex set and $E = E(G)$ denotes the edge set. Given a graph G, two distinct vertices $x, y \in G$ are called adjacent — written $x \sim y$ — when an edge joins them; when no such edge exists we write $x \nsim y$.

A subgraph G' of G is any graph satisfying $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For a subset $U \subseteq V(G)$, the induced subgraph $G(U)$ is formed by taking U as its vertex set and declaring two vertices of U adjacent precisely when they are adjacent in G. A graph is termed complete when every pair of distinct vertices is adjacent to one another.

A path is a finite sequence of pairwise distinct vertices in which consecutive vertices in the sequence are adjacent. The graph G is called connected if every pair of vertices can be joined by some path. For vertices x and y in G, the distance $d(x, y)$ (also written $d_G(x, y)$) is defined as the number of edges contained in a shortest path connecting x and y.

Throughout the remainder of this work, our attention is restricted to finite commutative rings with identity for which the associated nil clean graph is connected.

2. Nil clean graph as a generalized composition of graphs

Our goal in this section is to write a nil clean graph as a generalized composition of graphs, a viewpoint that will prove convenient later when we compute its Wiener index. We also use this representation to work out the distance between any pair of vertices once the nil clean graph has been expressed in this composed form.

We start by recalling the definition of the nil clean graph attached to a commutative ring with unity.

Definition 1. Ref. [1] Let R be a ring. We define the nil clean graph of R, denoted $G_N(R)$, to be the simple graph whose vertex set is R itself, where two distinct vertices are joined by an edge precisely when their sum is a nil clean element of the ring — that is, whenever their sum can be written as the sum of some nilpotent element and some idempotent element of R.

The generalized composition of graphs, denoted by $H[G_1, G_2, \ldots, G_k]$, was introduced by Schwenk in Ref. [16].

Definition 2. Ref. [16] Consider a graph H having k vertices $\{v_1, v_2, \ldots, v_k\}$, and let $G_1, G_2, \ldots, G_k$ be k graphs that are pairwise disjoint from one another. The *generalized composition graph* $G = H[G_1, G_2, \ldots, G_k]$ is then constructed by substituting each vertex v_i of H with the graph G_i, and subsequently connecting every vertex of G_i to every vertex of G_j exactly when v_i and v_j are adjacent in H. Here H is referred to as the base graph of G, while the graphs G_i are called its factors.

Now let R denote an arbitrary finite commutative ring with identity for which $G_N(R)$ is connected. Our aim is to represent such a ring in terms of a generalized graph composition. For every element $x \in R$, we define the nil clean set of x, written $N(x)$, to consist of all elements $y \in R$ such that $x + y$ is a nil clean element. In particular, if $2x$ happens to be nil clean, then x itself belongs to $N(x)$. We denote the set of nil clean elements of R by $NC(R)$.

Next, we introduce a relation $\sim$ on R by declaring $x \sim y$ whenever $N(x) = N(y)$. One can check that this relation is indeed an equivalence relation on R. Denote by $C_1, C_2, \ldots, C_k$ the resulting equivalence classes, with $c_1, c_2, \ldots, c_k$ chosen as representative elements of each class respectively. These classes will be referred to as the equivalence nil clean classes of $G_N(R)$.

We now record a few basic facts about the equivalence nil clean classes just introduced.

Lemma 1. *Two representatives c_i and c_j satisfy $c_i + c_j \in NC(R)$ precisely when every element of C_i is adjacent to every element of C_j.*

Proof. Assume first that $c_i + c_j \in NC(R)$, and pick arbitrary $x \in C_i$, $y \in C_j$. From $c_i + c_j \in NC(R)$ it follows that $c_j \in N(c_i)$ and $c_i \in N(c_j)$. Because $x \in C_i$ and $y \in C_j$, we have $N(x) = N(c_i)$ and $N(y) = N(c_j)$, so $c_j \in N(x)$ and $c_i \in N(y)$; the former gives $x \in N(c_j)$, and combining this with $N(c_j) = N(y)$ yields $x \in N(y)$. Hence $x + y \in NC(R)$, as required.

For the converse, suppose every element of C_i is adjacent to every element of C_j. In particular this holds for $c_i \in C_i$ and $c_j \in C_j$, so c_i and c_j are adjacent in $G_N(R)$, i.e. $c_i + c_j \in NC(R)$. □

Corollary 2. For each i, the subgraph of $G_N(R)$ induced on C_i is either complete or totally disconnected, being complete exactly when $2c_i \in NC(R)$. Moreover, whenever $i \neq j$, either every vertex of C_i is adjacent to every vertex of C_j in $G_N(R)$, or none is.

Denote by H the subgraph of $G_N(R)$ induced on the representative set $\{c_1, c_2, \ldots, c_k\}$; thus two distinct representatives c_i and c_j are adjacent in H exactly when $c_i + c_j \in NC(R)$.

Lemma 3. *The graph H is connected.*

Proof. Take any two distinct vertices c_i, c_j of H. If $c_i + c_j \in NC(R)$ they are already adjacent, so suppose instead $c_i + c_j \notin$

$NC(R)$. Since $G_N(R)$ itself is connected, some path joins c_i to c_j there, say $c_i \sim d_1 \sim d_2 \sim \cdots \sim d_k \sim c_j$, with each intermediate vertex d_i belonging to some class C_k.

Replacing every intermediate vertex by the representative of its class – say $u_1, u_2, \ldots, u_k$ in the order induced by the path – produces a c_i-c_j walk $c_i \sim u_1 \sim u_2 \sim \cdots \sim u_m \sim u_{n+1} \sim \cdots \sim u_k \sim c_j$ inside H (Lemma 1 guarantees the required adjacencies survive this substitution). A walk between two vertices always contains a path between them, so H contains a c_i-c_j path, proving H is connected. □

We now show that $G_N(R)$ can itself be realized as a generalized composition built from certain complete and empty graphs.

Proposition 4. Let G_i denote the subgraph of $G_N(R)$ induced on the vertex set C_i. Then

$$G_N(R) = H[G_1, G_2, \ldots, G_k]. \tag{1}$$

Proof. This is an immediate consequence of Corollary 2. □

3. Wiener index of a nil clean graph

Having represented a connected nil clean graph as a generalized composition of graphs, we now determine the distance between any two of its vertices, and use this to arrive at a closed formula for its Wiener index.

Lemma 5. *Let G_i be the subgraph of $G_N(R)$ induced by C_i and let $G_N(R) = H[G_1, G_2, \ldots, G_k]$. If $x \in C_i$, $y \in C_j$ and $i \neq j$, then $d_R(x, y) = d_H(c_i, c_j)$.*

Proof.

Write $n = d_H(c_i, c_j)$; since $i \neq j$ we have $c_i \neq c_j$, so $n \geq 1$. Let $c_i = f_0 \sim f_1 \sim f_2 \sim \cdots \sim f_{n-1} \sim f_n = c_j$ be a shortest c_i-c_j path in H; thus $f_0, f_1, \ldots, f_n$ are pairwise distinct representatives, each $f_t = c_m$ for a suitable index m.

Step 1: $d_R(x, y) \leq n$. We first dispose of the case $n = 1$: then $c_i \sim c_j$ in H, i.e. $c_i + c_j \in NC(R)$, so $c_j \in N(c_i) = N(x)$, giving $x \sim c_j$ in R. Since $N(y) = N(c_j)$, and adjacency is symmetric, $y \sim z$ iff $c_j \sim z$ for every z; taking $z = x$ and using $x \sim c_j$ gives $x \sim y$. Thus $d_R(x, y) \leq 1 = n$.

Now suppose $n \geq 2$. Since $f_0 \sim f_1$ in H, we have $f_0 + f_1 \in NC(R)$, i.e. $f_1 \in N(f_0) = N(c_i)$. As $x \in C_i$, we have $N(x) = N(c_i)$, so $f_1 \in N(x)$ and hence $x \sim f_1$ in R. Symmetrically, $f_{n-1} \sim f_n = c_j$ in H gives $f_{n-1} \in N(c_j) = N(y)$, so $f_{n-1} \sim y$ in R. Since $n \geq 2$, f_1 and f_{n-1} are among the interior vertices $f_1, \ldots, f_{n-1}$ of the path (possibly $f_1 = f_{n-1}$ if $n = 2$, which causes no issue). Together with the edges $f_1 \sim f_2 \sim \cdots \sim f_{n-1}$ already present in H (hence in R, since H is an induced subgraph of R on the representative set), this produces the sequence

$$x \sim f_1 \sim f_2 \sim \cdots \sim f_{n-1} \sim y, \tag{2}$$

of n consecutive edges in R. The vertices $f_1, \ldots, f_{n-1}$ represent classes other than C_i and C_j (as $f_0, \ldots, f_n$ are pairwise distinct representatives), so none of them can coincide with x or y; thus the displayed sequence is a genuine path in R of length n joining x and y. Consequently $d_R(x, y) \leq n = d_H(c_i, c_j)$ in this case as well.

Step 2: $d_H(c_i, c_j) \leq d_R(x, y)$. Let $x = g_0 \sim g_1 \sim \cdots \sim g_m = y$ be a shortest x-y path in R, so $m = d_R(x, y)$. For each t, let $\bar{g}_t$ denote the representative of the class containing g_t; in particular $\bar{g}_0 = c_i$ and $\bar{g}_m = c_j$. By Lemma 1, adjacency in $G_N(R)$ depends only on the classes of the two vertices involved: since $g_t \sim g_{t+1}$, either g_t and g_{t+1} already lie in the same class, or every vertex of that class is adjacent to every vertex of the other, in which case $\bar{g}_t \sim \bar{g}_{t+1}$ in H. Form the sequence $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_m$ and delete each term that repeats its predecessor; by the observation just made, consecutive *distinct* terms of the resulting sequence are adjacent in H, so what remains is a walk in H from c_i to c_j of length at most m. Since a walk between two vertices contains a path between them, H has a c_i-c_j path of length at most m, whence $d_H(c_i, c_j) \leq m = d_R(x, y)$.

Combining Steps 1 and 2 gives $d_R(x, y) = d_H(c_i, c_j)$. □

Continuing with this notation, write $V(H) = \{c_1, c_2, \ldots, c_k\}$, and denote the vertex set of the subgraph G_i induced by C_i as $\{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$. We now compute the distance between an arbitrary pair of vertices of $G_N(R)$ under this generalized-composition description.

Lemma 6. *Let v_i^p and v_j^q, where $1 \leq p \leq n_i$ and $1 \leq q \leq n_j$, be any two vertices of $G_N(R)$. Then*

$$d_{G_N(R)}(v_i^p, v_j^q) = \begin{cases} 1, & i = j,\ G_i \text{ complete}, \\ 2, & i = j,\ G_i \text{ totally disconnected}, \\ d_H(i, j), & i \neq j, \end{cases} \tag{3}$$

where $d_{G_N(R)}(v_i^p, v_j^q)$ represents the distance between v_i^p and v_j^q inside $G_N(R)$.

Proof. First, suppose $i = j$ and G_i happens to be complete. In this case v_i^p and v_j^q are directly joined by an edge, giving $d_{G_N(R)}(v_i^p, v_j^q) = 1$.

Next, suppose $i = j$ but G_i is totally disconnected (an empty graph). Then v_i^p and v_j^q share no edge within G_i. Since H is connected, we can find some vertex s of H adjacent to i; consequently, every vertex belonging to G_i is joined in $G_N(R)$ to every vertex belonging to G_s. Choosing any vertex x from G_s, we obtain $v_i^p \sim x \sim v_j^q$, a path of length two joining v_i^p to v_j^q in $G_N(R)$. Hence $d_{G_N(R)}(v_i^p, v_j^q) = 2$.

Finally, assume $i \neq j$, so that v_i^p lies in G_i and v_j^q lies in G_j. Because H is connected, its vertices i and j are joined by a path in H. Take a shortest such path $i \sim d_1 \sim d_2 \sim \cdots \sim d_{k-1} \sim d_k = j$ in H. Correspondingly, we can build a path

$$v_i^p \sim v_{d_1}^{p_1} \sim \cdots \sim v_{d_{k-1}}^{p_{k-1}} \sim v_j^q, \tag{4}$$

in $G_N(R)$ connecting v_i^p to v_j^q. As the underlying path between i and j in H is of minimal length, the corresponding path between v_i^p and v_j^q in $G_N(R)$ is likewise minimal. Therefore $d(v_i^p, v_j^q) = d_H(i, j)$. □

Take R to be any finite commutative ring with unity for which the nil clean graph is connected. Proposition 4 then gives $G_N(R) = H[G_1, G_2, \ldots, G_k]$, with each factor G_i either complete or empty.

Theorem 7. *Under the notation established above, the Wiener index of the nil clean graph $G_N(R)$ satisfies*

$$W(G_N(R)) = \sum_{2c_i \in NC(R)} \binom{|C_i|}{2} + \sum_{2c_i \notin NC(R)} 2\binom{|C_i|}{2} + \sum_{1 \le i<j \le k} |C_i|\,|C_j|\, d_H(c_i, c_j). \tag{5}$$

Proof.

Recall that $G_N(R) = H[G_1, G_2, \ldots, G_k]$, where each G_i is the subgraph of $G_N(R)$ induced on C_i. Whenever $2c_i \in NC(R)$, the corresponding G_i is complete and every pair of its vertices sits at distance 1; summing over all such classes contributes $\sum \binom{|C_i|}{2}$ to the total. When instead $2c_i \notin NC(R)$, the induced subgraph G_i is totally disconnected, so by Lemma 6 every internal pair is at distance 2, contributing $\sum 2\binom{|C_i|}{2}$.

Finally, for $x \in C_i$ and $y \in C_j$ with $i \neq j$, Lemma 5 gives $d_{G_N(R)}(x, y) = d_H(c_i, c_j)$. Each unordered pair of distinct classes $\{C_i, C_j\}$ contributes exactly $| C_i || C_j |$ cross-class vertex pairs (one vertex from each class), and every cross-class vertex pair belongs to exactly one such unordered class-pair; summing $| C_i || C_j |\ d_H(c_i, c_j)$ once over each unordered pair $\{C_i, C_j\}$, i.e. over $1 \le i < j \le k$, therefore contributes $\sum_{1 \le i<j \le k} | C_i || C_j |\ d_H(c_i, c_j)$ to the total (summing instead over all ordered pairs $1 \le i, j \le k$ would count each such cross-class pair twice, since d_H is symmetric). Adding the three contributions establishes the stated formula. □

Remark 8. As an illustration of the preceding theorem, consider the ring $(\mathbb{Z}_{12}, \bigoplus, \bigodot)$, whose nil clean elements are $\{0, 1, 3, 4, 6, 7, 9, 10\}$. Here the equivalence-nil clean classes turn out to be $C_1 = \{0, 3, 6, 9\}$, $C_2 = \{1, 4, 7, 10\}$, $C_3 = \{2, 5, 8, 11\}$, with respective representatives $0, 1, 2$. The base graph H is a path on 3 vertices, with edges $0 \sim 1$ and $1 \sim 2$ (so $d_H(0, 1) = d_H(1, 2) = 1$ and $d_H(0, 2) = 2$), and among the factors $G_1 = G_3 = K_4$ while $G_2 = \overline{K}_4$, giving $G = H[G_1, G_2, G_3]$.

Applying the theorem, the Wiener index of $G_N(\mathbb{Z}_{12})$ works out to

$$W(G_N(\mathbb{Z}_{12})) = \binom{4}{2} + \binom{4}{2} + 2\binom{4}{2} + \sum_{i<j} | C_i || C_j |\ d_H(c_i, c_j), \tag{6}$$

where the first two binomial terms come from the complete classes C_1 and C_3 (since $2 \cdot 0 = 0$ and $2 \cdot 2 = 4$ are both nil clean) and the middle term from the totally disconnected class C_2 (since $2 \cdot 1 = 2$ is not nil clean). Since $| C_i |= 4$ for every i, the cross-class sum is

$$16(d_H(c_1, c_2)+d_H(c_1, c_3)+d_H(c_2, c_3)) = 16(1+2+1) = 64, \tag{7}$$

so $W(G_N(\mathbb{Z}_{12})) = (6 + 6 + 12) + 64 = 24 + 64 = 88$. Hence $W(G_N(\mathbb{Z}_{12})) = 88$.

4. Wiener matrix of a nil clean graph

We open this section by recalling the notions of the Wiener matrix and total trace of a graph, and then show that for $G_N(R)$ the Wiener index coincides with the total trace of this matrix.

Let G be a generalized composition built from graphs $G_1, G_2, \ldots, G_k$ over a connected base graph H on k vertices, where every factor G_i – with vertex set $\{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$ – is either complete or totally disconnected. The Wiener matrix of such a graph is defined below.

Definition 3. Ref. [17]. The Wiener matrix $W(G)$ of the graph G is the $k \times k$ real symmetric matrix whose entries are given by

$$(W(G))_{i,j} = \begin{cases} \binom{|G_i|}{2}, & i = j,\ G_i \text{ complete}, \\ 2\binom{|G_i|}{2}, & i = j,\ G_i \text{ totally disconnected}, \\ |G_i|\,|G_j|\, d_H(i, j), & i \neq j. \end{cases} \tag{8}$$

We also recall the total trace of a square matrix.

Definition 4. Ref. [17]. For a real symmetric $k \times k$ matrix $A = (a_{i,j})$, its total trace $tt(A)$ is given by $tt(A) = \frac{1}{2}(\sum_{i \neq j} a_{i,j}) + \sum_{1 \le i \le k} a_{i,i}$.

Remark 9. Ref. [17] Looking back at how the Wiener matrix of $G = H[G_1, G_2, \ldots, G_k]$ is built, its diagonal entries record the sum of pairwise distances taken within a single equivalence class, while an off-diagonal (i, j) entry (for $i \neq j$) records the sum of distances between vertices of G_i and vertices of G_j – which is symmetric in i, j, so $W(G)$ is indeed a symmetric matrix. Summing the diagonal entries together with the entries strictly above the diagonal therefore reproduces the Wiener index of G exactly, which is to say the total trace of the Wiener matrix equals the Wiener index of the graph.

We illustrate below how the Wiener index of a connected nil clean graph can be recovered from its Wiener matrix by taking the total trace.

Example 10. Take the ring $(\mathbb{Z}_{10}, \bigoplus, \bigodot)$, whose nil clean elements are $\{0, 1, 5, 6\}$. Its equivalence classes are $C_1 = \{0, 5\}$, $C_2 = \{1, 6\}$, $C_3 = \{2, 7\}$, $C_4 = \{3, 8\}$, $C_5 = \{4, 9\}$, with representatives $0, 1, 2, 3, 4$ respectively.

We first identify which classes induce a complete subgraph by checking, for each representative c, whether $2c \in NC(\mathbb{Z}_{10}) = \{0, 1, 5, 6\}$: $2 \cdot 0 = 0 \in NC(\mathbb{Z}_{10})$ and $2 \cdot 3 = 6 \in NC(\mathbb{Z}_{10})$, while $2 \cdot 1 = 2$, $2 \cdot 2 = 4$, $2 \cdot 4 = 8$ are not nil clean. Hence, by Corollary 2, only C_1 and C_4 induce complete graphs K_2; the remaining three classes induce totally disconnected graphs $\overline{K}_2$.

This is exactly the pattern anticipated by Theorem 13 and Remark 14. Indeed $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ (here $k = 1$, $p_1 = 5$), and under the isomorphism $x \mapsto (x \bmod 2, x \bmod 5)$ the two classes predicted to be complete are $\{(0, 0), (1, 0)\}$ and $\{(0, \frac{p_1+1}{2}), (1, \frac{p_1+1}{2})\} = \{(0, 3), (1, 3)\}$. Translating back to $\mathbb{Z}_{10}$, $\{(0, 0), (1, 0)\}$ corresponds to $\{0, 5\} = C_1$ (the class of 0, i.e. the all-zero/idempotent class) and $\{(0, 3), (1, 3)\}$ corresponds to $\{3, 8\} = C_4$, confirming the two classes found above directly.

Next we determine the base graph H on the representative set $\{0, 1, 2, 3, 4\}$: two representatives c, c' are adjacent exactly when $c+c' \in NC(\mathbb{Z}_{10})$. Checking all pairs, $0+1 = 1$, $1+4 = 5$, $4 + 2 = 6$, $2 + 3 = 5$ all lie in $NC(\mathbb{Z}_{10})$, while every other pair sums to 2, 3, 4, 7 or 9, none of which is nil clean. Thus H is the path

$$0 \sim 1 \sim 4 \sim 2 \sim 3, \tag{9}$$

a path on 5 vertices (equivalently, of length 4; consistent with the general fact, evident from Lemma 20 and Lemma 21 for $\mathbb{Z}_{p^2}$-type constructions, that the two complete classes sit at the two ends of this path), with $C_1 = \{0, 5\}$ and $C_4 = \{3, 8\}$ occupying the endpoints. Reading off distances along this path gives

$$\begin{aligned} &d_H(0,1) = 1, \quad d_H(0,4) = 2, \quad d_H(0,2) = 3,\\ &d_H(0,3) = 4, \quad d_H(1,4) = 1, \end{aligned} \tag{10}$$

$$\begin{aligned} &d_H(1,2) = 2, \quad d_H(1,3) = 3, \quad d_H(4,2) = 1,\\ &d_H(4,3) = 2, \quad d_H(2,3) = 1. \end{aligned} \tag{11}$$

Using Definition 3, the diagonal entries of the Wiener matrix are $\binom{2}{2} = 1$ for the complete classes C_1, C_4 and $2\binom{2}{2} = 2$ for the totally disconnected classes C_2, C_3, C_5, while each off-diagonal entry is $|C_i||C_j|\, d_H(c_i, c_j) = 4\, d_H(c_i, c_j)$. This yields

$$W(G_N(\mathbb{Z}_{10})) = \begin{bmatrix} 1 & 4 & 12 & 16 & 8\\ 4 & 2 & 8 & 12 & 4\\ 12 & 8 & 2 & 4 & 4\\ 16 & 12 & 4 & 1 & 8\\ 8 & 4 & 4 & 8 & 2 \end{bmatrix}, \tag{12}$$

with rows/columns ordered C_1, C_2, C_3, C_4, C_5; note that the two diagonal entries equal to 1 (the complete classes) occur precisely in positions 1 and 4, i.e. at C_1 and C_4, as predicted above.

Its total trace equals 88, so we conclude $W(G_N(\mathbb{Z}_{10})) = 88$.

Remark 11. Recall that the distance matrix D of a simple graph G is the $| V(G) | \times | V(G) |$ matrix with $D_{i,j} = d_G(i, j)$ for $i \neq j$ and zero on the diagonal; its total trace is, by definition, exactly the Wiener index of G. What makes the Wiener matrix advantageous by comparison is its much smaller order, equal to the number of equivalence nil clean classes rather than $| V(G) |$. In Example 10, for instance, $| V(G_N(\mathbb{Z}_{10})) |= 10$, so the distance matrix has order 10×10, whereas the Wiener matrix of $\mathbb{Z}_{10}$ needs only order 5×5.

5. Wiener index of nil clean graphs of products of rings of integers modulo p

We begin this section by turning to the nil clean graph of a reduced ring. As the example below shows, such a graph need not be connected, so the Wiener index cannot always be defined for reduced rings in general. On the other hand, Ref. [1] shows that the nil clean graph of any $\mathbb{Z}_n$ is connected, from which one can deduce connectedness for the nil clean graph of a product of such rings as well. Note also that whenever every equivalence nil clean class is a singleton, the Wiener matrix reduces to the ordinary distance matrix. With this in mind, we now examine the equivalence nil clean classes of the nil clean graph of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ for distinct primes p_i.

Theorem 12. *If $R \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, where $p_i > 2$ are distinct primes, then there are $\prod_{i=1}^{k} p_i$ equivalence nil clean classes, each of which is a singleton.*

Proof. As each $\mathbb{Z}_{p_i}$ is a field, the nil clean elements of R are precisely the tuples $\{(x_1, x_2, \ldots, x_k) : x_i = 0 \text{ or } 1\}$. Fix $x = (x_1, x_2, \ldots, x_k) \in R$, so that $N(x) = \{y \in R : x + y \in NC(R)\}$, i.e. $N(x) = \{y = (y_1, y_2, \ldots, y_k) \in R : x_i + y_i = 0 \text{ or } 1\}$, which rewrites as $N(x) = \{y = (y_1, y_2, \ldots, y_k) \in R : y_i = p_i - x_i \text{ or } y_i = p_i - x_i + 1\}$. Our task is to show no $z \neq x$ in R can share this same set $N(x)$.

So suppose, for contradiction, that some $z = (z_1, z_2, \ldots, z_k) \neq x$ in R satisfies $N(z) = N(x)$; then also $N(z) = \{y = (y_1, y_2, \ldots, y_k) \in R : y_i = p_i - x_i \text{ or } y_i = p_i - x_i + 1\}$. Every $y \in N(z)$ must satisfy $y + z \in NC(R)$, meaning $y_i + z_i = 0$ or 1 coordinatewise, which translates to $p_i - x_i + z_i = 0$ or 1 or $p_i - x_i + 1 + z_i = 0$ or 1.

Working through the four cases: $p_i - x_i + z_i = 0$ forces $z_i = x_i$; $p_i - x_i + z_i + 1 = 1$ again forces $z_i = x_i$; $p_i - x_i + z_i + 1 = 1$ (the alternate branch) gives $z_i = x_i + 1$; and $p_i - x_i + z_i + 1 = 0$ gives $z_i = x_i - 1$. So each coordinate z_i can only be x_i, $x_i + 1$, or $x_i - 1$. If $z_i = x_i$ holds for every i, then $z = x$, contradicting $z \neq x$.

So assume instead that some coordinate differs from x. If the j-th coordinate is $z_j = x_j + 1$, pick $y \in N(x)$ with $y_j = p_j - x_j + 1$; then $y_j + z_j = 2$, so $y + z \notin NC(R)$, meaning $y \notin N(z)$ even though $y \in N(x)$ – contradicting $N(z) = N(x)$. Likewise, if the k-th coordinate is $z_k = x_k - 1$, choosing $y \in N(x)$ with $y_k = p_k - x_k$ gives $y_k + z_k = p_k - x_k + x_k - 1 = -1$, so again $y + z \notin NC(R)$ while $y \in N(x)$, the same contradiction. Both alternatives are therefore impossible, so no such $z \neq x$ exists, and every equivalence class in R is a singleton – giving exactly $\prod_{i=1}^{k} p_i$ equivalence classes. □

Next we turn to the case where the product of integer-modulo-p_i rings includes a factor $\mathbb{Z}_2$, and examine how the equivalence nil clean classes change.

Theorem 13. *If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, where $p_i > 2$ are distinct primes, then there are $\prod_{i=1}^{k} p_i$ equivalence nil clean classes, each containing exactly two elements.*

Proof. Here the nil clean elements of R are the tuples $\{(x_1, x_2, \ldots, x_k) : x_i = 0 \text{ or } 1\}$. Fix $x = (x_1, x_2, \ldots, x_k) \in R$; then $N(x) = \{y = (y_1, y_2, \ldots, y_k) \in R : x_i + y_i = 0 \text{ or } 1\}$, equivalently $N(x) = \{y = (y_1, y_2, \ldots, y_k) \in R : y_1 = 0 \text{ or } 1,\ y_i = p_i - x_i \text{ or } p_i - x_i + 1 \text{ for } i \geq 2\}$. We look for $z \neq x$ in R with $N(z) = N(x)$.

Suppose such a z exists, so $N(z)$ takes the same form: $N(z) = \{y = (y_1, y_2, \ldots, y_k) \in R : y_1 = 0 \text{ or } 1,\ y_i = p_i - x_i \text{ or } p_i - x_i + 1\}$.

Every y in this set must satisfy $y+z \in NC(R)$, i.e. $y_i+z_i = 0$ or 1 coordinatewise – in particular $y_1+z_1 = 0$ or 1, and $y_i+z_i = 0$ or 1 for $2 \le i \le k$. From $y_1 = 0$ we get $z_1 \in \{0, 1\}$, and the same follows if $y_1 = 1$; either way $z_1 \in \{0, 1\}$.

For the remaining coordinates, $y_i + z_i = 0$ or 1 expands to $p_i - x_i + z_i = 0$ or 1, or $p_i - x_i + 1 + z_i = 0$ or 1, for $2 \le i \le k$. Each of these four sub-cases pins down z_i:

$p_i-x_i+z_i = 0$ gives $z_i = x_i$; $p_i-x_i+1+z_i = 0$ gives $z_i = x_i-1$; $p_i - x_i + z_i = 1$ gives $z_i = 1 + x_i$; and $p_i - x_i + 1 + z_i = 1$ gives $z_i = x_i$. Hence, for $2 \le i \le k$, each z_i must equal x_i, $x_i + 1$, or $x_i - 1$.

Suppose $z_i = x_i$ holds for every i with $2 \le i \le k$; then $z = (x_1, x_2, \ldots, x_k)$ where only the first coordinate is free to be 0 or 1.

Now suppose instead some coordinate $j \ge 2$ has $z_j = x_j+1$; choosing $y \in N(x)$ with $y_j = p_j - x_j + 1$ gives $y_j + z_j = 2$, so $y + z \notin NC(R)$ and hence $y \notin N(z)$ despite $y \in N(x)$ – contradicting $N(z) = N(x)$. A symmetric contradiction arises if some coordinate k has $z_k = x_k - 1$: taking $y \in N(x)$ with $y_k = p_k - x_k$ yields $y_k + z_k = p_k - x_k + x_k - 1 = -1$, so again $y+z \notin NC(R)$ while $y \in N(x)$, contradicting $N(z) = N(x)$. Both alternatives being ruled out, z must be either $(0, x_2, x_3, \ldots, x_k)$ or $(1, x_2, x_3, \ldots, x_k)$, one of which is x itself and the other its unique partner. Thus, for every $x \in R$, there is exactly one $z \ne x$ with $N(z) = N(x)$, so each equivalence class contains precisely two elements, and there are $\prod_{i=1}^{k} p_i$ such classes in total. □

Remark 14. Theorem 13 tells us that whenever a factor of 2 appears among the p_i's in $\mathbb{Z}_2 \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_k}$, the number of equivalence classes is $p_1p_2 \ldots p_k$ (exactly as in Theorem 12), with every class of size exactly 2. Among these classes (of which there are $2p_1p_2 \ldots p_k$ elements in total, i.e. all of R, grouped into $p_1p_2 \ldots p_k$ pairs), only two induce a complete subgraph, namely $\{(0, 0, \ldots, 0), (1, 0, \ldots, 0)\}$ and $\{(0, \frac{p_1+1}{2}, \frac{p_2+1}{2}, \ldots, \frac{p_k+1}{2}), (1, \frac{p_1+1}{2}, \frac{p_2+1}{2}, \ldots, \frac{p_k+1}{2})\}$; every remaining class induces a totally disconnected subgraph.

Remark 15. Consider $R \cong \mathbb{Z}_2 \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_k}$. Since exactly 2 of its equivalence classes induce complete (hence connected) subgraphs while the remaining $p_1p_2 \ldots p_k - 2$ induce empty subgraphs, Theorem 7 yields

$$W(R) = 1 + 1 + 2(p_1p_2 \ldots p_k - 2) + 4 \sum_{1 \le i < j \le p_1p_2 \ldots p_k} d_H(c_i, c_j), \tag{13}$$

which simplifies to

$$W(R) = 2(p_1p_2 \ldots p_k - 1) + 4 \sum_{1 \le i < j \le p_1p_2 \ldots p_k} d_H(c_i, c_j). \tag{14}$$

The sum runs over the $\binom{p_1p_2 \ldots p_k}{2}$ unordered pairs of the $p_1p_2 \ldots p_k$ equivalence classes, each pair counted once, as in Theorem 7.

6. Spectrum of the nil clean graph

This section works out the spectrum of the nil clean graph of $\mathbb{Z}_n$ for several specific choices of n.

Lemma 16. *For a prime p, the nil clean graph of $(\mathbb{Z}_p, \oplus, \odot)$ is a path on p vertices.*

Proof. Since $\mathbb{Z}_p$ is a field, its only nil clean elements are $\{0, 1\}$. Following the definition of the nil clean graph, one checks that $0 \sim 1 \sim p-1 \sim 2 \sim p-2 \sim \ldots \sim \frac{p-1}{2} \sim \frac{p+1}{2}$ traces out a path in $G_N(\mathbb{Z}_p)$ visiting every vertex, along which each element m is adjacent to either $p-m$ or $p-m+1$. To see this is the full edge set, suppose m is adjacent to some $r \in \mathbb{Z}_p$; then $m+r \in NC(\mathbb{Z}_p)$ forces $m+r \in \{0, 1\}$, so r equals $p-m$ or $p-m+1$ accordingly. Hence $G_N(\mathbb{Z}_p)$ is exactly a path on p vertices. □

Remark 17. Recall that a path on p vertices has spectrum $\{\lambda_i : \lambda_i = 2\cos(\frac{\pi}{p+1}), i = 1, 2, \ldots p\}$; by the previous lemma, this is likewise the spectrum of $G_N(\mathbb{Z}_p)$.

Remark 18. It was shown in Ref. [1] that for any positive integer n, the nil clean graph of $\mathbb{Z}_{2^n}$ is the complete graph K_{2^n}, whose spectrum is therefore $\{2^n - 1, \underbrace{-1, -1, \ldots - 1}_{2^n-1 \text{ times}}\}$.

We now turn to the spectrum of the nil clean graph of $(\mathbb{Z}_{p^2}, \oplus, \odot)$ for a prime $p > 2$. The nilpotents of $\mathbb{Z}_{p^2}$ are $\{0, p, 2p, 3p, \ldots (p-1)p\}$ and its idempotents are $\{0, 1\}$, so its nil clean elements are $\{0, 1, p, p+1, 2p, 2p+1, 3p, 3p+1, \ldots, (p-1)p, (p-1)p+1\}$. As before, $G_N(\mathbb{Z}_{p^2})$ can be written as a generalized composition of complete and totally disconnected graphs. A direct computation shows $N(0) = N(p) = N(2p) = \ldots = N((p-1)p)$,

$N(1) = N(p+1) = N(2p+1) = \ldots = N(p^2-p+1)$, and so on for the remaining residues. This yields the equivalence classes $\{0, p, 2p, 3p, \ldots (p-1)p\}$, $\{1, p+1, 2p+1, 3p+1, \ldots p^2 - p+1\}$, $\{2, p+2, 2p+2, 3p+2, \ldots p^2-p+2\}, \ldots \{p-1, 2p-1, 3p-1, \ldots p^2-1\}$ – that is, p equivalence classes, each of size p. We next describe the base graph H underlying $G_N(\mathbb{Z}_{p^2})$.

Lemma 19. *In the generalized composition of $G_N(\mathbb{Z}_{p^2})$, the base graph H is a path on p vertices.*

Proof. Recall the equivalence classes of $\mathbb{Z}_{p^2}$: $\{0, p, 2p, 3p, \ldots (p-1)p\}$, $\{1, p+1, 2p+1, 3p+1, \ldots p^2 - p+1\}$, $\{2, p+2, 2p+2, 3p+2, \ldots p^2-p+2\}, \ldots \{p-1, 2p-1, 3p-1, \ldots p^2-1\}$, with respective representatives $0, 1, 2, \ldots p-1$, so $V(H) = \{0, 1, 2, \ldots p-1\}$. One readily checks that $0 \sim 1 \sim p-1 \sim 2 \sim p-2 \sim \ldots \frac{p-1}{2} \sim \frac{p+1}{2}$ is a path in H passing through every vertex.

Our task is to show this exhausts H's edges, i.e. that H is precisely this path on p vertices. Along it, each vertex $m(\ne 0, \ne 1)$ is adjacent to $p-m$ or $p-m+1$. To confirm no further edges exist, suppose $m \in V(H)$ with $m \ne 0, 1$ is adjacent to some $r \in V(H)$; then $m + r \in NC(\mathbb{Z}_{p^2})$, and since the nil clean elements of $\mathbb{Z}_{p^2}$ all take the form $0, 1, sp, sp+1$ for $s \in \{1, 2, \ldots p-1\}$, we examine each possibility for $m+r$ in turn.

If $m + r = 0$, then $r = -m = p^2 - m$, which does not lie in $V(H)$ – a contradiction. If $m+r = 1$, then $r = 1-m = p^2+1-m$, again outside $V(H)$, another contradiction. If $m + r = sp$ for some $s \in \{1, 2, \ldots p-1\}$, then $r = sp - m$; taking $s = 1$ gives the valid $r = p - m$, but any $s \ne 1$ pushes r outside $V(H)$, a contradiction. Similarly, if $m+r = sp+1$, then $r = sp+1-m$; only $s = 1$ (giving $r = p+1-m$) keeps r inside $V(H)$, while $s \neq 1$ again contradicts $r \in V(H)$. So indeed every $m(\neq 0, \neq 1)$ in H is adjacent only to $p-m$ or $p-m+1$.

It remains to check the neighbours of 0 and 1. Since 0 and 1 are the only nil clean elements represented in H, vertex 0 can only be adjacent to 1. As for 1, suppose it is adjacent to some m in H; then $m+1 \in NC(\mathbb{Z}_{p^2})$, so $m+1$ takes one of the forms $0, 1, sp$, or $sp+1$ for $s \in \{1, 2, \dots p-1\}$. The case $m+1 = 0$ gives $m = p^2-1 \notin V(H)$, a contradiction; $m+1 = 1$ gives $m = 0$, so 1 is adjacent to 0; $m+1 = sp$ gives $m = sp-1$, which lies in $V(H)$ only when $s = 1$ (yielding $m = p-1$, so 1 is adjacent to $p-1$), any other s being a contradiction; and $m+1 = sp+1$ gives $m = sp$, which never lies in $V(H)$ for $s \in \{1, \dots, p-1\}$, a contradiction. Hence the only neighbours of 1 are 0 and $p-1$, and altogether H is exactly the path on p vertices described above. □

We next examine the subgraphs induced by these p equivalence classes in $G_N(\mathbb{Z}_{p^2})$, which will feed into our computation of its spectrum.

Lemma 20. *Among the p equivalence classes $C(0), C(1), \dots C(p-1)$ of $\mathbb{Z}_{p^2}$, the subgraphs of $G_N(\mathbb{Z}_{p^2})$ induced by $C(0)$ and $C(\frac{p+1}{2})$ are complete and all other induced subgraphs are totally disconnected.*

Proof. Within $C(0) = \{0, p, 2p, 3p, \dots, (p-1)p\}$ we find $0+p = p \in NC(\mathbb{Z}_{p^2})$, so 0 and p are adjacent, making the subgraph induced by $C(0)$ is complete. Similarly, in $C(\frac{p+1}{2}) = \{\frac{p+1}{2}, p+\frac{p+1}{2}, 2p+\frac{p+1}{2}, \dots (p-1)p+\frac{p+1}{2}\}$ we have $\frac{p+1}{2}+p+\frac{p+1}{2} = 2p+1 \in NC(\mathbb{Z}_{p^2})$, so this class also induces a complete subgraph. To see that no other class does so, suppose some $m \in \mathbb{Z}_{p^2}$ with $0 < m < p$ and $m \neq \frac{p+1}{2}$ has $C(m)$ inducing a complete subgraph; then $m+p+m \in NC(\mathbb{Z}_{p^2})$, i.e. $m+p+m$ equals $0,\ 1,\ sp$, or $sp+1$ for some $s \in \{1, 2, \dots p-1\}$.

Checking each case: $m+p+m = 0$ gives $m = \frac{p^2-p}{2} \geq p$, contradicting $m < p$; $m+p+m = 1$ gives $m = \frac{p^2-p+1}{2} \notin \mathbb{Z}_{p^2}$, also impossible. If $m+p+m = sp$, then $s = 1$ forces $m = 0$ (a contradiction), while an even s keeps $m \in \mathbb{Z}_{p^2}$ but an odd s pushes $m \geq p$, again a contradiction. If instead $m+p+m = sp+1$: taking $s = 1$ gives $m \notin \mathbb{Z}_{p^2}$, taking $s = 2$ gives $m = \frac{p+1}{2}$ (excluded by assumption), an odd s gives $m \notin \mathbb{Z}_{p^2}$, and an even s forces $m > p$ – every branch a contradiction.

Hence no such m with $0 < m < p$ and $m \neq \frac{p+1}{2}$ can induce a complete subgraph, confirming the claim. □

To derive the spectrum of $G_N(\mathbb{Z}_{p^2})$ we now invoke the following tool from Ref. [18].

Lemma 21. *Ref. [18]. Let $G = H[G_1, G_2, \dots G_k]$ be the H-join of the graphs $G_1, G_2, \dots G_k$ and let H be a graph of order k. Assume that ρ_{ij} is the (i, j)th entry of the adjacency matrix of H. Let n_i be the order of G_i, A_i be the adjacency matrix of G_i and each G_i be r_i-regular, for $1 \leq i \leq k$. Then r_i is an eigenvalue of G_i and $Spec(A(G)) = (\cup(spec(A_i) \setminus \{r_i\})) \cup spec(\tilde{A}(G))$, where*

$$\tilde{A}(G) = \begin{bmatrix} r_1 & \sqrt{n_1 n_2}\rho_{1,2} & \cdots & \sqrt{n_1 n_k}\rho_{1,k} \\ \sqrt{n_2 n_1}\rho_{2,1} & r_2 & \cdots & \sqrt{n_2 n_k}\rho_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{n_k n_1}\rho_{k,1} & \sqrt{n_k n_2}\rho_{k,2} & \cdots & r_k \end{bmatrix}. \tag{15}$$

Remark 22. The spectrum of $G_N(\mathbb{Z}_{p^2})$ consists of $\{\underbrace{-1, -1, \dots, -1}_{p-1 \text{ times}},\ \underbrace{-1, -1, \dots, -1}_{p-1 \text{ times}}, \underbrace{\underbrace{0, 0, \dots, 0}_{p-1 \text{ times}}, \dots, \underbrace{0, 0, \dots, 0}_{p-1 \text{ times}}}_{(p-2) \text{ times}}\}$ and the spectrum of a $p \times p$ matrix.

Proof. Combining Lemmas 19 and 20, $G_N(\mathbb{Z}_{p^2})$ decomposes as the generalized composition $G_N(\mathbb{Z}_{p^2}) = P_p[K_p,\ \overline{K}_p, \overline{K}_p, \dots \overline{K}_p, K_p]$, with P_p a path on p vertices and K_p, $\overline{K}_p$ the complete and totally disconnected graphs on p vertices, whose spectra are respectively $\{p-1,\ -1, \dots, -1\}$ and $\{0, 0, \dots, 0\}$. Invoking Lemma 21, the spectrum is

$$\begin{aligned} \operatorname{Spec} G_N(\mathbb{Z}_{p^2}) = \{&\underbrace{-1, -1, \dots, -1}_{p-1 \text{ times}}, \underbrace{-1, -1, \dots, -1}_{p-1 \text{ times}}, \\ &\underbrace{\underbrace{0, 0, \dots, 0}_{p-1 \text{ times}}, \dots, \underbrace{0, 0, \dots, 0}_{p-1 \text{ times}}}_{(p-2) \text{ times}}\} \\ &\cup \operatorname{Spec} \tilde{A}(G), \end{aligned} \tag{16}$$

where

$$\tilde{A}(G) = \begin{bmatrix} p-1 & p & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p & 0 & p & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 0 & p & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & p & 0 & p & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & p & 0 & p \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & p & p-1 \end{bmatrix}. \tag{17}$$

□

Proposition 23.

The p eigenvalues contributed by $\tilde{A}(G)$ above – so that, together with Remark 22, the full spectrum of $G_N(\mathbb{Z}_{p^2})$ is characterized explicitly – are precisely the p real roots of the following explicitly given Chebyshev polynomial equation:

$$\begin{aligned} p^p U_p\left(\frac{\lambda}{2p}\right) - 2(p-1)p^{p-1} U_{p-1}\left(\frac{\lambda}{2p}\right) \\ + (p-1)^2 p^{p-2} U_{p-2}\left(\frac{\lambda}{2p}\right) = 0, \end{aligned} \tag{18}$$

where U_n denotes the Chebyshev polynomial of the second kind. Equivalently, writing $\lambda = 2p\cos\theta$ with $\theta \in (0, \pi)$, these are exactly the values $2p\cos\theta$ for which

$$p^2\sin((p+1)\theta) - 2p(p-1)\sin(p\theta) + (p-1)^2\sin((p-1)\theta) = 0. \tag{19}$$

Proof.

Let $f_k(\lambda)$ be the characteristic polynomial of the leading $k\times k$ principal submatrix of $\lambda I - \tilde{A}(G)$. Expanding the tridiagonal determinant along the last row gives

$$\begin{aligned} f_0 &= 1, \qquad f_1(\lambda) = \lambda-(p-1), \\ f_k(\lambda) &= \lambda f_{k-1}(\lambda) - p^2 f_{k-2}(\lambda), \qquad 2\le k\le p-1, \end{aligned} \tag{20}$$

since rows $2,\dots,p-1$ of $\tilde{A}(G)$ have diagonal entry 0, and

$$f_p(\lambda) = (\lambda-(p-1))f_{p-1}(\lambda) - p^2 f_{p-2}(\lambda), \tag{21}$$

since row p has diagonal entry $p-1$ as well. Let $D_n(\lambda) = p^n U_n(\lambda/2p)$ be the characteristic polynomial of the unperturbed $n\times n$ path matrix pT_n (so $D_0 = 1$, $D_1 = \lambda$, and $D_n = \lambda D_{n-1} - p^2 D_{n-2}$ for $n\ge 2$). Since f_k and D_k satisfy the same linear recursion for $2\le k\le p-1$ and agree in the sense $f_0 = D_0$, $f_1 = D_1-(p-1)D_0$, linearity of the recursion gives $f_k(\lambda) = D_k(\lambda)-(p-1)D_{k-1}(\lambda)$ for $0\le k\le p-1$. Substituting $k = p-1, p-2$ into the formula for f_p,

$$\begin{aligned} f_p &= \lambda f_{p-1} - p^2 f_{p-2} - (p-1)f_{p-1} \\ &= [D_p-(p-1)D_{p-1}] \\ &\quad -(p-1)[D_{p-1}-(p-1)D_{p-2}], \end{aligned} \tag{22}$$

using that $\lambda f_{p-1} - p^2 f_{p-2} = D_p-(p-1)D_{p-1}$ by the same linearity. This simplifies to

$$f_p(\lambda) = D_p(\lambda) - 2(p-1)D_{p-1}(\lambda) + (p-1)^2 D_{p-2}(\lambda), \tag{23}$$

which is the characteristic polynomial of $\tilde{A}(G)$; the eigenvalues of $\tilde{A}(G)$ are exactly its roots, giving the first stated equation. Setting $\lambda = 2p\cos\theta$ and using $D_n(2p\cos\theta) = p^n\sin((n+1)\theta)/\sin\theta$, then clearing the common factor $p^{p-2}/\sin\theta$, gives the trigonometric form. □

Remark 24.

The spectrum of $G_N(\mathbb{Z}_{2p})$ consists of $\{-1,-1,\underbrace{0,0,\dots,0}_{p-2 \text{ times}}\}$ and the spectrum of a $p\times p$ matrix.

Proof. Similarly, $G_N(\mathbb{Z}_{2p})$ decomposes as

$G_N(\mathbb{Z}_{2p}) = P_p[K_2,\overline{K}_2,\overline{K}_2,\dots,\overline{K}_2,K_2]$, with P_p a path on p vertices and K_2, $\overline{K}_2$ the complete and totally disconnected graphs on 2 vertices.

Here $\mathrm{spec}(K_2) = \{1,-1\}$ with $r_i = 1$, and $\mathrm{spec}(\overline{K}_2) = \{0,0\}$ with $r_i = 0$. Removing r_i once from each of the two complete factors leaves -1 with multiplicity 2, and removing r_i once from each of the $p-2$ empty factors leaves 0 with multiplicity $p-2$. Hence, by Lemma 21, the spectrum of $G_N(\mathbb{Z}_{2p})$ is

$$\{-1,-1,\underbrace{0,0,\dots,0}_{p-2 \text{ times}}\}\cup \mathrm{Spec}\tilde{A}(G), \tag{24}$$

where

$$\tilde{A}(G) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 & 1 \end{bmatrix}. \tag{25}$$

□

Proposition 25.

The p eigenvalues contributed by $\tilde{A}(G)$ above are precisely the p real roots of the following explicitly given Chebyshev polynomial equation:

$$4\,U_p\Big(\frac{\lambda}{4}\Big) - 4\,U_{p-1}\Big(\frac{\lambda}{4}\Big) + U_{p-2}\Big(\frac{\lambda}{4}\Big) = 0, \tag{26}$$

where U_n denotes the Chebyshev polynomial of the second kind. Equivalently, writing $\lambda = 4\cos\theta$ with $\theta\in(0,\pi)$, these are exactly the values $4\cos\theta$ for which

$$4\sin((p+1)\theta) - 4\sin(p\theta) + \sin((p-1)\theta) = 0. \tag{27}$$

Proof.

This is the special case $q = 2, \alpha = 1$ of the same computation used above for $\tilde{A}(G)$ in the $\mathbb{Z}_{p^2}$ case (with the path-edge weight p there replaced by $q = 2$ and the corner weight $p-1$ there replaced by $\alpha = 1$): with $D_n(\lambda) = 2^n U_n(\lambda/4)$ the characteristic polynomial of the unperturbed path matrix $2T_n$, the characteristic polynomial of $\tilde{A}(G)$ works out to

$$\begin{aligned} &D_p(\lambda) - 2D_{p-1}(\lambda) + D_{p-2}(\lambda) \\ &\quad = 2^p U_p(\lambda/4) - 2\cdot 2^{p-1}U_{p-1}(\lambda/4) + 2^{p-2}U_{p-2}(\lambda/4), \end{aligned} \tag{28}$$

which, upon dividing by 2^{p-2}, is the first stated equation; the trigonometric form follows exactly as before. □

7. Conclusion

In this paper, we studied the Wiener index and adjacency spectrum of nil clean graphs associated with finite commutative rings. We obtained the Wiener index for $(G_N(\mathbb{Z}_n))$ and finite direct products of rings of the form $(\mathbb{Z}_n)$, and described the spectrum for several classes of rings. These results highlight the connection between the algebraic structure of a ring and the corresponding graph-theoretic properties, providing scope for further study of nil clean graphs.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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