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On the construction and structural analysis of Bell-enriched Appell-λ polynomials via quasi-monomiality

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Abstract

A fresh class of hybrid special polynomials, termed the Bell-enriched Appell-λ-polynomials, is constructed through the discrete convolution of Bell-based λ-polynomials with the Appell sequence. A generating function for this class is established, from which closed-form expansions, convolution identities, and Stirling-number factorizations follow. The multiplicative and derivative operators that give this family its quasi-monomial character are identified, and the associated differential equation is recorded as an immediate corollary. A determinant representation à la Wang is also obtained. To illustrate the scope of the general theory, three subfamilies are investigated: the Bell-enriched Bernoulli-λ and Euler-λ polynomials, which are admissible Appell specializations ($\mathcal{A}(0) \neq 0$), and the Bell-enriched Genocchi-λ polynomials, which belong to the associated-Appell class ($\mathcal{A}(0) = 0$) and are treated separately. For the first two subfamilies, operational, quasi-monomial, and determinantal properties are fully established; for the Genocchi subfamily, generating-function and operational results are obtained within the associated-Appell framework. A numerical exploration of polynomial values, zero distributions, and graphical profiles supplements the algebraic development.

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1. Background and motivation

Special functions occupy a central place in mathematical physics, serving as ready-made building blocks for the description of diverse physical phenomena. Their importance is particularly evident in the context of differential equations, where the availability of a well-understood function class can turn an otherwise intractable problem into one that admits a complete closed-form solution [1–3]. Mastery of these function families is therefore inseparable from the ability to analyze the differential models that pervade science and engineering.

The relationship between a physical model and the search for new polynomial families is, by its nature, reciprocal. On the one hand, a given application may demand extensions or modifications of classical sequences; on the other hand, the new classes that emerge from such extensions frequently reveal algebraic or combinatorial structures whose reach extends well beyond the original application. Reformulating a problem in the language of special polynomials thus carries a double reward: the analytic treatment gains in clarity, and unexpected connections between disparate mathematical areas often come to the surface.

Within this broader programme, the Appell sequences [4, 5]

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have shown exceptional flexibility, appearing in approximation theory, quantum mechanics, number theory, and numerous other disciplines. Every Appell sequence is governed by a generating function of the form

$$\mathcal{A}(u,\rho) := \mathcal{A}(\rho)e^{u\rho} = \sum_{n=0}^{\infty} \mathcal{A}_n(u)\frac{\rho^n}{n!}, \qquad \mathcal{A}_n := \mathcal{A}_n(0). \tag{1}$$

where $\mathcal{A}(\rho)$ is analytic at the origin with the expansion

$$\mathcal{A}(\rho) = \sum_{n=0}^{\infty} \mathcal{A}_n\frac{\rho^n}{n!}, \qquad \mathcal{A}_0 \neq 0, \qquad \mathcal{A}_i\ (i=0,1,2,\ldots) \quad \text{being real coefficients.} \tag{2}$$

The individual members $\mathcal{A}_n(u)$ admit the explicit expansion

$$\mathcal{A}_n(u) = \sum_{k=0}^{n} \binom{n}{k}\mathcal{A}_{n-k}\, u^k, \quad \mathcal{A}_n'(u) = n\,\mathcal{A}_{n-1}(u).$$

Choosing $\mathcal{A}(\rho)$ appropriately produces the classical members of the Appell family. Table 1 collects the most prominent ones.

For quick reference, the first few Bernoulli, Euler, and Genocchi numbers are gathered in Table 2.
It should be observed that $G_n(u)$ has degree $n-1$ instead of n, a feature that distinguishes it from genuine Appell polynomials and places it in the broader class of associated Appell sequences (cf. [8, 9] for a thorough discussion).

Dattoli and collaborators [10] brought into play the two-variable λ-polynomials $\lambda_n(u,v)$, defined through

$$e^{v\rho}\cos(\sqrt{u\,\rho}) = \sum_{n=0}^{\infty} \lambda_n(u,v)\frac{\rho^n}{n!}, \tag{3}$$

whose explicit form reads

$$\lambda_n(u,v) = n!\sum_{k=0}^{n} \frac{(-1)^k u^k v^{n-k}}{(2k)!(n-k)!}. \tag{4}$$

A powerful lens through which to study such families is the monomiality principle. Rooted in Steffensen's notion of a poweroid [11], this framework was substantially developed by Dattoli and co-workers [12] and has since been applied systematically to general Appell sequences [13]. A polynomial sequence $\{\mathcal{D}_n(u)\}$ is called quasi-monomial whenever there exist a raising operator $\mathcal{M}$ and a lowering operator $\mathcal{P}$ such that

$$\mathcal{D}_{n+1}(u) = \mathcal{M}\{\mathcal{D}_n(u)\}, \tag{5}$$

$$n\,\mathcal{D}_{n-1}(u) = \mathcal{P}\{\mathcal{D}_n(u)\}, \tag{6}$$

where the operators $\mathcal{M}$ (multiplicative) and $\mathcal{P}$ (derivative) satisfy the Weyl commutation relation

$$[\mathcal{P},\mathcal{M}] = \hat{1}.$$

Whenever such a pair exists, the following structural identities are immediate consequences:

$$\mathcal{M}\mathcal{P}\{\mathcal{D}_n(u)\} = n\,\mathcal{D}_n(u), \tag{7}$$

$$\mathcal{D}_n(u) = \mathcal{M}^n\{1\}, \tag{8}$$

$$\exp(\tau\mathcal{M})\{1\} = \sum_{n=0}^{\infty} \mathcal{D}_n(u)\frac{\tau^n}{n!}. \tag{9}$$

Recall that the Stirling numbers of the second kind express ordinary powers in terms of falling factorials [14]: for every $j \in \mathbb{N}_0$,

$$\varkappa^j = \sum_{q=0}^{j} S_2(j,q)(\varkappa)_q. \tag{10}$$

The quantity $S_2(j,q)$ enumerates the partitions of a j-element set into exactly q nonempty blocks. Its exponential generating function is readily deduced from Eq. (10):

$$\frac{1}{k!}(e^{\rho}-1)^k = \sum_{j=k}^{\infty} S_2(j,k)\frac{\rho^j}{j!}, \tag{11}$$

a relation that pervades the theory of partition-related polynomial sequences.

Named after E. T. Bell, the Bell polynomials $Bel_n(u)$ play a unifying role across combinatorics, number theory, and special-function theory. They are determined by the exponential generating function [15–18]

$$e^{u(e^{\rho}-1)} = \sum_{n=0}^{\infty} Bel_n(u)\frac{\rho^n}{n!}. \tag{12}$$

Setting $u=1$ recovers the Bell numbers Bel_n [15, 16]:

$$e^{e^{\rho}-1} = \sum_{n=0}^{\infty} Bel_n\frac{\rho^n}{n!}. \tag{13}$$

Combining Eq. (11) with Eq. (12) gives the well-known expansion

$$Bel_n(u) = \sum_{k=0}^{n} u^k S_2(n,k). \tag{14}$$

The two-variable Bell polynomials $Bel_n(u,v)$ are governed by [15, 16]

$$e^{u\rho+v(e^{\rho}-1)} = \sum_{n=0}^{\infty} Bel_n(u,v)\frac{\rho^n}{n!}, \quad n \geqslant 0, \tag{15}$$

which yields the binomial relation

$$Bel_n(u,v) = \sum_{k=0}^{n} \binom{n}{k} u^k Bel_{n-k}(v). \tag{16}$$

During the last two decades there has been an intense activity directed at combining well-known polynomial systems including those of Gould–Hopper, Hermite, Legendre, Laguerre, Appell, and Sheffer type into hybrid classes that possess a richer algebraic and analytic framework [19–35]. The underlying

Table 1: Notable polynomials of the Appell family.

S.No.	Name of polynomials	$\mathcal{R}(\rho)$	Generating function	Series definition
I.	Bernoulli polynomials and numbers [6]	$\frac{\rho}{e^{\rho}-1}$	$\left(\frac{\rho}{e^{\rho}-1}\right)e^{u\rho}=\sum_{n=0}^{\infty}B_n(u)\frac{\rho^n}{n!}$; $\left(\frac{\rho}{e^{\rho}-1}\right)=\sum_{n=0}^{\infty}B_n\frac{\rho^n}{n!}$; $B_n:=B_n(0)$; note $B_n(0)=B_n(1)$ for $n\neq 1$	$B_n(u)=\sum_{k=0}^{n}\binom{n}{k}B_k u^{n-k}$
II.	Euler polynomials and numbers [6]	$\frac{2}{e^{\rho}+1}$	$\left(\frac{2}{e^{\rho}+1}\right)e^{u\rho}=\sum_{n=0}^{\infty}E_n(u)\frac{\rho^n}{n!}$; $\frac{2e^{\rho}}{e^{2\rho}+1}=\sum_{n=0}^{\infty}E_n\frac{\rho^n}{n!}$; $E_n:=2^nE_n\left(\frac{1}{2}\right)$	$E_n(u)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(u-\frac{1}{2}\right)^{n-k}$
III.	Genocchi polynomials and numbers [7]	$\frac{2\rho}{e^{\rho}+1}$	$\left(\frac{2\rho}{e^{\rho}+1}\right)e^{u\rho}=\sum_{n=0}^{\infty}G_n(u)\frac{\rho^n}{n!}$; $\frac{2\rho}{e^{\rho}+1}=\sum_{n=0}^{\infty}G_n\frac{\rho^n}{n!}$; $G_n:=G_n(0)$	$G_n(u)=\sum_{k=0}^{n}\binom{n}{k}G_k u^{n-k}$

Table 2: Initial values of B_n, E_n, and G_n.

n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

motivation is clear: merging the generating-function machinery and operational identities of two or more classical families yields polynomial sequences capable of capturing structural features that no single classical family can accommodate. Such hybrid families have found their way into areas as diverse as quantum-mechanical matrix elements, wave-propagation models, signal-processing algorithms, and stochastic modelling.

In parallel, the operational calculus anchored in the monomiality principle has matured into a systematic device for deriving identities, differential equations, and integral representations for hybrid polynomial classes. When the polynomials depend on several variables, the joint use of operational and monomiality methods proves especially powerful for tackling the attendant partial differential equations [1].

With this perspective in mind, the present work is organized as follows. Section 2 introduces the Bell-type Appell-λ polynomials, records their generating function, and establishes several algebraic identities through series-rearrangement arguments. Section 3 develops the quasi-monomial characterization, derives the governing differential equation, and provides a determinantal formula. Section 4 specializes the general framework to the Bernoulli-λ, Euler-λ, and Genocchi-λ cases. Finally, Section 5 presents a numerical study of polynomial values, zeros, and graphs.

2. Bell-enriched Appell-λ-polynomials

The starting point for the entire development is the generating function for the new polynomial class. Once this is secured, all further results identities, operational formulas, and matrix representations follow from it in a systematic fashion.

As a preparatory step, we recall the Bell-based λ-polynomials. Forming the product of the generating functions in Eqs. (3) and (12), the polynomials ${}_{Bel}\lambda_n(u,v,w)$ are given by

$$e^{v\rho+w(e^{\rho}-1)}\cos(\sqrt{u\rho})=\sum_{n=0}^{\infty}{}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!},\quad n\geqslant 0. \tag{17}$$

From Eq. (17) one reads off the binomial-type expansion

$${}_{Bel}\lambda_n(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}Bel_k(w)\lambda_{n-k}(u,v). \tag{18}$$

The family ${}_{Bel}\lambda_n(u,v,w)$ turns out to be quasi-monomial with respect to the operators

$$\hat{M}=v+we^{D_v}-\frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\Big(\sqrt{u(D_v)}\Big), \tag{19}$$

$$\hat{P}=D_v, \tag{20}$$

and the standard monomiality identities [10, 12] therefore hold:

$$\hat{M}_{{}_{Bel}\lambda_n(u,v,w)}={}_{Bel}\lambda_{n+1}(u,v,w), \tag{21}$$

$$\hat{P}_{{}_{Bel}\lambda_n(u,v,w)}=n\ {}_{Bel}\lambda_{n-1}(u,v,w), \tag{22}$$

$$\hat{M}_{{}_{Bel}\lambda}\hat{P}_{{}_{Bel}\lambda}\{{}_{Bel}\lambda_n(u,v,w)\}=n\ {}_{Bel}\lambda_n(u,v,w), \tag{23}$$

$$e^{(\rho\hat{M}_{{}_{Bel}\lambda})}\{1\}=\sum_{n=0}^{\infty}{}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!},\quad(|\,\rho\,|<\infty). \tag{24}$$

With these ingredients at hand, we are ready to define the central objects of the paper.

Theorem 2.1. *The Bell-enriched Appell-λ-polynomials ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$ possess the generating function*

$$\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u}\,\rho)=\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}. \tag{25}$$

Proof. Replace u in Eq. (1) by the multiplicative operator $\hat{M}_{Bel\lambda}$ from Eq. (19). This gives

$$\mathcal{A}(\rho)e^{(\hat{M}_{Bel\lambda})}=\sum_{n=0}^{\infty}\mathcal{A}_n(\hat{M}_{Bel\lambda})\frac{\rho^n}{n!}. \tag{26}$$

Inserting Eq. (24) and writing $\mathcal{A}_n(\hat{M}_{Bel\lambda})$ for the resulting Bell-enriched Appell-λ-polynomials ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$, we arrive at

$$\mathcal{A}(\rho)\left(\sum_{n=0}^{\infty}{}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!}\right)=\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\times\frac{\rho^n}{n!}. \tag{27}$$

Recognising the left-hand side through Eq. (17) yields Eq. (25). □

We next write down a closed-form series for these polynomials.

Theorem 2.2. *The Bell-enriched Appell-λ-polynomials satisfy*

$${}_{Bel}\mathcal{A}\lambda_n(u,v,w)=n!\sum_{k=0}^{n}\frac{(-1)^k u^k\,{}_{Bel}\mathcal{A}_{n-k}(v,w)}{(2k)!(n-k)!}. \tag{28}$$

Proof. Combining Eq. (1), Eq. (4) and Eq. (25) produces

$$\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k u^k\,{}_{Bel}\mathcal{A}_j(v,w)}{(2k)!\,j!}\rho^{j+k}. \tag{29}$$

Applying the standard series rearrangement [9], taking $n=j+k$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{30}$$

$$\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k u^k\,{}_{Bel}\mathcal{A}_{n-k}(v,w)}{(2k)!(n-k)!}\times\rho^n.$$

and matching the coefficients of $\rho^n/n!$ on both sides, we obtain Eq. (28). □

Theorem 2.3. *For the Bell-enriched Appell-λ-polynomials the following addition formula holds:*

$${}_{Bel}\mathcal{A}\lambda_n(u,v+\xi,w)=\sum_{k=0}^{n}\binom{n}{k}\xi^{n-k}\,{}_{Bel}\mathcal{A}\lambda_k(u,v,w). \tag{31}$$

Proof. Set $v\mapsto v+\xi$ in Eq. (25). Writing $e^{(v+\xi)\rho}=e^{\xi\rho}e^{v\rho}$ and expanding $e^{\xi\rho}=\sum_{k=0}^{\infty}\xi^k\rho^k/k!$, the Cauchy product with Eq. (25) yields Eq. (31). □

Theorem 2.4. *There holds the Stirling-type decomposition*

$${}_{Bel}\mathcal{A}\lambda_n(u,v,w)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}S_2(k,m)w^m\times{}_{\mathcal{A}}\lambda_{n-k}(u,v). \tag{32}$$

Proof. Starting from Eq. (25) and expanding the Bell factor in Stirling numbers via Eq. (11), we get

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}&=\mathcal{A}(\rho)e^{v\rho}\cos(\sqrt{u}\,\rho)\sum_{m=0}^{\infty}w^m\frac{(e^\rho-1)^m}{m!}\\&=\mathcal{A}(\rho)e^{v\rho}\cos(\sqrt{u}\,\rho)\sum_{m=0}^{\infty}w^m\sum_{n=m}^{\infty}S_2(n,m)\frac{\rho^n}{n!}\\&=\sum_{n=0}^{\infty}{}_{\mathcal{A}}\lambda_n(u,v)\frac{\rho^n}{n!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}w^mS_2(k,m)\frac{\rho^k}{k!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}w^mS_2(k,m)\,{}_{\mathcal{A}}\lambda_{n-k}(u,v)\frac{\rho^n}{n!}.\end{aligned} \tag{33}$$

Comparing the coefficients of $\rho^n/n!$ gives Eq. (32). □

Theorem 2.5. *For every $n\geq 0$,*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}{}_{Bel}\mathcal{A}\lambda_{n-k}(u,v,w)\xi^k&=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}{}_{Bel}\mathcal{A}\lambda_{n-k}(u,v,w)\\&\quad\times S_2(k,m)\xi^{\underline{m}}.\end{aligned} \tag{34}$$

$$\xi^{\underline{m}}=\xi(\xi-1)\cdots(\xi-m+1),\qquad \xi^{\underline{0}}=1.$$

Proof. Multiply both sides of Eq. (25) by $e^{\xi\rho}$ and appeal to Eq. (11):

$$\begin{aligned}e^{\xi\rho}\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}&=\left(\sum_{k=0}^{\infty}\frac{\xi^k\rho^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{Bel}\mathcal{A}\lambda_{n-k}(u,v,w)\\&\quad\times\xi^k\frac{\rho^n}{n!}.\end{aligned} \tag{35}$$

On the other hand, using the Stirling-number identity

$$\xi^k=\sum_{m=0}^{k}S_2(k,m)\xi^{\underline{m}},$$

where

$$\xi^{\underline{m}} = \xi(\xi-1)\cdots(\xi-m+1), \qquad \xi^{\underline{0}} = 1,$$

we have

$$\begin{aligned} e^{\xi\rho}\sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k} {}_{Bel}\mathcal{A}\lambda_{n-k}(u,v,w) \\ &\quad \times S_2(k,m)\xi^{\underline{m}}\frac{\rho^n}{n!}. \end{aligned} \tag{36}$$

Equating coefficients yields Eq. (34). □

Theorem 2.6. *The partial derivatives of ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$ obey*

$$\frac{\partial}{\partial v} {}_{Bel}\mathcal{A}\lambda_n(u,v,w) = n\ {}_{Bel}\mathcal{A}\lambda_{n-1}(u,v,w), \tag{37}$$

$$\frac{\partial}{\partial w} {}_{Bel}\mathcal{A}\lambda_n(u,v,w) = \triangle_v\ \left({}_{Bel}\mathcal{A}\lambda_n(u,v,w)\right), \tag{38}$$

where

$$\triangle_v f(u,v,w) = f(u,v+1,w) - f(u,v,w).$$

Proof. Differentiate Eq. (25) with respect to v and w in turn:

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial v}\left\{ {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\right\}\frac{\rho^n}{n!} = \rho\left\{\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho})\right\}.$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\partial}{\partial w}\left\{ {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\right\}\frac{\rho^n}{n!} \\ &= (e^\rho-1)\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &= \mathcal{A}(\rho)e^{(v+1)\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\quad - \mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &= \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v+1,w)\frac{\rho^n}{n!} \\ &\quad - \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}. \end{aligned}$$

Extracting the coefficient of $\rho^n/n!$ in each identity completes the proof. □

3. Quasi-monomial structure and determinantal calculus

With the generating function and the fundamental series identities at our disposal, the next step is to place ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$ inside the monomiality framework. This single embedding simultaneously supplies the raising and lowering operators, the governing differential equation, and a determinant representation.

Theorem 3.1. *The polynomials ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$ are quasi-monomial with respect to*

$$\hat{M} = v + we^{D_v} + \frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)} - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\left(\sqrt{u(D_v)}\right), \tag{39}$$

and

$$\hat{P} = D_v. \tag{40}$$

Proof. Differentiating each side of the generating function Eq. (25) with respect to ρ gives

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_{n+1}(u,v,w)\frac{\rho^n}{n!} &= \frac{\mathcal{A}'(\rho)}{\mathcal{A}(\rho)}\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\quad + v\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\quad + we^\rho\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\quad - \frac{1}{2}\sqrt{\frac{u}{\rho}}\sin(\sqrt{u\rho})\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}. \end{aligned}$$

Collecting terms,

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_{n+1}(u,v,w)\frac{\rho^n}{n!} &= \mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\quad \times\left[\frac{\mathcal{A}'(\rho)}{\mathcal{A}(\rho)} + v + we^\rho \right. \\ &\quad \left. - \frac{1}{2}\sqrt{\frac{u}{\rho}}\tan(\sqrt{u\rho})\right]. \end{aligned} \tag{41}$$

Since

$$\begin{aligned} D_v\left(\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho})\right) &= \rho\mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)} \\ &\quad \times\cos(\sqrt{u\rho}), \end{aligned} \tag{42}$$

the variable ρ appearing multiplicatively on the right-hand side of Eq. (41) can be traded for D_v:

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_{n+1}(u,v,w)\frac{\rho^n}{n!} \\ &= \mathcal{A}(\rho)e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\rho}) \\ &\times\left[\frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)} + v + we^{D_v} \right. \\ &\quad \left. - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\left(\sqrt{uD_v}\right)\right]. \end{aligned} \tag{43}$$

Invoking Eq. (25) on the right-hand side,

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_{n+1}(u,v,w)\frac{\rho^n}{n!} &= \sum_{n=0}^{\infty}\left[\frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)} + v + we^{D_v}\right. \\ &\quad \left. - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\left(\sqrt{uD_v}\right)\right] \\ &\quad \times {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}. \end{aligned} \tag{44}$$

□

Matching the coefficient of ρ^n on both sides produces

$$\begin{aligned} {}_{Bel}\mathcal{A}\lambda_{n+1}(u,v,w) &= \left[\frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)} + v + we^{D_v}\right. \\ &\quad \left. - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\left(\sqrt{uD_v}\right)\right] \\ &\quad \times {}_{Bel}\mathcal{A}\lambda_n(u,v,w), \end{aligned} \tag{45}$$

which is precisely Eq. (5) with the operator $\hat{M}$ given by Eq. (39).

Turning to the derivative operator, we apply D_v to Eq. (25):

$$D_v \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!} = \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^{n+1}}{n!}, \tag{46}$$

so that, upon equating powers of ρ,

$$D_v\left\{{}_{Bel}\mathcal{A}\lambda_n(u,v,w)\right\} = n\ {}_{Bel}\mathcal{A}\lambda_{n-1}(u,v,w), \quad n \geqslant 1, \tag{47}$$

confirming Eq. (40) via Eq. (6).

Remark 3.1. *Eqs. (8) and (39) together imply the following explicit representation.*

Corollary 3.1. *The Bell-enriched Appell-λ-polynomials admit the operational form*

$$\begin{aligned} {}_{Bel}\mathcal{A}\lambda_n(u,v,w) = \Bigg(& v + we^{D_v} + \frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)} \\ & - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\Big(\sqrt{uD_v}\Big)\Bigg)^n\{1\}. \end{aligned} \tag{48}$$

The differential equation governing the family follows immediately from the multiplicative and derivative operators.

Theorem 3.2. *The polynomials ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$ satisfy*

$$\begin{aligned} \Bigg(& vD_v + we^{D_v}D_v + \frac{\mathcal{A}'(D_v)}{\mathcal{A}(D_v)}D_v \\ & - \frac{1}{2}\sqrt{\frac{u}{D_v}}\tan\Big(\sqrt{uD_v}\Big)D_v - n\Bigg) \\ & \times {}_{Bel}\mathcal{A}\lambda_n(u,v,w) = 0. \end{aligned} \tag{49}$$

Proof. Insert Eq. (39) and Eq. (40) into the monomiality identity Eq. (7). □

A further useful identity expresses ${}_{Bel}\mathcal{A}\lambda_n$ in terms of the Appell coefficients and the Bell-based λ-polynomials.

Theorem 3.3. *There holds*

$${}_{Bel}\mathcal{A}\lambda_n(u,v,w) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{A}_k\ {}_{Bel}\lambda_{n-k}(u,v,w). \tag{50}$$

Proof. From Eq. (17) and Eq. (25),

$$\mathcal{A}(\rho)\sum_{n=0}^{\infty} {}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!} = \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}. \tag{51}$$

Expanding $\mathcal{A}(\rho)$ via Eq. (1) and applying the Cauchy product,

$$\begin{aligned} &\left(\sum_{n=0}^{\infty}\mathcal{A}_k\frac{\rho^k}{k!}\right) \\ &\times\left(\sum_{n=0}^{\infty} {}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}. \end{aligned} \tag{52}$$

Re-indexing $n \mapsto n-k$ on the left and comparing powers of ρ yields Eq. (50). □

3.1. Determinantal representation

Determinant-based formulas have demonstrated their value in numerous physical and mathematical contexts, including the propagation of electromagnetic waves through layered dielectrics, the solution structure of the Korteweg–de Vries equation, and problems in relativistic gravitation. Following the determinantal approach to Appell polynomials due to Costabile and Longo [36] and Wang's determinantal treatment of Sheffer sequences [37], which highlighted the computational and interpolation-theoretic merits of such representations, we derive a determinant expression for ${}_{Bel}\mathcal{A}\lambda_n(u,v,w)$.

Theorem 3.4. *The Bell-enriched Appell-λ-polynomials carry the determinantal representation*

$$\begin{aligned} {}_{Bel}\mathcal{A}\lambda_n(u,v,w) = {} & \frac{(-1)^n}{(\beta_0)^{n+1}} \\ & \times\begin{vmatrix} 1 & {}_{Bel}\lambda_1(u,v,w) & {}_{Bel}\lambda_2(u,v,w) & \cdots & {}_{Bel}\lambda_{n-1}(u,v,w) & {}_{Bel}\lambda_n(u,v,w) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}. \end{aligned} \tag{53}$$

where $\sum_{n=0}^{\infty} {}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!} = e^{v\rho+w(e^\rho-1)}\cos\big(\sqrt{u\,\rho}\big)$, $\frac{1}{\mathbb{A}(\rho)} = \sum_{k=0}^{\infty}\beta_k\frac{\rho^k}{k!}$, $\beta_0 \neq 0$. Assume that the Appell factor $A(\rho)$ satisfies $A(0) \neq 0$.

Proof. Dividing both sides of Eq. (25) by $\mathcal{A}(\rho)$ and writing the reciprocal as a power series,

$$\begin{aligned} e^{v\rho+w(e^\rho-1)}\cos(\sqrt{u\,\rho}) = {} & \left(\sum_{k=0}^{\infty}\beta_k\frac{\rho^k}{k!}\right) \\ & \times\left(\sum_{n=0}^{\infty} {}_{Bel}\mathcal{A}\lambda_n(u,v,w)\frac{\rho^n}{n!}\right). \end{aligned}$$

The Cauchy product on the right gives

$$\sum_{n=0}^{\infty} {}_{Bel}\lambda_n(u,v,w)\frac{\rho^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_k\ {}_{Bel}\mathcal{A}\lambda_{n-k}(u,v,w)\frac{\rho^n}{n!}.$$

Equating the coefficients of $\rho^n/n!$ and unfolding for $n = 0, 1, 2, \ldots$, we obtain the triangular system

$$\begin{aligned} {}_{Bel}\lambda_0(u,v,w) &= \beta_0\ {}_{Bel}\mathcal{A}\lambda_0(u,v,w), \\ {}_{Bel}\lambda_1(u,v,w) &= \beta_0\ {}_{Bel}\mathcal{A}\lambda_1(u,v,w) + \beta_1\ {}_{Bel}\mathcal{A}\lambda_0(u,v,w), \\ {}_{Bel}\lambda_2(u,v,w) &= \beta_0\,{}_{Bel}\mathcal{A}\lambda_2(u,v,w) + \binom{2}{1}\beta_1\,{}_{Bel}\mathcal{A}\lambda_1(u,v,w) \\ &\quad + \beta_2\,{}_{Bel}\mathcal{A}\lambda_0(u,v,w), \\ &\ \ \vdots \\ {}_{Bel}\lambda_{n-1}(u,v,w) &= \beta_0\,{}_{Bel}\mathcal{A}\lambda_{n-1}(u,v,w) \\ &\quad + \binom{n-1}{1}\beta_1\,{}_{Bel}\mathcal{A}\lambda_{n-2}(u,v,w) \\ &\quad + \cdots + \beta_{n-1}\,{}_{Bel}\mathcal{A}\lambda_0(u,v,w). \end{aligned}$$

$$
{}_{Bel}\lambda_n(u,v,w) = \beta_0\, {}_{Bel}\mathcal{A}\lambda_n(u,v,w) + \binom{n}{1}\beta_1\, {}_{Bel}\mathcal{A}\lambda_{n-1}(u,v,w) + \cdots + \beta_n\, {}_{Bel}\mathcal{A}\lambda_0(u,v,w).
$$

Solving by Cramer's rule, transposing the resulting determinant, and simplifying gives Eq. (53). This determinant form is valid for Appell factors satisfying $A(0) \neq 0$. In particular, the Genocchi-type Appell factor is excluded from this result because $A(0) = 0$; hence it does not admit an ordinary series expansion. □

4. Illustrative special cases

The Appell umbrella shelters a wide array of polynomial families Hermite, Laguerre, Bernoulli, Poisson–Charlier, Hahn, and many more each dictated by a particular choice of $\mathcal{A}(\rho)$. Below we select three classical specifications and work out the consequences in each case. Such concrete instantiations are important not only as tests of the general machinery but also because the resulting differential equations are prototypical of models arising in oscillatory and boundary-value problems [1, 2].

Example 4.1. With $\mathcal{A}(\rho) := \dfrac{\rho}{e^\rho - 1}$ (Bernoulli polynomials $\mathcal{B}_n(u)$) in Eq. (25), the generating function for the Bell-enriched Bernoulli-λ-polynomials ${}_{Bel}\mathcal{B}\lambda_n(u,v,w)$ takes the form

$$
\frac{\rho}{e^\rho - 1} e^{v\rho + w(e^\rho - 1)} \cos(\sqrt{u}\,\rho) = \sum_{n=0}^{\infty} {}_{Bel}\mathcal{B}\lambda_n(u,v,w) \frac{\rho^n}{n!}, \tag{54}
$$

with the series representation

$$
{}_{Bel}\mathcal{B}\lambda_n(u,v,w) = n! \sum_{k=0}^{n} \frac{(-1)^k u^k\, {}_{Bel}\mathcal{B}_{n-k}(v,w)}{(2k)!(n-k)!}. \tag{55}
$$

The corresponding series definition reads

$$
{}_{Bel}\mathcal{B}\lambda_n(u,v,w) = \sum_{k=0}^{n} \binom{n}{k} {}_{Bel}\lambda_{n-k}(u,v,w)\mathcal{B}_k. \tag{56}
$$

For $\mathcal{A}(\rho) = \dfrac{\rho}{e^\rho - 1}$, we have $\frac{\mathcal{A}'(\rho)}{\mathcal{A}(\rho)} = \dfrac{e^\rho(1-\rho) - 1}{\rho(e^\rho - 1)}$, so the family is quasi-monomial under

$$
\begin{aligned}
\hat{M} = v + we^{D_v} + \frac{e^{D_v}(1 - D_v) - 1}{D_v(e^{D_v} - 1)} \\
- \frac{1}{2}\sqrt{\frac{u}{D_v}} \tan\left(\sqrt{uD_v}\right),
\end{aligned} \tag{57}
$$

and

$$
\hat{P} = D_v, \tag{58}
$$

and satisfies the differential equation

$$
\begin{aligned}
\Bigg(vD_v + we^{D_v}D_v + \frac{e^{D_v}(1 - D_v) - 1}{e^{D_v} - 1} \\
- \frac{1}{2}\sqrt{\frac{u}{D_v}} \tan\left(\sqrt{uD_v}\right)D_v - n\Bigg) \\
\times\, {}_{Bel}\mathcal{B}\lambda_n(u,v,w) = 0.
\end{aligned} \tag{59}
$$

Substituting $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$, $(i = 1, 2, \cdots, n)$ into Eq. (53) produces the determinant form

$$
{}_{Bel}\mathcal{B}\lambda_n(u,v,w) = (-1)^n \times \begin{vmatrix}
1 & {}_{Bel}\lambda_1(u,v,w) & {}_{Bel}\lambda_2(u,v,w) & \cdots & {}_{Bel}\lambda_{n-1}(u,v,w) & {}_{Bel}\lambda_n(u,v,w) \\
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\
0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\
0 & 0 & 1 & \cdots & \binom{n-1}{1}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2}
\end{vmatrix}. \tag{60}
$$

where ${}_{Bel}\lambda_n(u,v,w)$, $(n = 0, 1, 2, \cdots, n)$, are the Bell-based λ-polynomials of degree n.

Example 4.2. Setting $\mathcal{A}(\rho) := \dfrac{2}{e^\rho + 1}$ (Euler polynomials $\mathcal{E}_n(u)$) in Eq. (25) yields the Bell-enriched Euler-λ-polynomials ${}_{Bel}\mathcal{E}\lambda_n(u,v,w)$ with generating function

$$
\frac{2}{e^\rho + 1} e^{v\rho + w(e^\rho - 1)} \cos(\sqrt{u}\,\rho) = \sum_{n=0}^{\infty} {}_{Bel}\mathcal{E}\lambda_n(u,v,w) \frac{\rho^n}{n!}, \tag{61}
$$

series expansion

$$
{}_{Bel}\mathcal{E}\lambda_n(u,v,w) = n! \sum_{k=0}^{n} \frac{(-1)^k u^k\, {}_{Bel}\mathcal{E}_{n-k}(v,w)}{(2k)!(n-k)!}, \tag{62}
$$

and series definition

$$
{}_{Bel}\mathcal{E}\lambda_n(u,v,w) = \sum_{k=0}^{n} \binom{n}{k} {}_{Bel}\lambda_{n-k}(u,v,w)\mathcal{E}_k. \tag{63}
$$

For $\mathcal{A}(\rho) = \dfrac{2}{e^\rho + 1}$, we have $\frac{\mathcal{A}'(\rho)}{\mathcal{A}(\rho)} = -\dfrac{e^\rho}{e^\rho + 1}$, so the quasi-monomial operators are

$$
\hat{M} = v + we^{D_v} - \frac{e^{D_v}}{e^{D_v} + 1} - \frac{1}{2}\sqrt{\frac{u}{D_v}} \tan\left(\sqrt{u(D_v)}\right), \tag{64}
$$

and

$$
\hat{P} = D_v, \tag{65}
$$

and the differential equation reads

$$
\begin{aligned}
\Bigg(vD_v + we^{D_v}D_v - \frac{e^{D_v}}{e^{D_v} + 1}D_v \\
- \frac{1}{2}\sqrt{\frac{u}{D_v}} \tan\left(\sqrt{uD_v}\right)D_v - n\Bigg) \\
\times\, {}_{Bel}\mathcal{E}\lambda_n(u,v,w) = 0.
\end{aligned} \tag{66}
$$

Setting $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$, $(i = 1, 2, \cdots, n)$ in Eq. (53) gives the determinant

$$
{}_{Bel}\mathcal{E}\lambda_n(u,v,w) = (-1)^n \times \begin{vmatrix} 1 & {}_{Bel}\lambda_1(u,v,w) & {}_{Bel}\lambda_2(u,v,w) & \cdots & {}_{Bel}\lambda_{n-1}(u,v,w) & {}_{Bel}\lambda_n(u,v,w) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{1}\frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}. \tag{67}
$$

where ${}_{Bel}\lambda_n(u,v,w)$, $(n = 0,1,2,\cdots,n)$, are the Bell-based λ-polynomials of degree n.

Example 4.3. The choice $\mathcal{A}(\rho) := \dfrac{2\rho}{e^\rho+1}$ (Genocchi polynomials $\mathcal{G}_n(u)$) leads to the Bell-enriched Genocchi-λ-polynomials ${}_{Bel}\mathcal{G}\lambda_n(u,v,w)$ whose generating function is

$$
\frac{2\rho}{e^\rho+1} e^{v\rho+w(e^\rho-1)} \cos(\sqrt{u\rho}) = \sum_{n=0}^{\infty} {}_{Bel}\mathcal{G}\lambda_n(u,v,w) \times \frac{\rho^n}{n!}. \tag{68}
$$

Remark 4.1. *The Genocchi case requires separate treatment because* $\mathcal{A}(\rho) = \dfrac{2\rho}{e^\rho+1}$ *satisfies* $\mathcal{A}(0) = 0$*, so it does* not *belong to the standard Appell family defined by Eq. (2). Instead, it belongs to the broader class of* associated Appell sequences *(see [8, 9]), for which the general Appell framework of Theorems 3.1–3.4 does not directly apply without modification. In particular, the sequence* $\{{}_{Bel}\mathcal{G}\lambda_n\}$ *is not of degree n: from Table 2 we see* $G_0 = 0$ *and Table 5 confirms* ${}_{Bel}\mathcal{G}\lambda_0(u,v,w) = 0$*, while* ${}_{Bel}\mathcal{G}\lambda_1(u,v,w) = 1$*, so the ordinary quasi-monomial relation* $\hat{M}\{1\} = {}_{Bel}\mathcal{G}\lambda_1$ *does not hold for* $\hat{M}$ *as in Eq. (39) with the Genocchi choice. The following results are therefore derived directly from the generating function Eq. (68) and are stated in the associated-Appell framework.*

The series representations follow directly from Eq. (68):

$$
{}_{Bel}\mathcal{G}\lambda_n(u,v,w) = n! \sum_{k=0}^{n} \frac{(-1)^k u^k \, {}_{Bel}\mathcal{G}_{n-k}(v,w)}{(2k)!(n-k)!}, \tag{69}
$$

$$
{}_{Bel}\mathcal{G}\lambda_n(u,v,w) = \sum_{k=0}^{n} \binom{n}{k} {}_{Bel}\lambda_{n-k}(u,v,w)\mathcal{G}_k. \tag{70}
$$

The factor $\mathcal{A}(\rho) = \dfrac{2\rho}{e^\rho+1}$ vanishes at $\rho = 0$, which is precisely why the standard quasi-monomial raising operator of Theorem 3.1 is unavailable: the ratio $\mathcal{A}'(\rho)/\mathcal{A}(\rho) = 1/\rho - e^\rho/(e^\rho+1)$ contains a simple pole at the origin, and no well-defined operator substitution $\rho \mapsto D_v$ can absorb the $1/\rho$ term on the polynomial space in a way that reproduces the correct recurrence. Consequently, no raising operator of the form Eq. (39) exists for this family, and the results of Theorems 3.1 and 3.3 do not apply.

Remark 4.2. *The obstruction just noted can be circumvented by a reindexing. Since the generating function Eq. (68) carries an explicit factor of ρ, define the* reindexed Bell-enriched Genocchi-λ-polynomials *by*

$$
\widetilde{\mathcal{G}}_n(u,v,w) := \frac{1}{n+1} {}_{Bel}\mathcal{G}\lambda_{n+1}(u,v,w), \quad n = 0,1,2,\ldots \tag{71}
$$

Their generating function follows immediately from Eq. (68):

$$
\sum_{n=0}^{\infty} \widetilde{\mathcal{G}}_n(u,v,w)\frac{\rho^n}{n!} = \frac{2}{e^\rho+1} e^{v\rho+w(e^\rho-1)} \cos(\sqrt{u\rho}). \tag{72}
$$

The right-hand side of Eq. (72) is of the form $\mathcal{A}(\rho)\, e^{v\rho+w(e^\rho-1)} \cos\left(\sqrt{u\rho}\right)$ *with* $\mathcal{A}(\rho) = \frac{2}{e^\rho+1}$*, which satisfies* $\mathcal{A}(0) = 1 \neq 0$*. The reindexed family therefore belongs to the admissible Appell framework, and the quasi-monomial operators, differential equation, and determinantal formula of Theorems 3.1–3.4 apply to* $\widetilde{\mathcal{G}}_n(u,v,w)$ *with* $\mathcal{A}(\rho) = \dfrac{2}{e^\rho+1}$ *(i.e. precisely as in Example 4.2 for the Euler-λ polynomials). These properties of the original sequence* ${}_{Bel}\mathcal{G}\lambda_n$ *are then recovered from Eq. (71).*

Remark 4.3. *Since* $\mathcal{A}(0) = 0$ *for the Genocchi factor, the reciprocal* $1/\mathcal{A}(\rho) = (e^\rho+1)/(2\rho)$ *is a Laurent series containing a pole at* $\rho = 0$*, and hence does not admit an ordinary power-series expansion with* $\beta_0 \neq 0$ *as required in Theorem 3.4. Consequently, the determinantal formula Eq. (53) is* not *valid for the Genocchi case, and no determinant representation of the form Eq. (53) is claimed for* ${}_{Bel}\mathcal{G}\lambda_n(u,v,w)$*.*

5. Numerical values, zeros, and graphical profiles

To complement the algebraic results developed above, we now turn to a concrete numerical exploration. To give the algebraic framework a concrete numerical underpinning, we now evaluate selected polynomial values, catalogue the zeros of the three Bell-enriched subfamilies, and plot their graphs. Every numerical entry has been obtained by truncating the generating-function expansion Eq. (25) symbolically (equivalently, via Eq. (28) combined with Eq. (4) and Eq. (18)) and has been cross-checked with Wolfram Mathematica 14.0 and SymPy 1.13 (Python); Tables 3–8 and Figures 1–5 are therefore mutually consistent.

We start with the Bell-enriched Bernoulli-λ-polynomials ${}_{Bel}\mathcal{B}\lambda_n(u,v,w)$, evaluated from Eq. (54) at $u = 0$ (so that $\cos\left(\sqrt{u\rho}\right) = 1$) and $w = 1$.

The corresponding Euler-λ values at $u = 0$, $w = 1$ (from Eq. (61)) are as follows.

The Genocchi-λ analogues (from Eq. (68)) are collected next.

Figures 1–3 plot ${}_{Bel}\mathcal{B}\lambda_n(0,v,1)$, ${}_{Bel}\mathcal{E}\lambda_n(0,v,1)$, and ${}_{Bel}\mathcal{G}\lambda_n(0,v,1)$ for $n = 0,1,2,3,4$ as functions of v over $[-2.5, 2.5]$.

Figure 4 isolates the influence of the Bell parameter w by plotting ${}_{Bel}\mathcal{B}\lambda_3(0,v,w)$ for $w = 0,1,2,3$, thereby showing how the Bell convolution reshapes the Bernoulli-λ polynomial.

Table 3: First five values of ${}_{Bel}\mathcal{B}\lambda_n(u, v, w)$ at $u = 0$, $w = 1$.

n	${}_{Bel}\mathcal{B}\lambda_n(0, v, 1)$
0	1
1	$v + \frac{1}{2}$
2	$v^2 + v + \frac{7}{6}$
3	$v^3 + \frac{3}{2}v^2 + \frac{7}{2}v + \frac{5}{2}$
4	$v^4 + 2v^3 + 7v^2 + 10v + \frac{209}{30}$

Table 4: First five values of ${}_{Bel}\mathcal{E}\lambda_n(u, v, w)$ at $u = 0$, $w = 1$.

n	${}_{Bel}\mathcal{E}\lambda_n(0, v, 1)$
0	1
1	$v + \frac{1}{2}$
2	$v^2 + v + 1$
3	$v^3 + \frac{3}{2}v^2 + 3v + \frac{9}{4}$
4	$v^4 + 2v^3 + 6v^2 + 9v + 6$

Table 5: First five values of ${}_{Bel}\mathcal{G}\lambda_n(u, v, w)$ at $u = 0$, $w = 1$.

n	${}_{Bel}\mathcal{G}\lambda_n(0, v, 1)$
0	0
1	1
2	$2v + 1$
3	$3v^2 + 3v + 3$
4	$4v^3 + 6v^2 + 12v + 9$

Table 6: Zeros of ${}_{Bel}\mathcal{B}\lambda_n(0, v, 1)$ for small n.

n	Degree in v	Zeros (real and complex)
1	1	$v = -0.5000$
2	2	$v \approx -0.5000 \pm 0.9574\, i$
3	3	$v \approx -0.8483, \quad v \approx -0.3259 \pm 1.6855\, i$
4	4	$v \approx -0.9088 \pm 0.6863\, i, \quad v \approx -0.0912 \pm 2.3159\, i$

Table 7: Zeros of ${}_{Bel}\mathcal{E}\lambda_n(0, v, 1)$ for small n.

n	Degree in v	Zeros (real and complex)
1	1	$v = -0.5000$
2	2	$v \approx -0.5000 \pm 0.8660\, i$
3	3	$v \approx -0.9131, \quad v \approx -0.2934 \pm 1.5421\, i$
4	4	$v \approx -0.9732 \pm 0.6006\, i, \quad v \approx -0.0268 \pm 2.1418\, i$

Table 8: Zeros of ${}_{Bel}\mathcal{G}\lambda_n(0, v, 1)$ for small n.

n	Degree in v	Zeros (real and complex)
2	1	$v = -0.5000$
3	2	$v \approx -0.5000 \pm 0.8660\, i$
4	3	$v \approx -0.9131, \quad v \approx -0.2934 \pm 1.5421\, i$

Finally, Figure 5 displays the real-zero counts from Tables 6–8 as grouped bars, giving a side-by-side comparison across the three subfamilies.

Inspecting the computed data for $n \leq 4$ at the parameter values $u = 0$, $w = 1$, a clear regularity stands out. In the Bernoulli-λ and Euler-λ families, every odd-degree polynomial ($n = 1, 3$) possesses precisely one real zero, while the remaining roots appear as conjugate complex pairs; the even-degree polynomials ($n = 2, 4$) are free of real zeros altogether. The Genocchi-λ family displays an analogous alternation, shifted by one unit in the degree: polynomials whose actual degree is odd ($n = 2, 4$, recalling that $\deg \mathcal{G}_n = n - 1$) have a single real zero, and those of

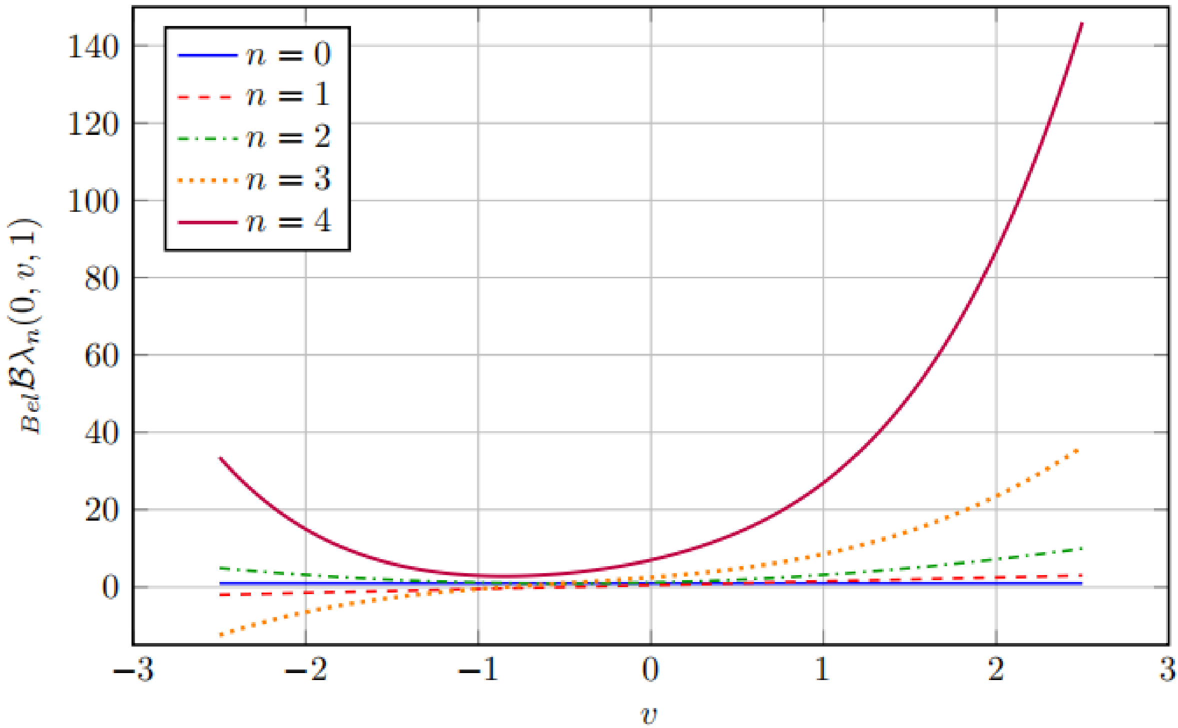

Figure 1: Profiles of $_{Bel}\mathcal{B}\lambda_n(0, v, 1)$ for $n = 0, 1, 2, 3, 4$.

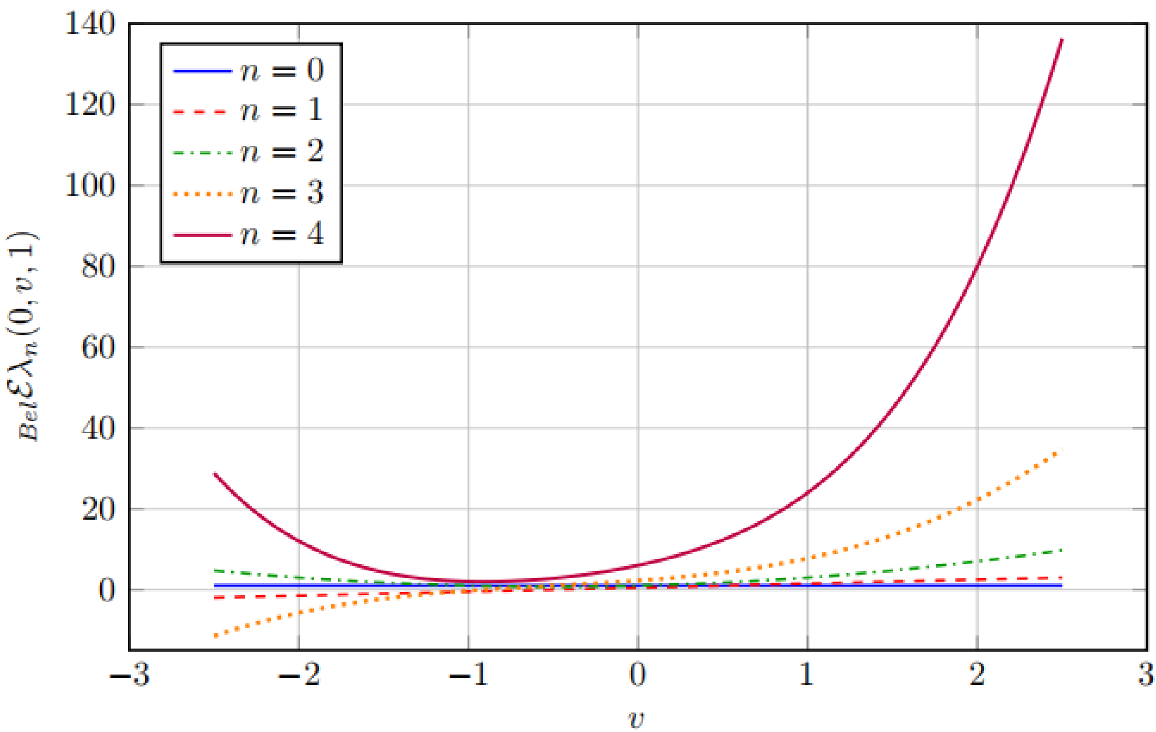

Figure 2: Profiles of $_{Bel}\mathcal{E}\lambda_n(0, v, 1)$ for $n = 0, 1, 2, 3, 4$.

even degree ($n = 3$) have none. It remains an open problem to determine whether this parity pattern extends to all orders. The variation of the Bell parameter w visibly displaces the zero loci and modulates the polynomial amplitude (Figure 4), confirming

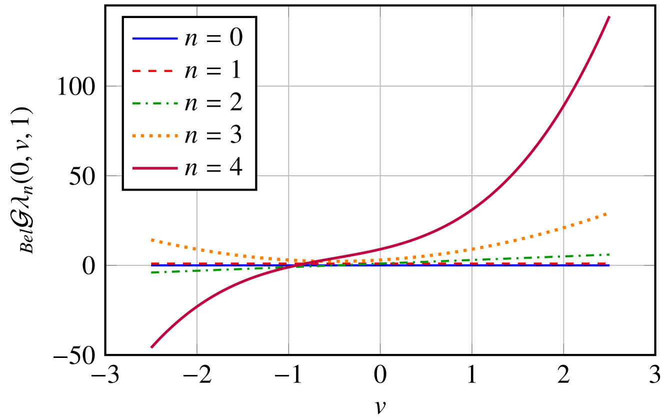

Figure 3: Profiles of $_{Bel}\mathcal{G}\lambda_n(0, v, 1)$ for $n = 0, 1, 2, 3, 4$.

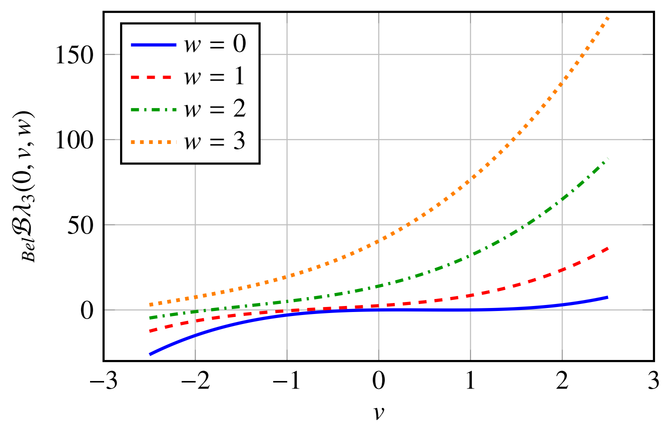

Figure 4: How the Bell parameter w shapes $_{Bel}\mathcal{B}\lambda_3(0, v, w)$.

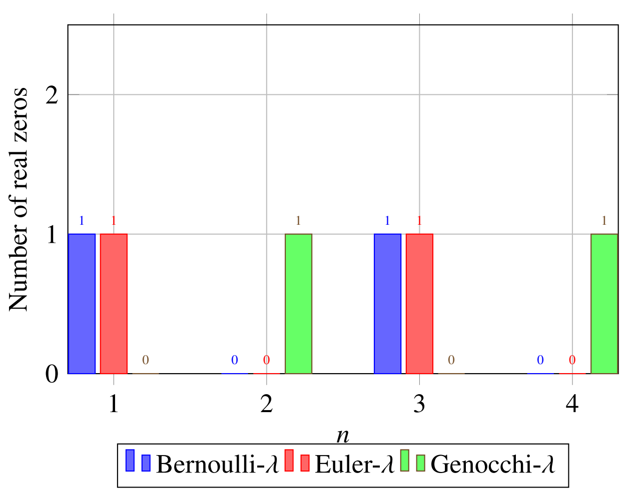

Figure 5: Grouped real-zero counts for the three Bell-enriched polynomial subfamilies.

the non-trivial combinatorial effect of the Bell convolution.

6. Concluding remarks

This work has introduced and systematically investigated a composite polynomial class—the Bell-enriched Appell-λ-polynomials—within the unified setting of operational calculus and the monomiality principle. Starting from a generating function obtained by convoluting Bell-based λ-polynomials with an arbitrary Appell sequence, we derived closed-form series representations, convolution and Stirling-type identities, and a compact determinantal formula. The multiplicative and derivative operators were pinpointed, yielding the governing differential equation as a direct by-product.

The general machinery was tested against three prominent choices of the generating-function factor. For the Bernoulli and Euler cases, which satisfy $\mathcal{A}(0) \neq 0$, the complete quasi-monomial, algebraic, and determinantal apparatus of Theorems 3.1–3.4 applies and the results were recorded in full. The Genocchi case, for which $\mathcal{A}(0) = 0$, falls outside the standard Appell framework; it was treated separately within the associated-Appell setting, where the generating-function and series results remain valid but the quasi-monomial raising operator of Theorem 3.1 and the determinantal formula of Theorem 3.4 do not apply. It was shown, however, that a simple reindexing of the Genocchi-λ sequence absorbs the zero at the origin and produces a family $\{\widetilde{\mathcal{G}}_n\}$ that is admissible in the full Appell framework, so that the quasi-monomial structure is recovered at that level. A supplementary numerical study confirmed the theoretical predictions and revealed a suggestive parity pattern in the zero distributions, whose persistence at higher orders remains an open question.

Several natural continuations of this programme present themselves. On the algebraic side, one may explore connections with Riordan arrays, umbral calculus, and combinatorial matrix theory. On the analytic side, multivariate and fractional generalizations as well as q-deformations merit investigation. The role that these polynomials could play in furnishing explicit solutions to differential and integral equations encountered in oscillatory systems and mathematical physics [1, 2, 38, 39] is another direction worth pursuing.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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