



Explicit prime labelings for families of unicyclic graphs with star and path attachments

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Abstract

A graph G admits a prime labeling if its vertices can be assigned the integers $1, 2, \dots, |V(G)|$ so that every pair of adjacent vertices receives coprime labels; a graph with this property is called a prime graph. This paper studies prime labeling for unicyclic graphs formed by attaching stars or pairs of pendant paths to every vertex of a cycle. Here $C_n * S_m$ denotes the graph obtained by attaching m new pendant vertices to each vertex of C_n . We give explicit constructions showing that $C_n * S_2$, $C_n * S_4$, and $C_n * S_6$ – cycles with two, four, or six pendant vertices at each cycle vertex – are all prime graphs, extending known results for three, five, and seven pendant vertices. We then consider unicyclic graphs in which every cycle vertex is identified with one endpoint of each of two paths P_{k_1} and P_{k_2} , where $k_1 \geq k_2$, and prove that such a graph is prime whenever $\gcd(k_1 - 1, k_2) = 1$ and $\gcd(k_2, n - 1) = 1$, or whenever $\gcd(k_1, k_2 - 1) = 1$ and $\gcd(k_1, n - 1) = 1$. A direct consequence is that a unicyclic graph whose two pendant paths both have k vertices is prime whenever $\gcd(k, n - 1) = 1$. Worked examples accompany each construction, so the labelings are straightforward to verify and to extend to other unicyclic families.

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1. Introduction

In graph theory, labeling the vertices of a graph G in a systematic way often reveals structural properties that are not otherwise apparent. Vertex labelings connect graph structure to number theory and have applications in network design, coding theory, and cryptography. One well-studied type is prime labeling (PL): each vertex of G receives a distinct label from $\{1, 2, \dots, |V(G)|\}$, and adjacent vertices must receive coprime labels. A graph that admits such a labeling is called a prime graph (PG). The idea goes back to Entringer and was formalized in Ref. [1].

Around 1980, Entringer conjectured that every tree is prime. The conjecture is still open. Partial results exist: paths,

stars, caterpillars, complete binary trees, spiders, and all trees on fewer than 16 vertices have been shown to be prime (see Ref. [2]), but no proof covers trees in general.

Prime labeling has since been extended to other graph families. All cycles are prime, and so is the disjoint union $C_{2k} \cup C_n$ [3]. The complete graph K_n is not prime for $n \geq 4$, while the wheel graph W_n is prime exactly when n is even [1]. These examples show that primality depends on structure, not merely on the order of the graph.

Seoud *et al.* in Ref. [4] proved that fans, helms, flowers, $K_{2,n}$, and $K_{3,n}$ (except when $n = 3$ or 7) are prime, and that the join $P_n + K_m$ with $m \geq 3$ is not – an example of how denser constructions can break the coprimality condition.

Here we focus on graphs formed by attaching extra paths or star shapes to the vertices of a cycle. Such graphs are simple to describe, yet their structure is rich enough to raise nontrivial

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labeling questions. Related work spans trees, cycles, and combinations of simple graphs; see Refs. [5–11] for further results.

Prime labelings are useful not merely as existence results: an explicit labeling is a short arithmetic certificate that every edge satisfies a local coprimality constraint. Finding such a certificate by unrestricted search may require examining a factorial number of vertex permutations, whereas a closed formula exposes the number-theoretic mechanism and produces a labeling directly. Unicyclic graphs are a natural testing ground because they are the smallest connected extension of trees containing a cycle. Their pendant parts impose local constraints, while the closing cycle edge creates a genuinely global compatibility condition. Thus, constructions for these graphs help isolate techniques that may be reused in broader graph-labeling problems.

Recent studies continue to investigate prime labeling and related graph-labeling schemes, often through explicit constructions. Bedi *et al.* studied uniform cycle-snake graphs in 2023 [12], while Dayal *et al.* considered prime and prime-distance labelings of several elementary families [13]. Further explicit prime constructions appeared for wheel-based graphs, k -prime graphs, and unions of graphs in 2024 [5, 7, 8]. Ponraj *et al.* developed a related prime-modular cordial labeling in 2025 [14]. Within the broader literature on arithmetic graph labelings, two recent papers treated even-vertex odd-edge root-square-mean labelings, including cycle-related families [15, 16]. These studies emphasize constructive techniques, but they do not supply the prime labelings established here.

We are interested specifically in unicyclic graphs, that is, graphs containing exactly one cycle. Throughout this paper, $C_n * S_m$ denotes the graph obtained from a cycle C_n by attaching m new pendant vertices to each cycle vertex; thus, S_m refers here to the set of m new leaves in this construction. Diefenderfer *et al.* in Ref. [17] showed that $C_n * S_3$, $C_n * S_5$, and $C_n * S_7$ are prime. We extend this line of work by proving that $C_n * S_2$, $C_n * S_4$, and $C_n * S_6$ are also prime, and by studying the more general case in which two pendant paths are attached at each cycle vertex.

All constructions below are explicit and illustrated with examples, so they are straightforward to verify and to extend to other unicyclic families. Terms not defined here follow Refs. [18–20].

2. Prime vertex labeling of families of unicyclic graphs

In this section, we present explicit prime vertex labelings for several families of unicyclic graphs obtained from a cycle by attaching stars or pendant paths at each cycle vertex.

The common construction principle is to partition the available labels into consecutive local blocks, one block for each cycle vertex. The cycle vertex receives a carefully selected odd label near the middle of its block, and the pendant vertices receive the remaining labels. Small label differences reduce the edge checks to excluding only a few prime divisors. The final cycle edge is then checked separately because it joins the first and last blocks.

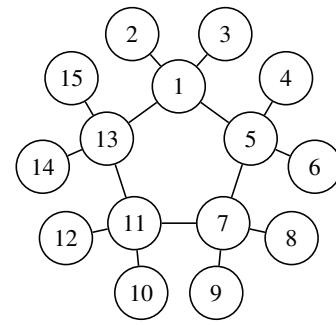


Figure 1. A PL of $C_5 * S_2$

Theorem 2.1. For every integer $n \geq 3$, the unicyclic graph $C_n * S_2$ is a PG.

Proof. The construction alternates the position of the cycle label inside each three-label block. This keeps every cycle label odd, while adjacent cycle labels differ by only 2 or 4.

Observe that $C_n * S_2$ has $3n$ vertices. Let $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be the vertices of the cycle and let the pendant vertices adjacent to α_i be denoted by β_i^j where $1 \leq j \leq 2$.

Define the labeling function $f: V \rightarrow \{1, 2, \dots, 3n\}$ by:

1. If i is even, where $0 \leq i \leq n-1$, then

$$\begin{aligned} f(\alpha_i) &= 3i + 1, \\ f(\beta_i^j) &= 3i + 1 + j, \quad \text{where } 1 \leq j \leq 2. \end{aligned}$$

Because $f(\alpha_i) = 3i + 1$ is odd,

$$\begin{aligned} \gcd(f(\alpha_i), f(\beta_i^2)) &= \gcd(3i + 1, 3i + 3) = 1, \\ \gcd(f(\alpha_i), f(\beta_i^1)) &= \gcd(3i + 1, 3i + 2) = 1. \end{aligned}$$

2. If i is odd, where $1 \leq i \leq n-1$, then

$$\begin{aligned} f(\alpha_i) &= 3i + 2, \\ f(\beta_i^1) &= 3i + 1, \quad f(\beta_i^2) = 3i + 3. \end{aligned}$$

The label of α_i is therefore relatively prime to the labels of β_i^j , where $j = 1, 2$.

Because $f(\alpha_i)$ is odd and $f(\alpha_{i+1}) - f(\alpha_i) = 2$ or 4 for all $0 \leq i \leq n-2$, $\gcd(f(\alpha_i), f(\alpha_{i+1}))$ is odd and divides 4; hence

$$\gcd(f(\alpha_i), f(\alpha_{i+1})) = 1 \text{ for all } 0 \leq i \leq n-2.$$

Also,

$$\gcd(f(\alpha_0), f(\alpha_{n-1})) = \gcd(1, f(\alpha_{n-1})) = 1$$

Thus, the labels of consecutive cycle vertices are relatively prime. So, f is a PL. Therefore, $C_n * S_2$ is a PG. $\square$

Example 2.2. Figure 1 illustrates the block construction for $C_5 * S_2$; the five inner labels are the cycle labels and each pair of outer labels is the remainder of the corresponding three-label block.

Theorem 2.3. For every integer $n \geq 3$, the unicyclic graph $C_n * S_4$ is a PG.

Proof. Each five-label block is arranged so that its cycle label is odd. When a possible difference is divisible by 3, the alternative position in the block is selected to keep the cycle label indivisible by 3.

Observe that $C_n * S_4$ has $5n$ vertices. Let $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be the vertices of the cycle, and let the pendant vertices adjacent to α_i be denoted by β_i^j where $1 \leq j \leq 4$.

Define the labeling function $f : V \rightarrow \{1, 2, \dots, 5n\}$ by:

1. For $i = 0$:

$$f(\alpha_i) = 1, \quad f(\beta_i^j) = j + 1 \quad \text{for } j = 1, 2, 3, 4.$$

2. For i is even where $2 \leq i \leq n - 1$:

$$f(\alpha_i) = 5i + 3$$

$$f(\beta_i^j) = \begin{cases} 5i + j & \text{for } j = 1, 2 \\ 5i + j + 1 & \text{for } j = 3, 4 \end{cases}$$

Since $f(\alpha_i) = 5i + 3$ is odd and $|f(\alpha_i) - f(\beta_i^j)| = 1$ or 2 . Then

$$\gcd(f(\alpha_i), f(\beta_i^j)) = 1 \quad \text{for all } j = 1, 2, 3, 4.$$

3. For i is odd, we have two cases:

Case 1: $3 \mid 5i + 1$. In this case, we get $3 \nmid 5i + 2$. The labeling function f is:

$$f(\alpha_i) = 5i + 2,$$

$$f(\beta_i^1) = 5i + 1,$$

$$f(\beta_i^j) = 5i + 1 + j, \quad \text{for } j = 2, 3, 4.$$

Since $f(\alpha_i) = 5i + 2$ is odd, $3 \nmid 5i + 2$ and $|f(\alpha_i) - f(\beta_i^j)| = 1, 2$, or 3 . Then

$$\gcd(f(\alpha_i), f(\beta_i^j)) = 1 \quad \text{for all } j = 1, 2, 3, 4.$$

Case 2: $3 \nmid 5i + 1$. The labeling function f is:

$$f(\alpha_i) = 5i + 4$$

$$f(\beta_i^4) = 5i + 5$$

$$f(\beta_i^j) = 5i + j \quad \text{for } j = 1, 2, 3.$$

Similar to **Case 1**, we get

$$\gcd(f(\alpha_i), f(\beta_i^j)) = 1 \quad \text{for all } j = 1, 2, 3, 4.$$

Now, we want to show that the labels for consecutive cycle vertices are relatively prime.

Firstly, since $f(\alpha_0) = 1$, then

$$\gcd(f(\alpha_0), f(\alpha_1)) = 1,$$

$$\gcd(f(\alpha_0), f(\alpha_{n-1})) = 1.$$

When i is odd, then $f(\alpha_{i+1}) = 5(i + 1) + 3 = 5i + 8$ and for the label of α_i , we have two cases:

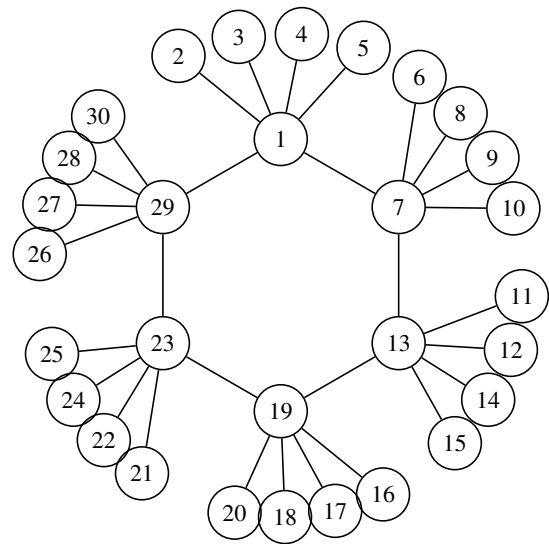


Figure 2. A PL of $C_6 * S_4$.

Case 1: $3 \mid 5i + 1$. Then

$$3 \nmid 5i + 2, \quad f(\alpha_i) = 5i + 2 \text{ is odd,}$$

$$f(\alpha_{i+1}) - f(\alpha_i) = 5i + 8 - (5i + 2) = 6.$$

Thus, $\gcd(f(\alpha_i), f(\alpha_{i+1}))$ is odd, divides 6 and does not divide 3. So, $\gcd(f(\alpha_i), f(\alpha_{i+1})) = 1$

Case 2: $3 \nmid 5i + 1$. Then

$$f(\alpha_i) = 5i + 4, \quad \text{which is odd,}$$

$$f(\alpha_{i+1}) - f(\alpha_i) = 5i + 8 - (5i + 4) = 4.$$

Thus, $\gcd(f(\alpha_i), f(\alpha_{i+1}))$ is odd and divides 4. So, $\gcd(f(\alpha_i), f(\alpha_{i+1})) = 1$. Now let $i \geq 2$ be even, so that $i + 1$ is odd. Then $f(\alpha_i) = 5i + 3$, whereas $f(\alpha_{i+1})$ is either $5i + 7$ or $5i + 9$. The corresponding differences are 4 and 6. In the first case, any common divisor is odd and divides 4; in the second case, the construction for the odd index $i + 1$ ensures that $3 \nmid f(\alpha_{i+1})$, so an odd common divisor of the two labels cannot divide 6. Hence $\gcd(f(\alpha_i), f(\alpha_{i+1})) = 1$.

We have therefore considered both possibilities for $1 \leq i \leq n - 2$, namely, i odd and i even. Together with the two cycle edges incident with α_0 , this accounts for every edge of the cycle.

Thus, the labels of consecutive cycle vertices are relatively prime.

□

Example 2.4. Figure 2 illustrates the labeling of $C_6 * S_4$. In particular, the closing cycle edge joins the vertices labeled 1 and 29, and hence its endpoint labels are coprime.

Theorem 2.5. For every integer $n \geq 3$, the unicyclic graph $C_n * S_6$ is a PG.

Proof. The seven-label construction follows the same idea. The label of each cycle vertex is selected to avoid the small odd primes 3 and 5 that may divide the relevant label differences.

Observe that $C_n * S_6$ has $7n$ vertices. Let $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be the vertices of the cycle and let the pendant vertices adjacent to α_i be denoted by β_i^j where $1 \leq j \leq 6$.

For each i , let

$$B_i = \{7i + 1, 7i + 2, \dots, 7i + 7\}.$$

First set

$$f(\alpha_0) = 1.$$

If $i > 0$ is even, set

$$f(\alpha_i) = \begin{cases} 7i + 3, & 3 \nmid 7i + 3, \\ 7i + 5, & 3 \mid 7i + 3. \end{cases}$$

If i is odd, choose $c_i \in \{2, 4, 6\}$ so that

$$3 \nmid 7i + c_i \quad \text{and} \quad 5 \nmid 7i + c_i,$$

and set $f(\alpha_i) = 7i + c_i$. Such a choice exists: among the three candidates, exactly one is divisible by 3 and at most one is divisible by 5, so at least one candidate is divisible by neither.

After the cycle label has been selected, let

$$c_i = f(\alpha_i) - 7i;$$

thus, c_i is the position of the selected cycle label in B_i . For $1 \leq j \leq 6$, define

$$f(\beta_i^j) = \begin{cases} 7i + j, & 1 \leq j < c_i, \\ 7i + j + 1, & c_i \leq j \leq 6. \end{cases}$$

Thus, the vertices associated with α_i receive every label in B_i exactly once. Since the blocks $B_0, B_1, \dots, B_{n-1}$ partition $\{1, 2, \dots, 7n\}$, the function f is a bijection.

It remains to verify coprimality. For $i = 0$, we have $f(\alpha_0) = 1$. If $i > 0$ is even, then $c_i \in \{3, 5\}$, the integer $f(\alpha_i)$ is odd and is not divisible by 3, and its nonzero differences from the six leaf labels belong to $\{1, 2, 3, 4\}$. Hence every centre-leaf gcd equals 1. If i is odd, then $f(\alpha_i)$ is odd and is divisible by neither 3 nor 5, while its nonzero differences from the leaf labels belong to $\{1, 2, 3, 4, 5\}$. Again, every centre-leaf gcd equals 1.

Finally, consider a cycle edge $\alpha_i \alpha_{i+1}$ with $1 \leq i \leq n - 2$. Both endpoint labels are odd, and

$$|f(\alpha_{i+1}) - f(\alpha_i)| \in \{4, 6, 8, 10\}.$$

If i is even, the odd-index endpoint α_{i+1} has a label divisible by neither 3 nor 5; if i is odd, the same is true of α_i . Therefore, a common divisor can be divisible by none of 2, 3, 5, although it must divide one of 4, 6, 8, 10. It follows that the common divisor is 1. The two cycle edges incident with α_0 are also coprime because $f(\alpha_0) = 1$. Consequently, f is a PL. $\square$

Example 2.6. Figure 3 gives the PL of $C_4 * S_6$.

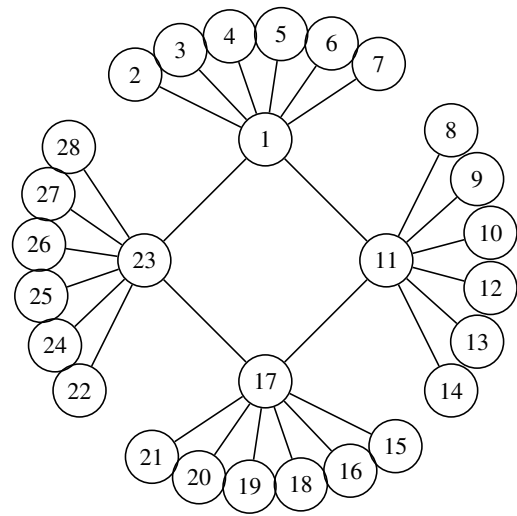


Figure 3. A PL of $C_4 * S_6$.

As a type of unicyclic graph, we study when a graph with two pendant paths at every cycle vertex is a PG.

For clarity, P_k denotes a path on k vertices (and hence $k - 1$ edges). Attaching P_k at a cycle vertex means identifying one endpoint of P_k with that cycle vertex, so the attachment contributes $k - 1$ new vertices.

The intuition for the two-path construction is to place the cycle label at a pivot inside each consecutive block. Labels then increase by one along one pendant path and decrease by one along the other, making every path edge automatically coprime. The pivot labels form an arithmetic progression; the hypotheses in Theorem 2.7 control its internal edges and, crucially, its closing edge.

Theorem 2.7. Let $n \geq 3$ and $k_1 \geq k_2 \geq 2$. The unicyclic graph obtained by attaching P_{k_1} and P_{k_2} at every vertex of C_n is a PG whenever

$$[\gcd(k_1 - 1, k_2) = 1 \text{ and } \gcd(k_2, n - 1) = 1]$$

or

$$[\gcd(k_1, k_2 - 1) = 1 \text{ and } \gcd(k_1, n - 1) = 1].$$

Proof. Suppose C_n is the cycle and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are its vertices. The first vertex of each pendant path at the i th cycle vertex is α_i . The vertices of the first pendant path corresponding to α_i are denoted by β_i^j , where $1 \leq j \leq k_1 - 1$, and the vertices of the second pendant path are denoted by γ_i^m , where $1 \leq m \leq k_2 - 1$. The vertices β_i^1 and γ_i^1 are adjacent to α_i . Define the labeling function $f : V \rightarrow \{1, 2, \dots, n(k_1 + k_2 - 1)\}$ by:

1. Suppose that $\gcd(k_1 - 1, k_2) = 1$ and $\gcd(k_2, n - 1) = 1$. Then

$$f(\alpha_i) = i(k_1 + k_2 - 1) + k_2,$$

$$f(\beta_i^j) = i(k_1 + k_2 - 1) + k_2 + j,$$

$$f(\gamma_i^m) = i(k_1 + k_2 - 1) + k_2 - m,$$

$$0 \leq i \leq n - 1, \quad 1 \leq j \leq k_1 - 1,$$

$$1 \leq m \leq k_2 - 1.$$

For each fixed i , these formulas assign all integers in $\{i(k_1+k_2-1)+1, \dots, (i+1)(k_1+k_2-1)\}$ exactly once. Moreover, adjacent vertices along either pendant path receive consecutive integers and hence receive coprime labels. We want to show that the labels of consecutive cycle vertices α_i and α_{i+1} are relatively prime. Let

$$t \mid f(\alpha_i) = i(k_1 + k_2 - 1) + k_2$$

and

$$\begin{aligned} t \mid f(\alpha_{i+1}) \\ = (i+1)(k_1 + k_2 - 1) + k_2. \end{aligned}$$

Then

$$t \mid f(\alpha_{i+1}) - f(\alpha_i) = k_1 + k_2 - 1$$

So,

$$t \mid f(\alpha_i) - i(k_1 + k_2 - 1) = k_2$$

Thus,

$$t \mid (k_1 + k_2 - 1) - k_2 = k_1 - 1$$

But,

$$\gcd(k_1 - 1, k_2) = 1 \text{ and hence } t = 1.$$

Thus, $f(\alpha_i)$ and $f(\alpha_{i+1})$ are relatively prime for all $i = 0, 1, \dots, n-2$.

For the closing edge, put $D = k_1 + k_2 - 1$. Then

$$\begin{aligned} \gcd(f(\alpha_0), f(\alpha_{n-1})) &= \gcd(k_2, (n-1)D + k_2) \\ &= \gcd(k_2, (n-1)D) = 1. \end{aligned}$$

Indeed, $\gcd(k_2, D) = \gcd(k_2, k_1 - 1) = 1$ and the added hypothesis gives $\gcd(k_2, n-1) = 1$. Thus, f is a PL.

2. Suppose that $\gcd(k_1, k_2 - 1) = 1$ and $\gcd(k_1, n-1) = 1$. Then

$$\begin{aligned} f(\alpha_i) &= i(k_1 + k_2 - 1) + k_1, \\ f(\beta_i^j) &= i(k_1 + k_2 - 1) + k_1 - j, \\ f(\gamma_i^m) &= i(k_1 + k_2 - 1) + k_1 + m, \\ 0 \leq i \leq n-1, \quad 1 \leq j \leq k_1 - 1, \\ 1 \leq m \leq k_2 - 1. \end{aligned}$$

As in the first case, every block uses each of its labels exactly once, and adjacent vertices on each pendant path receive consecutive labels. For consecutive cycle vertices,

$$\begin{aligned} \gcd(f(\alpha_i), f(\alpha_{i+1})) &= \gcd(k_1, k_1 + k_2 - 1) \\ &= \gcd(k_1, k_2 - 1) = 1 \end{aligned}$$

for $0 \leq i \leq n-2$. For the closing edge, with $D = k_1 + k_2 - 1$, we have

$$\begin{aligned} \gcd(f(\alpha_0), f(\alpha_{n-1})) &= \gcd(k_1, (n-1)D + k_1) \\ &= \gcd(k_1, (n-1)D) = 1, \end{aligned}$$

because $\gcd(k_1, D) = \gcd(k_1, k_2 - 1) = 1$ and $\gcd(k_1, n-1) = 1$. Hence this labeling is also a PL.

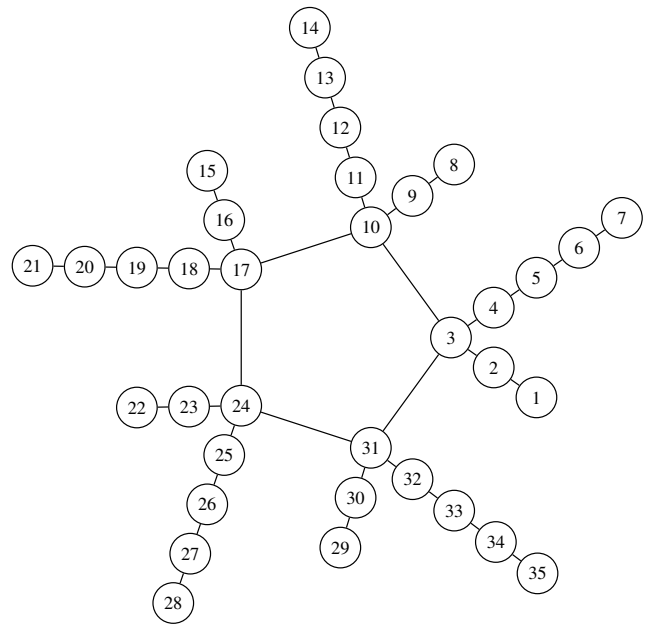


Figure 4. Prime labeling of a unicyclic graph with two pendant paths at the cycle vertices.

□

Example 2.8. Figure 4 illustrates a prime labeling of the unicyclic graph with two pendant paths P_5 and P_3 at every cycle vertex of C_5 . The two arms at each cycle vertex run consecutively away from the pivot label.

Corollary 2.9. Let $n \geq 3$ and $k \geq 2$. A unicyclic graph obtained by attaching two copies of P_k at every vertex of C_n is a PG whenever $\gcd(k, n-1) = 1$.

Proof. Since

$$\gcd(k-1, k) = 1$$

and, by hypothesis, $\gcd(k, n-1) = 1$, the first condition of Theorem 2.7 therefore applies. So, the graph is a PG. □

3. Constructive algorithms and complexity

The formulas in the preceding proofs give deterministic construction algorithms. The following pseudocode makes the procedure explicit. In each algorithm, output means assigning the indicated integer to the named vertex.

Algorithm 1: labeling $C_n * S_6$

Require: Integer $n \geq 3$.

- 1: **for** $i = 0, 1, \dots, n-1$ **do**
- 2: **if** $i = 0$ **then**
- 3: Set $r \leftarrow 1$.
- 4: **else if** i is even **then**
- 5: Set $r \leftarrow 3$ if $3 \nmid (7i+3)$; otherwise set $r \leftarrow 5$.
- 6: **else**
- 7: Choose the first $r \in \{2, 4, 6\}$ for which $3 \nmid (7i+r)$ and $5 \nmid (7i+r)$.

```

8:   end if
9:   Output  $f(\alpha_i) \leftarrow 7i + r$ .
10:  for  $j = 1, 2, \dots, 6$  do
11:    Output  $f(\beta_i^j) \leftarrow 7i + j$  if  $j < r$ , and  $7i + j + 1$  otherwise.
12:  end for
13: end for

```

Algorithm 2: labeling the two-path family

Require: Integers $n \geq 3$ and $k_1 \geq k_2 \geq 2$.

```

1: Put  $D \leftarrow k_1 + k_2 - 1$ .
2: if  $\gcd(k_1 - 1, k_2) = \gcd(k_2, n - 1) = 1$  then
3:   for  $i = 0, 1, \dots, n - 1$  do
4:     Output  $f(\alpha_i) \leftarrow iD + k_2$ .
5:     Output  $f(\beta_i^j) \leftarrow iD + k_2 + j$  for  $1 \leq j < k_1$ .
6:     Output  $f(\gamma_i^m) \leftarrow iD + k_2 - m$  for  $1 \leq m < k_2$ .
7:   end for
8: else if  $\gcd(k_1, k_2 - 1) = \gcd(k_1, n - 1) = 1$  then
9:   for  $i = 0, 1, \dots, n - 1$  do
10:    Output  $f(\alpha_i) \leftarrow iD + k_1$ .
11:    Output  $f(\beta_i^j) \leftarrow iD + k_1 - j$  for  $1 \leq j < k_1$ .
12:    Output  $f(\gamma_i^m) \leftarrow iD + k_1 + m$  for  $1 \leq m < k_2$ .
13:   end for
14: else
15:   Report that these sufficient constructions do not certify
     the input.
16: end if

```

The same direct implementation of the displayed formulas for $C_n * S_2$ and $C_n * S_4$ has identical asymptotic behavior. If N is the number of vertices, then $N = 7n$ for Algorithm 1 and $N = n(k_1 + k_2 - 1)$ for Algorithm 2. Every vertex label is written once, so the running time is $\Theta(N)$, which is optimal for explicitly outputting all labels. The preliminary gcd tests use the Euclidean algorithm and take $O(\log \max\{n, k_1, k_2\})$ time. Storing the full labeling requires $O(N)$ space, while a streaming implementation uses only $O(1)$ auxiliary space.

4. Conclusion

This paper studied prime vertex labeling for two families of unicyclic graphs: cycles with a star attached at every vertex, and cycles with two pendant paths attached at every vertex. We gave explicit prime labelings for $C_n * S_2$, $C_n * S_4$, and $C_n * S_6$, extending earlier results that covered only three, five, or seven pendant vertices per cycle vertex.

For the two-path case, where every cycle vertex is identified with one endpoint of paths P_{k_1} and P_{k_2} with $k_1 \geq k_2$, we showed that the graph is prime whenever $\gcd(k_1 - 1, k_2) = \gcd(k_2, n - 1) = 1$ or $\gcd(k_1, k_2 - 1) = \gcd(k_1, n - 1) = 1$. In particular, a unicyclic graph with two identical pendant paths P_k at every cycle vertex is prime whenever $\gcd(k, n - 1) = 1$. Each construction is explicit and based on simple arithmetic, so the labelings are easy to check and to adapt to other unicyclic families.

Several questions remain open. One is whether the two-path result can be extended to a full characterization of the triples (n, k_1, k_2) that admit a prime labeling. Another is whether cycles

with other fixed numbers of pendant vertices, or other labeling schemes such as graceful labeling, admit similar constructions. A more computational direction would be an algorithm that decides whether a general unicyclic graph is prime, together with possible applications to network design and coding theory.

Data availability

No datasets were generated or analyzed in this study. All results are theoretical, and all construction data are contained in the article; therefore, no external repository link is applicable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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