

On the inverse-order divisor graph of the cyclic group of order $2p$

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Abstract

This paper introduces the inverse-order divisor graph associated with a finite group G , a hybrid graph obtained by combining the adjacency conditions of the order-divisor graph and the classical inverse graph. Our primary focus is on the additive cyclic group of order $2p$, where p denotes an odd prime. By partitioning the vertices according to the element orders, we obtain an explicit characterization of the adjacency relations of this hybrid graph. Using this characterization, we derive exact formulas for the size of the graph and the degree of each vertex. We further establish that the resulting graph is connected without being complete. In addition, we determine several fundamental metric and combinatorial properties, including the domination number, radius, diameter, girth, clique number, and chromatic number. These results establish several theoretical properties of the inverse-order divisor graph associated with $\mathbb{Z}_{2p}$ for every odd prime p and contribute to the broader study of algebraic hybrid graphs.

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1. Introduction

Finite groups provide a natural framework for constructing and analyzing combinatorial structures, especially graphs. This interplay between group theory and graph theory has led to many graph constructions that reflect algebraic properties of groups. Classical examples include the non-commuting graph [1], the inverse graph [2], the order-divisor graph [3], the power graph [4, 5], and the nilpotent graph [6].

More recently, there has been growing interest in hybrid graphs formed by combining two or more graph definitions into a single construction [7–11]. These graphs are designed to capture multiple algebraic relations at once.

In this work, we introduce the inverse-order divisor graph of a finite group G , denoted by $\Gamma_{\text{IVOD}}(G)$. This construction combines the adjacency conditions of the order-divisor graph and the inverse graph. As a first systematic study of this construction, we focus on the additive cyclic group $\mathbb{Z}_{2p}$, where $p > 2$ is prime. The choice of $\mathbb{Z}_{2p}$ provides a natural starting point because its four distinct element orders, 1, 2, p , and $2p$, yield nontrivial order-divisor interactions, while its two self-inverse elements, 0 and p , ensure that the inverse condition also contributes essentially to the adjacency structure. At the same time, this relatively simple order structure makes $\mathbb{Z}_{2p}$ rich enough to exhibit the interaction between the two conditions yet simple enough for a complete analysis, so that the resulting hybrid graph and its fundamental parameters can be determined explicitly. The study of $\Gamma_{\text{IVOD}}(G)$ for other families of finite groups,

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including higher-order cyclic groups and non-abelian groups, is left for future work.

Our main aim is to investigate the structural, combinatorial, and metric properties of $\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$, where $p > 2$ is a prime. Specifically, we determine its degree distribution, size, connectivity, completeness, diameter, radius, girth, domination number, clique number, and chromatic number. These results provide a detailed structural characterization of $\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$ and establish a foundation for further investigations of this class of algebraic hybrid graphs.

The remaining sections are structured in the following order. Section 2 reviews the essential notions from group theory and graph theory, along with some definitions and auxiliary results. Section 3 establishes the structural and algebraic properties of $\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$, where $p > 2$ is a prime. Section 4 discusses our findings, and Section 5 provides a summary and proposed extensions of this study.

2. Preliminaries

Throughout this work, all graphs are finite, simple, and undirected, and all groups are finite.

2.1. Basic definitions

We begin by recalling some basic concepts in graph theory. Let $\Gamma = (V, E)$ be a graph, where V and E denote its vertex and edge sets, respectively. The *order* and *size* of Γ are denoted by $|V|$ and $|E|$, respectively. For a vertex $a \in V$, its degree is denoted by $\deg(a)$, while the maximum and minimum vertex degrees of Γ are denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$, respectively.

For any vertex $a \in V(\Gamma)$, let $\varepsilon(a)$ denote its eccentricity, that is, the greatest distance from a to any other vertex of Γ . The *radius* and *diameter* of Γ , denoted by $\text{rad}(\Gamma)$ and $\text{diam}(\Gamma)$, are the minimum and maximum values of $\varepsilon(a)$ over all vertices of Γ , respectively. If Γ is disconnected, both the radius and the diameter are taken to be infinite.

The minimum length among all cycles in Γ is called the *girth* of Γ , and is denoted by $g(\Gamma)$. If no cycle exists, then $g(\Gamma) = \infty$.

A subset of $V(\Gamma)$ is called a *clique* if every pair of its distinct vertices is joined by an edge. The maximum cardinality of a clique is denoted by $\omega(\Gamma)$ and is called the *clique number* of Γ .

A *proper coloring* of Γ is an assignment of colors to its vertices such that any two vertices joined by an edge receive distinct colors. The least number of colors required for such a coloring is called the *chromatic number* of Γ and is denoted by $\chi(\Gamma)$.

A *dominating set* in Γ is a subset $L \subseteq V(\Gamma)$ such that every vertex of Γ either belongs to L or is adjacent to at least one vertex of L . The minimum number of vertices needed to form such a set is the *domination number* of Γ , denoted by $\gamma(\Gamma)$.

We also recall the group-theoretic notation used throughout the paper. Let $(G, *)$ be a group with identity element e . Unless otherwise specified, the vertex set of every graph considered in this paper is the underlying set of G .

For each element $a \in G$, its inverse is denoted by a' and satisfies $a * a' = a' * a = e$. In additive notation, the inverse of a is $-a$, and $a + (-a) = -a + a = 0$.

For an element $a \in G$, we define its order $o(a)$ as the smallest positive integer s such that $a^s = e$. In additive notation, this becomes $s \cdot a = e$.

A group generated by a single element is called *cyclic*. Thus, if $G = \langle g \rangle$ has order n , every element of G has the form g^s (or $s \cdot g$ in an additive group), and its order is given by $o(a) = \frac{n}{N}$, where N denotes the greatest common divisor of s and n .

All graph-theoretic and group-theoretic terminology not explicitly introduced here is used in its standard sense.

Next, we introduce the graphs that form the basis for this study. For a group $(G, *)$, let $M(G)$ denote the set of elements that are not equal to their own inverses.

Definition 2.1. [2] The *inverse graph* of G , written $\Gamma_{\text{IV}}(G)$, is defined on the vertex set $V(\Gamma_{\text{IV}}) = G$, such that two distinct vertices $a, b \in V(\Gamma_{\text{IV}})$ are joined by an edge exactly when $a * b \in M(G)$ or $b * a \in M(G)$.

Definition 2.2. [3] The *order-divisor graph* of G , written $\Gamma_{\text{OD}}(G)$, is defined on the vertex set $V(\Gamma_{\text{OD}}) = G$, with the condition that two distinct vertices $a, b \in V(\Gamma_{\text{OD}})$ are joined by an edge if and only if $o(a) \neq o(b)$ and either $o(a) \mid o(b)$ or $o(b) \mid o(a)$.

We now define the inverse-order divisor graph, a hybrid graph that combines the two preceding graph constructions.

Definition 2.3. Let $M(G)$ be as above. The *inverse-order divisor graph* of G , written $\Gamma_{\text{IOD}}(G)$, is the graph with vertex set $V(\Gamma_{\text{IOD}}) = G$, where two distinct vertices $a, b \in V(\Gamma_{\text{IOD}})$ are joined by an edge if and only if both of the following conditions are satisfied:

1. $a * b \in M(G)$ or $b * a \in M(G)$;
2. $o(a) \neq o(b)$ and either $o(a) \mid o(b)$ or $o(b) \mid o(a)$.

Example 2.1. Consider the additive cyclic group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6. An element $x \in \mathbb{Z}_6$ is self-inverse if $x \equiv -x \pmod{6}$, or equivalently, $2x \equiv 0 \pmod{6}$.

Thus, the self-inverse elements are 0 and 3, and hence

$$M(\mathbb{Z}_6) = \mathbb{Z}_6 \setminus \{0, 3\} = \{1, 2, 4, 5\}. \quad (1)$$

The orders of the elements of $\mathbb{Z}_6$ are given by

$$\begin{array}{c|cccccc} x & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline o(x) & 1 & 6 & 3 & 2 & 3 & 6 \end{array}. \quad (2)$$

Since $\mathbb{Z}_6$ is written additively, the group operation $*$ in Definition 2.3 is represented by $+$. Moreover, addition in $\mathbb{Z}_6$ is commutative, so the first adjacency condition reduces to $a + b \in M(\mathbb{Z}_6)$.

Therefore, two distinct vertices $a, b \in \mathbb{Z}_6$ are adjacent if and only if

- $a + b \in M(\mathbb{Z}_6)$, and

- $o(a) \neq o(b)$ and either $o(a) \mid o(b)$ or $o(b) \mid o(a)$.

Applying these conditions to the elements of $\mathbb{Z}_6$, we obtain the edge set

$$E(\Gamma_{\text{IOD}}(\mathbb{Z}_6)) = \{\{0, 1\}, \{0, 2\}, \{0, 4\}, \{0, 5\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{3, 5\}\}. \quad (3)$$

The resulting graph is illustrated in Figure 1 alongside its two constituent graphs, the inverse graph and the order-divisor graph. Although all three graphs share the same vertex set $\mathbb{Z}_6$, their edge sets are determined by their respective adjacency conditions.

2.2. Order partition for $\mathbb{Z}_{2p}$

Lemma 2.1. [9] Let $p > 2$ be a prime, and consider the additive group $\mathbb{Z}_{2p}$ of integers modulo $2p$, whose elements are $0, 1, \dots, 2p - 1$. Then the elements of $\mathbb{Z}_{2p}$ can be partitioned according to their orders:

1. $V_1 = \{0\}$, where $o(0) = 1$ and $|V_1| = 1$;
2. $V_2 = \{p\}$, where $o(p) = 2$ and $|V_2| = 1$;
3. $V_3 = \{2, 4, \dots, 2p - 2\}$, where $o(x) = p$ for all $x \in V_3$, and $|V_3| = p - 1$;
4. $V_4 = \{1, 3, \dots, 2p - 1\} \setminus \{p\}$, where $o(x) = 2p$ for all $x \in V_4$, and $|V_4| = p - 1$.

Example 2.2. For $\mathbb{Z}_{10}$ with $p = 5$, Lemma 2.1 yields:

$$\begin{aligned} V_1 &= \{0\}, & o(0) &= 1, \\ V_2 &= \{5\}, & o(5) &= 2, \\ V_3 &= \{2, 4, 6, 8\}, & o(2) = o(4) = o(6) = o(8) &= 5, \\ V_4 &= \{1, 3, 7, 9\}, & o(1) = o(3) = o(7) = o(9) &= 10. \end{aligned} \quad (4)$$

2.3. Edge characterization

Proposition 2.2. Let $\Gamma = \Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$, where $p > 2$ is a prime. Let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ from Lemma 2.1. By Definition 2.3, the adjacencies between distinct vertices are as follows:

1. Every vertex in $V_3 \cup V_4$ is adjacent to $0 \in V_1$.
2. Every vertex in V_4 is adjacent to $p \in V_2$.
3. Vertices $u \in V_3$ and $v \in V_4$ are adjacent if and only if $u + v \not\equiv p \pmod{2p}$.
4. There are no edges within any V_i ($i = 1, 2, 3, 4$), nor between V_1 and V_2 , nor between V_2 and V_3 .

Proof. Recall that the set of non-self-inverse elements is $M(\mathbb{Z}_{2p}) = \mathbb{Z}_{2p} \setminus \{0, p\}$, since 0 and p are the unique elements in $\mathbb{Z}_{2p}$ satisfying $2a \equiv 0 \pmod{2p}$. By Definition 2.3, two distinct vertices a, b are joined by an edge if and only if $a + b \in M(\mathbb{Z}_{2p})$ and $o(a) \neq o(b)$ with either $o(a) \mid o(b)$ or $o(b) \mid o(a)$. We analyze the interactions between the partition sets:

Case 1 (Adjacencies with V_1): Let $u = 0 \in V_1$ and $v \in V_3 \cup V_4$. Since $o(0) = 1$, it divides $o(v) \in \{p, 2p\}$. Furthermore, $u + v = v \notin \{0, p\}$, so $u + v \in M(\mathbb{Z}_{2p})$. Thus, 0 is adjacent to all elements in $V_3 \cup V_4$.

Case 2 (Adjacencies with V_2): Let $u = p \in V_2$ and $v \in V_4$. Since $o(p) = 2$ and $o(v) = 2p$, the divisibility condition holds. As $v \in V_4$ is odd, $p + v$ is odd, ensuring $p + v \not\equiv 0 \pmod{2p}$ and $p + v \not\equiv p \pmod{2p}$. Thus, $u + v \in M(\mathbb{Z}_{2p})$, and adjacency holds.

Case 3 (Adjacencies between V_3 and V_4): Let $u \in V_3$ and $v \in V_4$. We have $o(u) = p$ and $o(v) = 2p$, satisfying the order divisibility condition. The condition $u + v \in M(\mathbb{Z}_{2p})$ fails if and only if $u + v \equiv p \pmod{2p}$, which is equivalent to $v \equiv p - u \pmod{2p}$. For all other pairs, adjacency holds.

Case 4 (Non-adjacencies): Pairs within any of the sets V_1, V_2, V_3 , or V_4 fail the distinct order-divisibility requirement. Furthermore, edges between V_1 and V_2 are prohibited because $0 + p = p \notin M(\mathbb{Z}_{2p})$. Finally, for $u \in V_2$ and $w \in V_3$, the order-divisor condition fails, as $o(u) = 2$ does not divide $o(w) = p$, rendering all such pairs non-adjacent. $\square$

3. Main results

3.1. Degree distribution and graph size

Theorem 3.1. Let $p > 2$ be a prime. The degree of a vertex v in $\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$ is given by

$$\deg(v) = \begin{cases} 2p - 2, & \text{if } o(v) = 1, \\ p - 1, & \text{if } o(v) = 2 \text{ or } o(v) = p, \\ p, & \text{if } o(v) = 2p. \end{cases} \quad (5)$$

Proof. Let $\Gamma = \Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$ and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ given in Lemma 2.1.

By Proposition 2.2, the adjacency relations are as follows. $0 \in V_1$ shares an edge with every vertex in $V_3 \cup V_4$, while $p \in V_2$ shares an edge with every vertex in V_4 .

Every $u \in V_3$ shares an edge with 0 and every vertex in V_4 , except for the unique vertex $v \in V_4$ satisfying $u + v \equiv p \pmod{2p}$. Every $v \in V_4$ shares an edge with 0, p , and every vertex in V_3 , except for the unique vertex u satisfying $u + v \equiv p \pmod{2p}$.

Since these are precisely the adjacencies in Γ , we compute the degrees as follows:

$$\begin{aligned} \deg(v) &= |V_3| + |V_4| = 2p - 2, & v \in V_1, \\ \deg(v) &= |V_4| = p - 1, & v \in V_2, \\ \deg(v) &= 1 + (|V_4| - 1) = p - 1, & v \in V_3, \\ \deg(v) &= 2 + (|V_3| - 1) = p, & v \in V_4. \end{aligned} \quad (6)$$

This yields the degrees stated in Theorem 3.1. $\square$

Corollary 3.2. For every prime $p > 2$,

$$\delta(\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})) = p - 1 \quad \text{and} \quad \Delta(\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})) = 2p - 2. \quad (7)$$

Proof. Let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ defined in Lemma 2.1. By Theorem 3.1, the possible vertex degrees in $\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$ are $2p - 2$, p , and $p - 1$. Since $p > 2$, we have

$$p - 1 < p < 2p - 2. \quad (8)$$

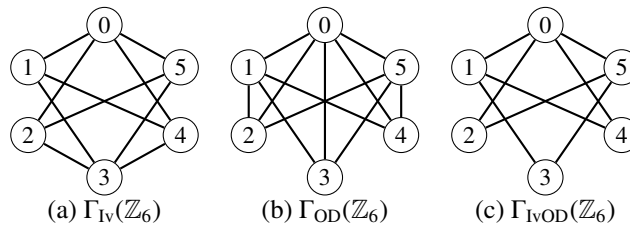


Figure 1: (a) Inverse graph of $\mathbb{Z}_6$, (b) Order-divisor graph of $\mathbb{Z}_6$, and (c) Inverse-order divisor graph of $\mathbb{Z}_6$.

Hence, the minimum degree is attained by the vertices in $V_2 \cup V_3$, whereas the maximum degree is attained by the vertex in V_1 . $\square$

Theorem 3.3. For every prime $p > 2$, the inverse-order divisor graph $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ has $p^2 - 1$ edges.

Proof. Let $\Gamma = \Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$. Recall from Lemma 2.1 that $|V_1| = |V_2| = 1$ and $|V_3| = |V_4| = p - 1$.

By Proposition 2.2, every edge of Γ belongs to exactly one of the following three disjoint edge sets.

$$\begin{aligned} E_1 &= \{\{0, a\} : a \in V_3 \cup V_4\}, \\ E_2 &= \{\{p, b\} : b \in V_4\}, \\ E_3 &= \{\{a, b\} : a \in V_3, b \in V_4, a + b \not\equiv p \pmod{2p}\}. \end{aligned} \quad (9)$$

The cardinalities of these edge sets are computed as follows. Since 0 is adjacent to every vertex in $V_3 \cup V_4$,

$$|E_1| = (p - 1) + (p - 1) = 2p - 2. \quad (10)$$

Since p is adjacent to every vertex in V_4 ,

$$|E_2| = p - 1. \quad (11)$$

For each $a \in V_3$, there is a unique $b \in V_4$ such that $a + b \equiv p \pmod{2p}$; this vertex is the only non-neighbor of a in V_4 . Hence each $a \in V_3$ has $p - 2$ neighbors in V_4 . Since every edge in E_3 joins a vertex of V_3 to a vertex of V_4 , each edge is counted exactly once, and therefore

$$|E_3| = (p - 1)(p - 2). \quad (12)$$

Therefore,

$$\begin{aligned} |E(\Gamma)| &= |E_1| + |E_2| + |E_3| \\ &= (2p - 2) + (p - 1) + (p - 1)(p - 2) \\ &= p^2 - 1. \end{aligned} \quad (13)$$

This completes the proof. $\square$

Example 3.1. Consider $\Gamma_{\text{IvOD}}(\mathbb{Z}_{22})$, where $2p = 22$, so that $p = 11$. By Theorem 3.3,

$$|E(\Gamma_{\text{IvOD}}(\mathbb{Z}_{22}))| = p^2 - 1 = 11^2 - 1 = 121 - 1 = 120. \quad (14)$$

We verify this directly using the edge sets established in the proof.

Indeed, for $p = 11$, the three disjoint edge sets in the proof have sizes

$$\begin{aligned} |E_1| &= 2p - 2 = 20, \\ |E_2| &= p - 1 = 10, \\ |E_3| &= (p - 1)(p - 2) = 10 \cdot 9 = 90. \end{aligned} \quad (15)$$

Therefore,

$$|E(\Gamma_{\text{IvOD}}(\mathbb{Z}_{22}))| = 20 + 10 + 90 = 120. \quad (16)$$

Hence, $\Gamma_{\text{IvOD}}(\mathbb{Z}_{22})$ has 120 edges as shown in Figure 2.

3.2. Connectivity and completeness

Theorem 3.4. For every prime $p > 2$, the inverse-order divisor graph $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ is connected.

Proof. Let $\Gamma = \Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ given in Lemma 2.1. We demonstrate that every vertex $v \in V(\Gamma)$ is in the same connected component as the vertex $0 \in V_1$.

By the adjacency relations established in Proposition 2.2, every vertex belonging to $V_3 \cup V_4$ shares an edge with 0. Thus, these vertices are connected to 0 by a path of length 1.

The vertex $p \in V_2$ shares an edge with every vertex in V_4 . Since $p > 2$, V_4 is non-empty. Choosing any $v \in V_4$ gives the path $p \sim v \sim 0$ from p to 0. Since all vertices in $V_3 \cup V_4$ are adjacent to 0, and p is connected to 0 via any $v \in V_4$, every vertex in $V(\Gamma)$ is connected to 0. Hence a path exists between every pair of vertices, proving that $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ is connected. $\square$

Theorem 3.5. Let $p > 2$ be a prime. Then the inverse-order divisor graph $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ is not complete.

Proof. By Lemma 2.1, the vertex set of $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ is partitioned into V_1, V_2, V_3 , and V_4 , where $|V_3| = p - 1$. Since $p > 2$, we have $|V_3| \geq 2$, so there exist two distinct vertices $u, v \in V_3$.

By Proposition 2.2, no two distinct vertices of V_3 are adjacent. Hence, $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ contains a pair of non-adjacent vertices and therefore is not complete. $\square$

3.3. Invariants

Theorem 3.6. Let $p > 2$ be a prime. Then $\chi(\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})) = 3$.

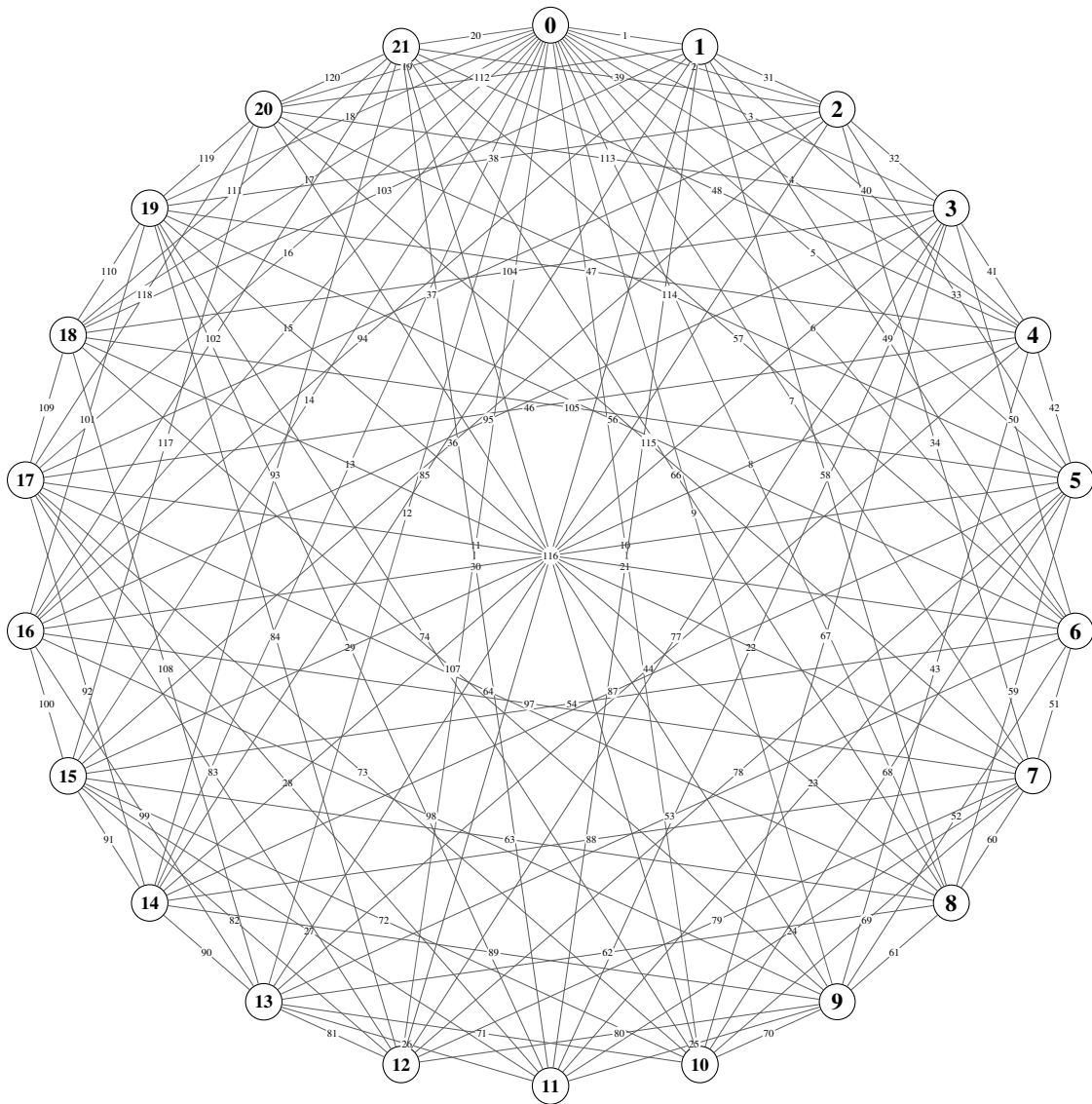


Figure 2: Inverse-order divisor graph of $\mathbb{Z}_{22}$ with 120 edges.

Proof. Let $\Gamma = \Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ given in Lemma 2.1.

To establish the upper bound, assign one color to the vertices of $V_1 \cup V_2$, a second color to the vertices of V_3 , and a third color to the vertices of V_4 . By Proposition 2.2, there are no edges within V_3 , no edges within V_4 , and no edge between the vertices of V_1 and V_2 . Hence, this is a proper coloring of Γ , and therefore $\chi(\Gamma) \leq 3$.

To establish the lower bound, suppose that two colors suffice. Since $p > 2$, every vertex in V_3 has a neighbor in V_4 . Let $a \in V_3$ and let $b \in V_4$ be such a neighbor. By Proposition 2.2, the vertex 0 is a neighbor of a and b . Thus, 0, a , and b form a triangle, contradicting the assumption that Γ is 2-colorable. Hence, $\chi(\Gamma) \geq 3$.

Combining the upper and lower bounds, we conclude that $\chi(\Gamma) = 3$. $\square$

The coloring used in the proof is illustrated in Figure 3.

Theorem 3.7. Let $p > 2$ be a prime. Then $\omega(\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})) = 3$.

Proof. Let $\Gamma = \Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$. Recall from Lemma 2.1 that $|V_1| = |V_2| = 1$ and $|V_3| = |V_4| = p - 1$. By Proposition 2.2, there are no edges within V_3 or within V_4 , the vertex 0 is adjacent to every vertex in $V_3 \cup V_4$, the vertex p shares an edge with every vertex in V_4 , while every vertex in V_3 shares an edge with all vertices in V_4 except one.

To establish the lower bound, let $a \in V_3$. Since $|V_4| = p - 1 \geq 2$, there exists a vertex $b \in V_4$ such that $a + b \not\equiv p \pmod{2p}$. Thus, a and b are joined by an edge. Since 0 is adjacent to both a and b , the set $\{0, a, b\}$ forms a clique of size 3. Therefore, $\omega(\Gamma) \geq 3$.

To establish the upper bound, observe that a clique contains at most one vertex from each partition set, since $|V_1| = |V_2| = 1$ and there are no edges within V_3 or within V_4 . Moreover, no clique can contain both a vertex from V_2 and a vertex from

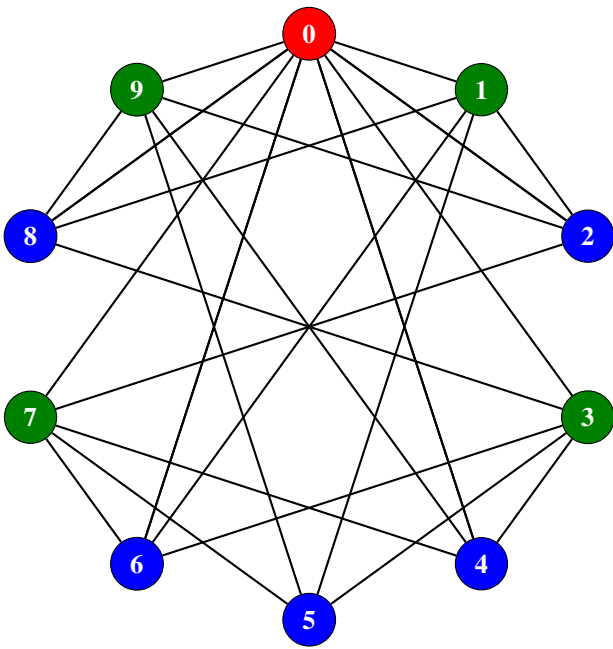


Figure 3: Inverse-order divisor graph of $\mathbb{Z}_{10}$ with $\chi(\Gamma) = 3$.

V_3 , because there are no edges between these partition sets. Hence, every clique contains at most three vertices, and therefore $\omega(\Gamma) \leq 3$.

Combining the upper and lower bounds, we conclude that $\omega(\Gamma) = 3$. $\square$

Theorem 3.8. Let $p > 2$ be a prime. Then $\gamma(\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})) = 2$.

Proof. Let $\Gamma = \Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$, and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ given in Lemma 2.1.

To establish the upper bound, consider the set $D = \{0, p\}$. By Proposition 2.2, the vertex 0 is adjacent to every vertex in $V_3 \cup V_4$, while p is adjacent to every vertex in V_4 . Since every vertex dominates itself, it follows that 0 dominates the vertices in $V_1 \cup V_3 \cup V_4$, and p dominates the vertices in $V_2 \cup V_4$. Hence, D dominates Γ , and therefore $\gamma(\Gamma) \leq 2$.

To establish the lower bound, suppose, for contradiction, that $\gamma(\Gamma) < 2$. Since the domination number is a positive integer, it follows that $\gamma(\Gamma) = 1$. Hence, there exists a vertex $a \in V(\Gamma)$ that dominates every vertex of Γ . By Proposition 2.2, we consider the following cases.

- If $a = 0$, then p is not adjacent to 0. Since $p \neq 0$, the vertex 0 does not dominate p , a contradiction.
- If $a = p$, then 0 is not adjacent to p . Since $0 \neq p$, the vertex p does not dominate 0, a contradiction.
- If $a \in V_3$, then p is not adjacent to any vertex in V_3 . Hence, a does not dominate p , a contradiction.
- If $a \in V_4$, then no two distinct vertices of V_4 are adjacent. Since $|V_4| = p - 1 \geq 2$, there exists $b \in V_4 \setminus \{a\}$. Thus, a does not dominate b , contradicting the assumption that a dominates every vertex of Γ .

Therefore, no single vertex dominates Γ , and so $\gamma(\Gamma) \geq 2$.

Combining the upper and lower bounds, we conclude that $\gamma(\Gamma) = 2$. $\square$

Theorem 3.9. Let $p > 2$ be a prime. Then

$$\text{diam}(\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})) = 2. \quad (17)$$

Proof. Let $\Gamma = \Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$, and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ given in Lemma 2.1. By Proposition 2.2, the adjacency relations in Γ are completely determined. In particular, the vertex 0 is adjacent to every vertex in $V_3 \cup V_4$, p shares an edge with every vertex in V_4 , there are no edges within V_3 or within V_4 , and each vertex in V_3 is adjacent to every vertex in V_4 except one.

Clearly, every pair of adjacent vertices is at distance 1. It therefore suffices to consider non-adjacent pairs of vertices.

- The vertices 0 and p are not adjacent. Since $|V_4| = p - 1 \geq 2$, let $c \in V_4$. Both 0 and p are adjacent to c . Hence, $d(0, p) = 2$.
- If $a, b \in V_3$, then both a and b are adjacent to 0. Thus, $d(a, b) = 2$.
- If $a, b \in V_4$, then both a and b are adjacent to 0. Thus, $d(a, b) = 2$.
- Suppose $a \in V_3$ and $b \in V_4$ are not joined by an edge. Then $a + b \equiv p \pmod{2p}$. Since each of a and b shares an edge with 0, the path $a \sim 0 \sim b$ has length 2. Consequently, $d(a, b) = 2$.
- If $a = p$ and $b \in V_3$, then a and b are not neighbors. Since $|V_4| = p - 1 \geq 2$ and each vertex in V_3 shares an edge with every vertex in V_4 except one, there exists a vertex $c \in V_4$ adjacent to b . Since p shares an edge with every vertex in V_4 , the path $p \sim c \sim b$ has length 2. Hence, $d(a, b) = 2$.

Since every pair of distinct vertices in Γ is at distance at most 2, it follows that $\text{diam}(\Gamma) = 2$. $\square$

Theorem 3.10. Let $p > 2$ be a prime. Then

$$\text{rad}(\Gamma_{\text{IOD}}(\mathbb{Z}_{2p})) = 2. \quad (18)$$

Proof. Let $\Gamma = \Gamma_{\text{IOD}}(\mathbb{Z}_{2p})$. By Theorem 3.9, $\text{diam}(\Gamma) = 2$.

Since the radius of a graph does not exceed its diameter, it follows that $\text{rad}(\Gamma) \leq 2$.

To establish the lower bound, it suffices to show that the eccentricity of every vertex is at least 2. By Proposition 2.2, every vertex has at least one non-neighbor:

- The vertex 0 is not a neighbor of p .
- The vertex p is not a neighbor of any vertex in V_3 .
- Each vertex in V_4 has a unique non-neighbor in V_3 .
- Each vertex in V_3 is a non-neighbor of p .

Table 1: Summary of properties of $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ for $p = 3, 5, 7, 11, 13, 17$.

Properties	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_6)$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{10})$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{14})$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{22})$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{26})$	$\Gamma_{\text{IvOD}}(\mathbb{Z}_{34})$
Order ($ V $)	$2p$	6	10	14	22	26	34
Size ($ E $)	$p^2 - 1$	8	24	48	120	168	288
Minimum degree (δ)	$p - 1$	2	4	6	10	12	16
Maximum degree (Δ)	$2p - 2$	4	8	12	20	24	32
Radius (rad)	2	2	2	2	2	2	2
Diameter (diam)	2	2	2	2	2	2	2
Girth (g)	3	3	3	3	3	3	3
Clique (ω)	3	3	3	3	3	3	3
Chromatic number (χ)	3	3	3	3	3	3	3
Domination number (γ)	2	2	2	2	2	2	2
Connected	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Complete	No	No	No	No	No	No	No

Since $\text{diam}(\Gamma) = 2$, every pair of vertices is at distance at most 2. Therefore, the distance from each vertex to any of its non-neighbors is exactly 2. Hence, every vertex has eccentricity at least 2, and therefore $\text{rad}(\Gamma) \geq 2$.

Combining the upper and lower bounds, we conclude that $\text{rad}(\Gamma) = 2$.

□

Theorem 3.11. Let $p > 2$ be a prime. Then $g(\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})) = 3$.

Proof. Let $\Gamma = \Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ and let $\{V_i\}_{i=1}^4$ be the partition of $V(\Gamma)$ defined in Lemma 2.1.

To prove that $g(\Gamma) = 3$, it suffices to show the existence of a triangle in Γ , since a simple graph contains no cycles of smaller length.

By Proposition 2.2, the vertex 0 is adjacent to every vertex in $V_3 \cup V_4$, and each vertex of V_3 is adjacent to all but one vertex of V_4 .

Now, consider any $u \in V_3$. Since $|V_4| = p - 1 \geq 2$ and at most one vertex of V_4 is nonadjacent to u , there exists a vertex $w \in V_4$ adjacent to u ; equivalently, $u + w \not\equiv p \pmod{2p}$. Therefore, 0 is adjacent to both u and w , while u is adjacent to w . Hence the vertices 0, u , and w form a triangle in Γ .

Since Γ contains a cycle of length 3 and no cycle of smaller length, we conclude that $g(\Gamma) = 3$.

□

4. Discussion

The results in Section 3 establish the principal structural properties of $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$. Table 1 summarizes these values for the primes $p = 3, 5, 7, 11, 13, 17$. The number of edges increases quadratically with p , and both minimum and maximum degrees increase linearly. In contrast, the radius, diameter, girth, clique number, chromatic number, and domination number are constant throughout the list. The family therefore exhibits a stable core of metric and combinatorial parameters, even when its size and degree sequence change with p .

A useful way to examine the effect of hybridization is to compare $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ with the two parent graphs (Γ_{OD} and Γ_{IV}) defined on the same vertex set. Table 2 presents the relevant values for $p = 3, 5, 7$, computed directly from the definitions of the three graphs. The most obvious difference is the size of the graph. For $p = 3$, $\Gamma_{\text{IvOD}}(\mathbb{Z}_6)$ has only 8 edges, compared with 11 for $\Gamma_{\text{OD}}(\mathbb{Z}_6)$ and 10 for $\Gamma_{\text{IV}}(\mathbb{Z}_6)$. For $p = 5$, the corresponding numbers are 24, 29, and 36, while for $p = 7$, they are 48, 55, and 78, respectively. In every case, the hybrid graph is strictly sparser. This is an immediate consequence of the definition: an edge exists only when both the inverse condition and the order-divisor condition hold simultaneously, so the edge set of the hybrid is a proper subset of each parent edge set.

The same restriction is seen in the degree sequence. In the examples in Table 2, the minimum degree of $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ is smaller than that of either parent graph, while the maximum degree coincides with that of the inverse graph. The reduction in minimum degree occurs because the inverse condition deletes one neighbor of every vertex of order p that lies among the vertices of order $2p$. On the other hand, the vertex 0 remains adjacent to every vertex in $V_3 \cup V_4$, giving it degree $2p - 2$. Hence, the hybrid graph retains the same maximum degree as the inverse graph.

The metric and domination parameters also show a combination of inherited and modified properties. As shown in Table 1, the diameter and radius of $\Gamma_{\text{IvOD}}(\mathbb{Z}_{2p})$ are both 2 for every prime listed, while its domination number is 2. Table 2 shows that all three graphs have diameter 2, but their radii differ. The order-divisor graph has radius 1, whereas the inverse graph and the hybrid graph have radius 2. This difference is associated with the absence of the edge $\{0, p\}$ in the inverse and hybrid graphs. Consequently, 0 is not a universal vertex, and the radius increases from 1 to 2. The same structural difference also changes the domination number from 1 in the order-divisor graph to 2 in the inverse and hybrid graphs. Thus, with respect to radius and domination, the hybrid graph exhibits the same behavior as the inverse graph.

Table 2: Comparison of properties of Γ_{IV} , Γ_{OD} , and Γ_{IVOD} on $\mathbb{Z}_6$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{14}$.

Graph	$ E $	δ	Δ	χ	ω	γ	g	rad	diam
$\Gamma_{IV}(\mathbb{Z}_6)$	10	3	4	3	3	2	3	2	2
$\Gamma_{OD}(\mathbb{Z}_6)$	11	3	5	3	3	1	3	1	2
$\Gamma_{IVOD}(\mathbb{Z}_6)$	8	2	4	3	3	2	3	2	2
$\Gamma_{IV}(\mathbb{Z}_{10})$	36	7	8	5	5	2	3	2	2
$\Gamma_{OD}(\mathbb{Z}_{10})$	29	5	9	3	3	1	3	1	2
$\Gamma_{IVOD}(\mathbb{Z}_{10})$	24	4	8	3	3	2	3	2	2
$\Gamma_{IV}(\mathbb{Z}_{14})$	78	11	12	7	7	2	3	2	2
$\Gamma_{OD}(\mathbb{Z}_{14})$	55	7	13	3	3	1	3	1	2
$\Gamma_{IVOD}(\mathbb{Z}_{14})$	48	6	12	3	3	2	3	2	2

The clique and chromatic numbers show a different pattern. As reported in Table 1, both parameters are equal to 3 for $\Gamma_{IVOD}(\mathbb{Z}_{2p})$ for all primes listed. The order-divisor graph also has the clique number and the chromatic number equal to 3, whereas the inverse graph has both equal to p . The hybrid graph therefore has the same small clique and chromatic numbers as the order-divisor graph, despite having fewer edges than either parent graph. This indicates that the order-divisor condition substantially limits the large cliques present in the inverse graph.

All three graphs are connected and have girth 3, while none is complete for $p > 2$. Overall, the hybrid graph combines structural features of both parent graphs: its radius and domination number coincide with those of the inverse graph, while its clique and chromatic numbers coincide with those of the order-divisor graph. At the same time, the requirement that both adjacency conditions be satisfied results in fewer edges and a different degree sequence. These features demonstrate that the hybrid construction yields a graph with its own structural characteristics rather than merely reproducing those of either parent graph.

5. Conclusion

This paper introduced the inverse-order divisor graph $\Gamma_{IVOD}(G)$ and established several of its structural properties for the family of cyclic groups $\mathbb{Z}_{2p}$, where $p > 2$ is a prime. These results extend the family of algebraic graphs associated with finite groups by combining the adjacency conditions of the inverse graph and the order-divisor graph.

The results obtained here provide a basis for investigating $\Gamma_{IVOD}(G)$ in other classes of groups. In particular, extending this analysis to cyclic groups of composite order beyond $2p$ and to non-abelian groups offers promising directions for future research.

Data availability

All data relevant to this study are contained in the submitted manuscript.

Declaration of competing interest

The authors declare that there are no conflicts of interest that could have influenced the conduct or reporting of this research.

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