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Negligible sets and the reconstruction of spacetime topology

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Abstract

When strong causality fails, the Alexandrov topology of a spacetime is strictly coarser than the manifold topology. Since pathologies confined to sets of measure zero are routinely disregarded in relativity, one might expect the deficit to be repairable up to a negligible set; we show that it is not, and determine how much volume must be discarded, as an infimum that need not be attained. Let $\sigma \subseteq \tau$ be topologies on a set X, let $\mathcal{I}$ be an ideal on X, and let $\sigma^*(\mathcal{I})$ be the topology generated by the differences $S \setminus N$ with $S \in \sigma$ and $N \in \mathcal{I}$. If $\mathcal{I}$ contains no nonempty τ-open set and $\tau \subseteq \sigma^*(\mathcal{I})$, then $\tau_\theta \subseteq \sigma$, where τ_θ is the topology of θ-open sets; for regular τ this forces $\sigma = \tau$, and the bound is attained. We characterize the admissible ideals: an open set O is the open core of an ideal recovering τ from σ precisely when $O \in \sigma$ and the two topologies agree on $X \setminus O$. Such refinements preserve the Hausdorff property in both directions, but preserve neither the T_0 nor the T_1 axiom. Consequently the strongly causal set of a spacetime is invariant under refinement by every codense ideal, and the region that must be discarded in order to recover the manifold topology from chronology is a union of chronological diamonds, of volume at least that of the chronology-violating set. For an explicit Lorentzian cylinder the infimum of these volumes is zero and is not attained.

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1. Introduction

A relativistic spacetime does not carry its topology as an independent datum. Zeeman [1], Hawking *et al.* [2] and Malament [3] showed, in complementary settings, that the manifold topology is recoverable from causal structure alone, and Kronheimer and Penrose [4] and Penrose [5] identified the exact condition under which the recovery is immediate: the Alexandrov topology $\mathcal{A}$, generated by the chronological diamonds $I^+(p) \cap I^-(q)$, coincides with the manifold topology τ_M precisely when the spacetime is strongly causal. When strong causality fails, $\mathcal{A}$ is strictly coarser and the reconstruction breaks down.

In relativity, sets of measure zero and nowhere dense sets are treated as negligible, and the reconstruction problem invites the same treatment: one might conjecture that a spacetime whose causal misbehavior is so confined admits reconstruction modulo a negligible set. The present paper determines whether this expectation is correct when the negligible set is removed within the topology rather than merely from the underlying point set, and, when it is not, determines the least cost of a successful reconstruction. The natural formalism is that of ideal topological spaces, introduced by Kuratowski [6] and Vaidyanathaswamy [7] and developed by Hayashi [8], Njåstad [9], Samuels [10] and, systematically, by Janković and Hamlett [11]: an ideal $\mathcal{I}$ selects the negligible sets, and the refinement $\sigma^*(\mathcal{I})$ declares $S \setminus N$ open whenever S is open and N is negligible. The question becomes whether $\tau_M \subseteq \mathcal{A}^*(\mathcal{I})$ for some

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ideal $\mathcal{I}$ containing no nonempty open set.

The answer is negative: the obstruction is point-set theoretic. Our first main result, Theorem 4.1, concerns an arbitrary topological space (X, τ), a coarser topology $\sigma \subseteq \tau$ and an ideal $\mathcal{I}$ containing no nonempty τ-open set: if $\tau \subseteq \sigma^*(\mathcal{I})$, then

$$\tau_\theta \subseteq \sigma \subseteq \tau,$$

where τ_θ is the topology of θ-open sets of Veličko [12]. A codense ideal can therefore recover at most the non-θ part of the ambient topology. Since $\tau_\theta = \tau$ exactly for regular τ, the rigidity statement that a strictly coarser topology is never restored by excising negligible sets (Corollary 4.1) is the regular case. Regularity is thus not an auxiliary hypothesis but the exact hypothesis under which Theorem 4.1 closes the gap between τ_θ and τ; it cannot be weakened to T_2, as Example 8 shows, although this single example does not determine which non-regular topologies admit a repairable strict coarsening. The bound is sharp: for the deleted sequence topology on $\mathbb{R}$ the Euclidean topology is repaired by the countable ideal and equals τ_θ exactly (Proposition 8.1).

Theorem 9.2 converts this qualitative obstruction into a quantitative characterization of repairability: for regular τ, a set O is the open core of some ideal restoring τ from σ if and only if O is open in τ, open in σ, and σ and τ agree on $X \setminus O$; the ideal of all subsets of O then suffices. The essential condition is that O is necessarily σ-open, a constraint absent from the qualitative theory. A third rigidity phenomenon, logically independent and requiring no regularity, appears in Theorem 5.2: codense ideal refinement preserves the Hausdorff property in both directions, whereas Example 6 shows that it preserves neither T_0 nor T_1.

Section 10 draws the relativistic consequences. The strongly causal set of a spacetime is unchanged by refinement along any codense ideal (Theorem 10.1), so no null, meager, nowhere dense, or countable modification of the chronological diamonds recovers the manifold topology; $\mathcal{A}^*(\mathcal{I})$ is Hausdorff exactly when the spacetime is strongly causal (Theorem 10.2), although it may be T_1 even in the Gödel universe (Proposition 10.1); and the chronological future topology alone is unrepairable in every spacetime, including the globally hyperbolic ones (Corollary 10.2).

The characterization theorem yields information that the negative results cannot. Interpreted causally, its two conditions state that a region whose removal restores the manifold topology must be a union of chronological diamonds, and that the spacetime must be strongly causal along the complement (Theorem 11.1). A causal pathology cannot therefore be excised along an arbitrarily thin set: the discarded region is necessarily causally saturated, and since chronological diamonds have nonempty interior, every reconstruction has positive cost.

The quantity

$$c(M) = \inf\{\mu_g(O) : O \text{ a discarded region}\},$$

which we call the *repair cost*, is accordingly a well-defined invariant of (M, g) and can be computed in examples. Theorem 11.2 gives $c(M) \geq \mu_g(\mathcal{V})$ for the chronology-violating set $\mathcal{V}$, while Section 12 exhibits a chronological Lorentzian cylinder, failing causality and strong causality on a single closed null geodesic, for which $c(M) = 0$ with the infimum unattained: the topology is recoverable after discarding a region of arbitrarily small volume, and never after discarding one of zero volume. Together these results yield a dichotomy (Corollary 13.1) between chronology violation, which carries an irreducible volume cost, and causality violation without it, which does not. The same cylinder separates the topological formulation of strong causality from its pointwise weakening, showing the latter to be strictly weaker.

The paper is organized as follows. Section 2 fixes notation and recalls the classical material on ideal refinements, the θ-topology and Alexandrov causality. Section 3 proves the rigidity theorems, Section 7 establishes sharpness and the exact characterization, Section 10 develops the relativistic application and the repair cost, and Section 12 constructs the Lorentzian cylinder.

The one-topology specialization $\sigma = \tau$ of Theorem 4.1 is vacuous, and the two-topology formulation studied here does not appear in the literature on ideal topological spaces, which is uniformly concerned with a single topology and an ideal on it [7, 9, 11, 13, 14]. The classical relativistic results invoked are stated as such in Section 2 with attribution. Theorems 4.1, 5.1, 9.1, 9.2, 10.1, 10.2, 11.1 and 11.2, Corollaries 4.1, 5.1, 10.1, 10.2, 11.1 and 13.1, Propositions 6.1, 8.1, 9.1, 10.1, 11.1 and 13.2, and the use made of Example 13, are new. Theorem 5.2 is elementary and in the spirit of known preservation results for codense ideals, for instance those of Dontchev, Ganster and Rose [14]; we could not locate it in this form and include its short proof for completeness without claiming priority.

2. Preliminaries

This section fixes notation and records the three bodies of classical material on which the paper rests: the elementary theory of ideal refinements of a topology, Veličko's θ-topology, and the standard facts about the Alexandrov topology of a spacetime. Results are stated with attribution, in the form in which they will be used, so as to keep the paper self-contained. Topological terminology not defined here follows Engelking [15].

2.1. Ideals and their refinements

Definition 2.1. *An* ideal *on a set X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ such that (i) $\emptyset \in \mathcal{I}$; (ii) if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$; and (iii) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$. If $\mathcal{I}$ is closed under countable unions it is a* σ-ideal. *For $O \subseteq X$ we write $\mathcal{P}(O)$ for the ideal of all subsets of O.*

Definition 2.2. *Let σ be a topology on X and $\mathcal{I}$ an ideal on X. Put*

$$\beta(\sigma, \mathcal{I}) = \{S \setminus N \ : \ S \in \sigma,\ N \in \mathcal{I}\}.$$

Since $(S_1 \setminus N_1) \cap (S_2 \setminus N_2) = (S_1 \cap S_2) \setminus (N_1 \cup N_2)$ and $\mathcal{I}$ is closed under finite unions, $\beta(\sigma, \mathcal{I})$ is closed under finite intersections; since also $X = X \setminus \emptyset \in \beta(\sigma, \mathcal{I})$, it is a base for a topology $\sigma^*(\mathcal{I})$ on X, the ideal refinement of σ by $\mathcal{I}$. *Taking $N = \emptyset$ gives $\sigma \subseteq \sigma^*(\mathcal{I})$.*

The topology $\sigma^*(\mathcal{I})$ is the one obtained from the local function $A \mapsto A^*(\mathcal{I},\sigma) = \{x \in X : U \cap A \notin \mathcal{I}$ for all $U \in \sigma$ with $x \in U\}$ through the Kuratowski closure operator $\mathrm{cl}^*(A) = A \cup A^*(\mathcal{I},\sigma)$; that $\beta(\sigma,\mathcal{I})$ is a base for it is due to Vaidyanathaswamy [7] and is recorded as such by Janković and Hamlett [11, §3]. We work throughout with the base description of Definition 2.2.

Definition 2.3. *Let τ be a topology on X. An ideal $\mathcal{I}$ on X is* τ-codense *if $\mathcal{I} \cap \tau = \{\emptyset\}$, that is, if no nonempty τ-open set is negligible. Spaces equipped with a codense ideal are also called Hayashi–Samuel spaces [8, 10, 14]; codense and completely codense ideals are characterized by Renuka Devi et al. [16].*

Codensity is the formalization of "negligible" that excludes triviality: the ideal $\mathcal{P}(X)$ refines every topology to the discrete one and is therefore uninformative. The standard examples of codense ideals on a smooth manifold M carrying a volume measure μ are the ideal $\mathcal{I}_\mu$ of μ-null sets, the ideal $\mathcal{I}_{\mathrm{mgr}}$ of meager sets, the ideal $\mathcal{I}_{\mathrm{nwd}}$ of nowhere dense sets, the ideal $\mathcal{I}_{\mathrm{cd}}$ of closed discrete sets, and the ideals $\mathcal{I}_{\mathrm{ctb}}$ and $\mathcal{I}_{\mathrm{fin}}$ of countable and of finite sets. Of these, $\mathcal{I}_\mu$, $\mathcal{I}_{\mathrm{mgr}}$ and $\mathcal{I}_{\mathrm{ctb}}$ are σ-ideals.

Definition 2.4. *The* open core *of an ideal $\mathcal{I}$ relative to a topology τ is*

$$O_\tau(\mathcal{I}) = \bigcup\{G \in \tau \;:\; G \in \mathcal{I}\}.$$

Thus $\mathcal{I}$ is τ-codense if and only if $O_\tau(\mathcal{I}) = \emptyset$. In general $O_\tau(\mathcal{I})$ need not itself belong to $\mathcal{I}$; Proposition 9.1 identifies a standard situation in which it does, and $O_\tau(\mathcal{P}(O)) = O$ for every τ-open O.

2.2. The θ-topology

Definition 2.5 (Veličko [12])**.** *Let (X,τ) be a topological space. A set $A \subseteq X$ is* θ-open *if for every $x \in A$ there is $U \in \tau$ with $x \in U \subseteq \mathrm{cl}_\tau(U) \subseteq A$.*

Lemma 2.1 (Veličko [12])**.** *The θ-open subsets of (X,τ) form a topology τ_θ with $\tau_\theta \subseteq \tau$, and $\tau_\theta = \tau$ if and only if τ is regular.*

Indeed, a θ-open set is a union of the corresponding sets U, hence τ-open; arbitrary unions of θ-open sets are θ-open; and if $x \in A \cap B$ with witnesses U, V then $x \in U \cap V \subseteq \mathrm{cl}(U \cap V) \subseteq \mathrm{cl}(U) \cap \mathrm{cl}(V) \subseteq A \cap B$. Regularity of τ is precisely the statement that every τ-open set is θ-open.

2.3. Causal structure and the Alexandrov topology

By a *spacetime* we mean a connected, second countable, time-oriented smooth Lorentzian manifold (M,g) of dimension at least two, with τ_M its manifold topology and μ_g its volume measure. We write $p \ll q$ if there is a future-directed timelike curve from p to q, and $p \leq q$ if $p = q$ or there is a future-directed causal curve from p to q. As usual $I^+(p) = \{q : p \ll q\}$ and $I^-(p) = \{q : q \ll p\}$, while $J^+(p) = \{q : p \leq q\}$ and $J^-(p) = \{q : q \leq p\}$; for $p \ll q$ we write

$$I(p,q) = I^+(p) \cap I^-(q),$$

for the *chronological diamond* determined by p and q. The relation $\ll$ is transitive and satisfies the push-up property

$$p \leq q \ll r \implies p \ll r, \qquad p \ll q \leq r \implies p \ll r; \tag{1}$$

the sets $I^\pm(p)$ are τ_M-open and nonempty; and τ_M is metrizable, hence regular and T_1, so that $(\tau_M)_\theta = \tau_M$ by Lemma 2.1. For these standard facts see Hawking and Ellis [17, Ch. 6], Beem, Ehrlich and Easley [18, Ch. 3], Geroch [19], and Minguzzi [20].

Definition 2.6. *The* Alexandrov topology *$\mathcal{A}$ of (M,g) is the topology on M with base $\{I(p,q) : p \ll q\}$; the* chronological future topology *$\tau^\uparrow$ is generated by $\{I^+(p) : p \in M\}$, and $\tau^\downarrow$ is defined dually.*

That $\{I(p,q)\}$ is a base follows from openness of the diamonds together with Eq. (1): if $z \in I(p_1,q_1) \cap I(p_2,q_2)$, choose $p \ll z \ll q$ with p, q inside the open set $I(p_1,q_1) \cap I(p_2,q_2)$; then $z \in I(p,q) \subseteq I(p_1,q_1) \cap I(p_2,q_2)$. Since $I^+(p) = \bigcup\{I(p,q) : q \in I^+(p)\}$ and dually, $\mathcal{A}$ is exactly the topology generated by all the sets $I^\pm(p)$, so $\mathcal{A} = \tau^\uparrow \vee \tau^\downarrow$ and

$$\tau^\uparrow \subseteq \mathcal{A} \subseteq \tau_M, \qquad \tau^\downarrow \subseteq \mathcal{A} \subseteq \tau_M. \tag{2}$$

A set is $\mathcal{A}$-open precisely when it is a union of chronological diamonds; this description is used repeatedly in Section 10.

Definition 2.7. *(M,g) is* strongly causal at *p if p possesses arbitrarily small τ_M-neighborhoods U such that no future-directed causal curve leaves U and returns to it. The set of such p is the* strongly causal set $\mathrm{SC}(M)$*, and (M,g) is* strongly causal *if $\mathrm{SC}(M) = M$.*

Theorem 2.1 (Kronheimer–Penrose [4]; Penrose [5, Ch. 4])**.** *(M,g) is strongly causal at p if and only if the family $\{I(q,r) : p \in I(q,r)\}$ is a τ_M-neighborhood base at p. Consequently the following are equivalent: (M,g) is strongly causal; $\mathcal{A} = \tau_M$; $\mathcal{A}$ is Hausdorff. Moreover $\mathrm{SC}(M)$ is τ_M-open.*

The following classical structure result is used below; for it see Penrose [5, Ch. 4] and Minguzzi [20, §4]: in a causal spacetime, if strong causality fails at p then there is an endless null geodesic through p at each of whose points strong causality fails. In particular $M \setminus \mathrm{SC}(M)$ is a union of endless null geodesics and can never be a single point. Strong causality is one rung of the causal ladder, whose higher rungs — stable causality, causal continuity [21], causal simplicity, and global hyperbolicity [22] — are surveyed by Minguzzi and Sánchez [23] and by García-Parrado and Senovilla [24]. Only the rung of strong causality is used below, since it is the one at which the Alexandrov topology becomes the manifold topology.

Finally, the *chronology-violating set* is $\mathcal{V} = \{p \in M : p \ll p\}$. For $p \in \mathcal{V}$ put $V_p = I(p,p) = I^+(p) \cap I^-(p)$, an open neighborhood of p contained in $\mathcal{V}$. If $q \in V_p$ then $q \ll q$ and $V_q = V_p$, so the sets V_p are the *chronology-violating components* and partition $\mathcal{V}$ into pairwise disjoint nonempty open sets [17, Prop. 6.4.2]. In particular $\mathcal{V}$ is τ_M-open, and since a pairwise disjoint family of nonempty open subsets of a second countable space is countable, there are at most countably many components. Spacetimes with $\mathcal{V} \neq \emptyset$ arise in exact solutions of the field equations, the Gödel universe [25] and the rotating dust cylinder of Tipler [26] among them, and the question of whether such regions can form from regular initial data is the subject of Hawking's chronology protection conjecture [27].

3. The θ-rigidity theorem

This section establishes the point-set core of the paper. The setting is a space (X, τ), a coarser topology $\sigma \subseteq \tau$ and an ideal $\mathcal{I}$ on X, and the question is whether $\sigma^*(\mathcal{I})$ can recover τ. The argument rests on two elementary lemmas: recovery of τ is equivalent to a local condition on the excess of a σ-approximation, and that excess always contains a τ-open set. Codensity then forces the excess to be empty, from which all three rigidity statements below follow.

Throughout, $\tau(p)$ denotes the family of τ-open sets containing p, and closures without further qualification are taken in τ.

Standing Hypothesis 4. *(X, τ) is a topological space, σ is a topology on X with $\sigma \subseteq \tau$, and $\mathcal{I}$ is an ideal on X.*

Lemma 4.1 (Excess criterion)**.** *Under Hypothesis 4 the following are equivalent.*

- *(i) $\tau \subseteq \sigma^*(\mathcal{I})$.*
- *(ii) For every $p \in X$ and every $W \in \tau(p)$ there is $S \in \sigma$ with $p \in S$ and $S \setminus W \in \mathcal{I}$.*

Proof. Assume (i) and let $p \in W \in \tau$. Then $W \in \sigma^*(\mathcal{I})$, so by Definition 2.2 there are $S \in \sigma$ and $N \in \mathcal{I}$ with $p \in S \setminus N \subseteq W$. From $S \setminus N \subseteq W$ we get $S \setminus W \subseteq N$, and $\mathcal{I}$ is hereditary, so $S \setminus W \in \mathcal{I}$; also $p \in S$. This is (ii).

Assume (ii) and let $W \in \tau$ and $p \in W$. Choose $S \in \sigma$ with $p \in S$ and $N := S \setminus W \in \mathcal{I}$. Then $S \setminus N = S \cap W$ lies in $\beta(\sigma, \mathcal{I})$, contains p and is contained in W. As $p \in W$ was arbitrary, $W \in \sigma^*(\mathcal{I})$. □

The set $S \setminus W$ of Lemma 4.1 is the *excess* of the σ-approximation S over the target W; the question is whether this excess can be made negligible (Figure 1).

Lemma 4.2. *Under Hypothesis 4, let $S \in \sigma$ and $U \in \tau$. Then $S \cap (X \setminus \overline{U})$ is τ-open and contained in $S \setminus U$.*

Proof. $S \in \sigma \subseteq \tau$ and $X \setminus \overline{U} \in \tau$, so the intersection lies in τ; it is contained in $S \setminus \overline{U} \subseteq S \setminus U$. □

Theorem 4.1 (θ-rigidity)**.** *Let (X, τ) be a topological space, $\sigma \subseteq \tau$ a topology on X, and $\mathcal{I}$ a τ-codense ideal on X. If $\tau \subseteq \sigma^*(\mathcal{I})$, then*

$$\tau_\theta \subseteq \sigma \subseteq \tau.$$

Proof. The inclusion $\sigma \subseteq \tau$ is Hypothesis 4. To establish $\tau_\theta \subseteq \sigma$, let $W \in \tau_\theta$ and $p \in W$. By Definition 2.5 there is $U \in \tau$ with

$$p \in U \subseteq \overline{U} \subseteq W.$$

Applying Lemma 4.1 to the pair (p, U) produces $S \in \sigma$ with $p \in S$ and $S \setminus U \in \mathcal{I}$. By Lemma 4.2 the set $G := S \cap (X \setminus \overline{U})$

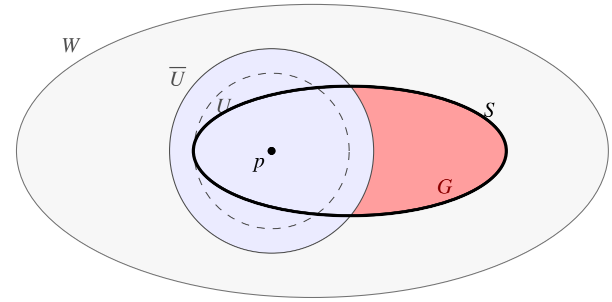

Figure 1: The configuration excluded in the proof of Theorem 4.1. For $p \in W \in \tau_\theta$ choose $U \in \tau$ with $p \in U \subseteq \overline{U} \subseteq W$; Lemma 4.1 then supplies $S \in \sigma$ with $p \in S$ and $S \setminus U \in \mathcal{I}$. The shaded region $G = S \cap (X \setminus \overline{U})$ is an intersection of two τ-open sets, hence τ-open, and lies in $\mathcal{I}$ by heredity; codensity therefore forces $G = \emptyset$. Thus $S \subseteq \overline{U} \subseteq W$, and W is a union of members of σ.

is τ-open and $G \subseteq S \setminus U$, so $G \in \mathcal{I}$ because $\mathcal{I}$ is hereditary. Codensity of $\mathcal{I}$ forces $G = \emptyset$, that is

$$S \subseteq \overline{U} \subseteq W,$$

so $p \in S \subseteq W$ with $S \in \sigma$. As $p \in W$ was arbitrary, W is a union of members of σ and hence $W \in \sigma$. □

Corollary 4.1 (Rigidity)**.** *Let (X, τ) be regular, $\sigma \subseteq \tau$ a topology on X, and $\mathcal{I}$ a τ-codense ideal. Then*

$$\tau \subseteq \sigma^*(\mathcal{I}) \iff \sigma = \tau,$$

and in that case $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$. Thus a strictly coarser topology is never restored by excising negligible sets.

Proof. If $\sigma = \tau$ then $\tau = \sigma \subseteq \sigma^*(\mathcal{I})$ by Definition 2.2, and the last assertion is immediate. Conversely, if $\tau \subseteq \sigma^*(\mathcal{I})$ then $\tau_\theta \subseteq \sigma \subseteq \tau$ by Theorem 4.1, and $\tau_\theta = \tau$ by Lemma 2.1. □

Remark 5. *The inclusion $\sigma \subseteq \tau$ of Hypothesis 4 is used twice in the proof of Theorem 4.1 and cannot be dispensed with, since it is what renders the excess τ-open and hence subject to codensity.*

Theorem 5.1 (Pointwise θ-rigidity)**.** *Let $\sigma \subseteq \tau$ and let $\mathcal{I}$ be τ-codense. Fix $p \in X$ and suppose that $\sigma^*(\mathcal{I})$ contains a τ-neighborhood base at p, that is, for every $W \in \tau(p)$ there is $G \in \sigma^*(\mathcal{I})$ with $p \in G \subseteq W$. Then for every θ-open W containing p there is $S \in \sigma$ with $p \in S \subseteq W$. If τ is regular, σ therefore contains a τ-neighborhood base at p.*

Proof. Let $W \in \tau_\theta$ with $p \in W$ and choose $U \in \tau$ with $p \in U \subseteq \overline{U} \subseteq W$. By hypothesis there is $G \in \sigma^*(\mathcal{I})$ with $p \in G \subseteq U$, and by Definition 2.2 there are $S \in \sigma$ and $N \in \mathcal{I}$ with $p \in S \setminus N \subseteq G \subseteq U$. Hence $S \setminus U \subseteq N$, so $S \setminus U \in \mathcal{I}$, and Lemma 4.2 with codensity gives $S \cap (X \setminus \overline{U}) = \emptyset$. Therefore $p \in S \subseteq \overline{U} \subseteq W$ with $S \in \sigma$. The last assertion follows from Lemma 2.1. □

Corollary 5.1. *Let (X, τ) be regular and $\sigma \subseteq \tau$. Define*

$$B_\sigma = \{p \in X : \ \sigma \text{ contains a } \tau\text{-neighborhood base at } p\},$$

and $B_{\sigma^*(\mathcal{I})}$ analogously. Then $B_\sigma = B_{\sigma^*(\mathcal{I})}$ for every τ-codense ideal $\mathcal{I}$; the set B_σ is an invariant of the pair (σ, τ) that no codense ideal can enlarge.

Proof. $B_\sigma \subseteq B_{\sigma^*(\mathcal{I})}$ because $\sigma \subseteq \sigma^*(\mathcal{I})$; the reverse inclusion is Theorem 5.1. □

The next result is a rigidity phenomenon of a different nature. It involves a single topology, requires no regularity, and needs only codensity relative to σ itself.

Theorem 5.2 (Hausdorff rigidity)**.** *Let σ be a topology on X and $\mathcal{I}$ an ideal on X with $\mathcal{I} \cap \sigma = \{\emptyset\}$. Then $\sigma^*(\mathcal{I})$ is Hausdorff if and only if σ is Hausdorff.*

Proof. If σ is Hausdorff then so is the finer topology $\sigma^*(\mathcal{I})$.

Conversely, suppose $\sigma^*(\mathcal{I})$ is Hausdorff and let $x \neq y$. Separate them by disjoint $\sigma^*(\mathcal{I})$-open sets and shrink each to a basic set: there are $S_1, S_2 \in \sigma$ and $N_1, N_2 \in \mathcal{I}$ with $x \in S_1 \setminus N_1$, $y \in S_2 \setminus N_2$ and

$$(S_1 \setminus N_1) \cap (S_2 \setminus N_2) = (S_1 \cap S_2) \setminus (N_1 \cup N_2) = \emptyset.$$

Hence $S_1 \cap S_2 \subseteq N_1 \cup N_2 \in \mathcal{I}$, so $S_1 \cap S_2 \in \mathcal{I}$ by heredity. But $S_1 \cap S_2 \in \sigma$ and $\mathcal{I} \cap \sigma = \{\emptyset\}$, so $S_1 \cap S_2 = \emptyset$. Since $x \in S_1$ and $y \in S_2$, the sets $S_1, S_2 \in \sigma$ separate x and y. □

Example 6 (T_0 and T_1 are not rigid)**.** *Let X be infinite, $\sigma = \{\emptyset, X\}$ the indiscrete topology and $\mathcal{I} = \mathcal{I}_{\text{fin}}$. Then $\mathcal{I} \cap \sigma = \{\emptyset\}$, and $\sigma^*(\mathcal{I})$ is the cofinite topology, which is T_1, whereas σ is not even T_0. Thus neither T_0 nor T_1 is preserved downwards by codense ideal refinement, and Theorem 5.2 is specific to T_2. In accordance with that theorem, the cofinite topology on an infinite set is not Hausdorff.*

Proposition 6.1 (Inert ideals)**.** *Let (X, τ) be regular, $\sigma \subseteq \tau$, and $\mathcal{I}$ a τ-codense ideal. Then $\sigma^*(\mathcal{I}) = \tau$ if and only if $\sigma = \tau$ and every member of $\mathcal{I}$ is τ-closed. In that case every member of $\mathcal{I}$ is closed and discrete.*

Proof. Suppose $\sigma^*(\mathcal{I}) = \tau$. Then $\tau \subseteq \sigma^*(\mathcal{I})$, so $\sigma = \tau$ by Corollary 4.1. Taking $S = X \in \sigma$ and $N \in \mathcal{I}$ arbitrary, $X \setminus N \in \sigma^*(\mathcal{I}) = \tau$ is open, so N is closed; by heredity every subset of N is closed, so N carries the discrete subspace topology and is closed. Conversely, if $\sigma = \tau$ and every $N \in \mathcal{I}$ is closed, each basic set $S \setminus N$ with $S \in \tau$ is τ-open, so $\sigma^*(\mathcal{I}) \subseteq \tau$; the reverse inclusion is automatic. □

7. Sharpness and an exact characterization of repairability

Theorem 4.1 bounds a repairable coarsening from below by τ_θ. This section shows that the bound is attained, and that once codensity is dropped, repairability admits a complete and checkable description in terms of the open core of the ideal. The witness for sharpness is the K-topology on the real line, where the repaired coarsening is exactly τ_θ (Figure 2).

Example 8. *Let $X = \mathbb{R}$, let σ be the Euclidean topology, let $K = \{1/n : n \in \mathbb{N}\}$, and let τ be the K-*topology *$\mathbb{R}_K$ on $\mathbb{R}$, also called the* deleted sequence topology *[28], generated by the base $\{(a, b)\} \cup \{(a, b) \setminus K\}$. It is classical that $\mathbb{R}_K$ is Hausdorff but not regular: K is τ-closed and cannot be separated from 0 by disjoint τ-open sets [29, §31]; in the terminology of Steen and Seebach [28] this is the deleted sequence topology on $\mathbb{R}$. Let $\mathcal{I} = \mathcal{I}_{\text{ctb}}$ be the σ-ideal of countable subsets of $\mathbb{R}$. Every nonempty τ-open set contains a set $(a, b) \setminus K$, which is uncountable, so $\mathcal{I}$ is τ-codense. Yet every basic τ-open set has the form $S \setminus N$ with $S \in \sigma$ and $N \in \mathcal{I}$, namely $(a, b) = (a, b) \setminus \emptyset$ and $(a, b) \setminus K$ with K countable. Hence*

$$\tau \subseteq \sigma^*(\mathcal{I}), \qquad \sigma \subsetneq \tau,$$

so the conclusion of Corollary 4.1 fails and regularity cannot be relaxed to T_2.

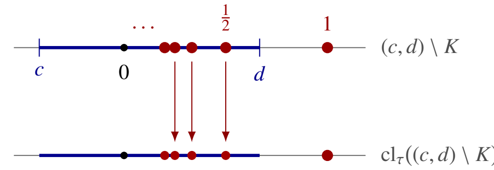

Figure 2: The K-topology $\mathbb{R}_K$ of Example 8, with $K = \{1/n\}$. A basic neighborhood of 0 is $(c, d) \setminus K$ (upper line, points of K hollow); its τ-closure (lower line) contains every $1/n \in (c, d)$ (arrows), because each such point has only genuine-interval neighborhoods, which meet $(c, d) \setminus K$. Hence a θ-open set containing 0 contains a full interval, so $(\mathbb{R}_K)_\theta$ is the Euclidean topology and the bound of Theorem 4.1 is attained.

Proposition 8.1 (The bound of Theorem 4.1 is attained)**.** *In the notation of Example 8, $(\mathbb{R}_K)_\theta$ is the Euclidean topology. Hence $\sigma = \tau_\theta$ there, and the inclusion $\tau_\theta \subseteq \sigma$ of Theorem 4.1 is an equality.*

Proof. Write $\tau = \mathbb{R}_K$ and let σ denote the Euclidean topology.

$\sigma \subseteq \tau_\theta$: let $x \in (a, b)$ and choose $\varepsilon > 0$ with $[x - \varepsilon, x + \varepsilon] \subseteq (a, b)$. Then $U = (x - \varepsilon, x + \varepsilon) \in \tau$, and since τ is finer than σ we have $\mathrm{cl}_\tau(U) \subseteq \mathrm{cl}_\sigma(U) = [x - \varepsilon, x + \varepsilon] \subseteq (a, b)$. So (a, b) is θ-open, and every Euclidean open set is a union of such intervals.

$\tau_\theta \subseteq \sigma$: let A be θ-open and $x \in A$, with $U \in \tau$ satisfying $x \in U \subseteq \mathrm{cl}_\tau(U) \subseteq A$. Three cases arise. If $x \notin K \cup \{0\}$, then $K \cup \{0\}$ is Euclidean closed, so x has a Euclidean neighborhood disjoint from $K \cup \{0\}$; intersecting it with a basic τ-set inside U gives a Euclidean neighborhood of x contained in A. If $x = 1/n \in K$, then every basic τ-set containing x is a genuine interval, since the sets $(a, b) \setminus K$ omit $1/n$; hence there is an interval (a, b) with $x \in (a, b) \subseteq U \subseteq A$, and x is a Euclidean interior point of A. If $x = 0$, then U contains a basic set $(c, d) \setminus K$ with $c < 0 < d$. Let $1/n \in (c, d)$. Every basic τ-neighborhood of $1/n$ is an interval $(a, b) \ni 1/n$, since sets of the form $(a, b) \setminus K$ omit $1/n$; such an interval meets $(c, d) \setminus K$, so $1/n \in \mathrm{cl}_\tau(U) \subseteq A$. Hence A contains $(c, d) \setminus K$ together with every point of $K \cap (c, d)$, that is $A \supseteq (c, d)$, and 0 is a Euclidean interior point of A. □

Remark 9. *Refinement by a codense ideal typically destroys regularity, so Corollary 4.1 cannot be applied iteratively by passing from τ to $\sigma^*(\mathcal{I})$: for σ Euclidean on $\mathbb{R}$, the topology*

$\sigma^*(\mathcal{I}_{\mathrm{ctb}})$, whose open sets are the differences $U \setminus A$ with U Euclidean open and A countable, is the countable complement extension topology, which is completely Hausdorff and Urysohn but not regular [28]. Regularity fails there because the countable set A is closed in $\sigma^*(\mathcal{I}_{\mathrm{ctb}})$ while being σ-dense, so a point outside A cannot be separated from A by disjoint $\sigma^*(\mathcal{I}_{\mathrm{ctb}})$-open sets.

Theorem 9.1 (Open core theorem)**.** *Let $\sigma \subseteq \tau$ be topologies on X, let $\mathcal{I}$ be an arbitrary ideal on X, and write $O = O_\tau(\mathcal{I})$. If $\tau \subseteq \sigma^*(\mathcal{I})$, then:*

(i) for every $p \in X$ and every θ-open W containing p there is $S \in \sigma$ with $p \in S \subseteq W \cup O$;

(ii) σ and τ_θ induce the same subspace topology on $X \setminus O$. If τ is regular, σ and τ induce the same subspace topology on $X \setminus O$.

Proof. (i) Fix $p \in W \in \tau_\theta$ and choose $U \in \tau$ with $p \in U \subseteq \overline{U} \subseteq W$. By Lemma 4.1 there is $S \in \sigma$ with $p \in S$ and $S \setminus U \in \mathcal{I}$. By Lemma 4.2, $G = S \cap (X \setminus \overline{U})$ is τ-open and contained in $S \setminus U$, hence $G \in \mathcal{I}$ by heredity, hence $G \subseteq O$ by Definition 2.4. Therefore

$$S = (S \cap \overline{U}) \cup G \subseteq \overline{U} \cup O \subseteq W \cup O.$$

(ii) Write $Y = X \setminus O$. Since $\sigma \subseteq \tau$ and $\tau_\theta \subseteq \tau$, it suffices to prove $\tau_\theta|_Y \subseteq \sigma|_Y \subseteq \tau|_Y$, the second inclusion being immediate. For the first, let $p \in Y$ and $W \in \tau_\theta$ with $p \in W$. By (i) there is $S \in \sigma$ with $p \in S \subseteq W \cup O$, hence $p \in S \cap Y \subseteq W \cap Y$ with $S \cap Y \in \sigma|_Y$. The regular case follows from Lemma 2.1, which gives $\tau_\theta = \tau$ and hence $\tau|_Y \subseteq \sigma|_Y \subseteq \tau|_Y$. □

The next theorem characterizes repairability completely. Its substantive assertion is that the open core is necessarily σ-open, a constraint that does not appear in Theorem 9.1 and on which the quantitative theory of Section 10 depends.

Theorem 9.2 (Exact characterization of repairable cores)**.** *Let (X, τ) be regular, let $\sigma \subseteq \tau$ be a topology on X, and let $O \subseteq X$. The following are equivalent.*

(a) There is an ideal $\mathcal{I}$ on X with $O_\tau(\mathcal{I}) = O$ and $\tau \subseteq \sigma^*(\mathcal{I})$.

(b) $O \in \tau$, $O \in \sigma$, and σ and τ induce the same subspace topology on $X \setminus O$.

When these hold, the ideal $\mathcal{P}(O)$ of all subsets of O satisfies $O_\tau(\mathcal{P}(O)) = O$ and $\tau \subseteq \sigma^*(\mathcal{P}(O))$.

Proof. (a) ⇒ (b). That $O \in \tau$ is immediate from Definition 2.4, and the agreement of subspace topologies on $X \setminus O$ is Theorem 9.1(ii). It remains to prove $O \in \sigma$. Let $p \in O$. Since $O \in \tau$ and τ is regular there is $W \in \tau$ with

$$p \in W \subseteq \overline{W} \subseteq O.$$

By Lemma 4.1 applied to (p, W) there is $S \in \sigma$ with $p \in S$ and $S \setminus W \in \mathcal{I}$. By Lemma 4.2, $G = S \cap (X \setminus \overline{W})$ is τ-open and contained in $S \setminus W$, hence $G \in \mathcal{I}$ and so $G \subseteq O$. Therefore

$$S = (S \cap \overline{W}) \cup G \subseteq \overline{W} \cup O = O.$$

Thus every $p \in O$ lies in a member of σ contained in O, so $O \in \sigma$.

(b) ⇒ (a). Put $\mathcal{I} = \mathcal{P}(O)$, an ideal with $O_\tau(\mathcal{I}) = O$ because $O \in \tau$ and every τ-open member of $\mathcal{I}$ is a subset of O. We verify condition (ii) of Lemma 4.1. Let $p \in W \in \tau$. If $p \in O$, take $S = O \in \sigma$; then $S \setminus W \subseteq O$, so $S \setminus W \in \mathcal{I}$. If $p \notin O$, put $Y = X \setminus O$; by the agreement of subspace topologies there is $S \in \sigma$ with $p \in S$ and $S \cap Y \subseteq W \cap Y$. If $z \in S \setminus W$ then $z \notin W$, so $z \notin S \cap Y$, so $z \notin Y$, that is $z \in O$; hence $S \setminus W \subseteq O$ and $S \setminus W \in \mathcal{I}$. By Lemma 4.1, $\tau \subseteq \sigma^*(\mathcal{I})$. □

Corollary 9.1. *Corollary 4.1 is the case $O = \emptyset$ of Theorem 9.2. More generally, if $\sigma \subsetneq \tau$ with τ regular, then every ideal $\mathcal{I}$ with $\tau \subseteq \sigma^*(\mathcal{I})$ has $O_\tau(\mathcal{I}) \neq \emptyset$, and $O_\tau(\mathcal{I})$ is a nonempty σ-open set.*

Proof. If $O = \emptyset$ then (b) reads $\sigma|_X = \tau|_X$, that is $\sigma = \tau$. The general statement follows from Theorem 9.2 applied to $O = O_\tau(\mathcal{I})$. □

Proposition 9.1. *Let (X, τ) be hereditarily Lindelöf and let $\mathcal{I}$ be a σ-ideal on X. Then $O_\tau(\mathcal{I}) \in \mathcal{I}$; consequently $\mathcal{I}$ is τ-codense if and only if $O_\tau(\mathcal{I}) = \emptyset$.*

Proof. By Definition 2.4, each $x \in O_\tau(\mathcal{I})$ lies in some $G_x \in \tau$ with $G_x \subseteq O_\tau(\mathcal{I})$ and $G_x \in \mathcal{I}$. The family $\{G_x\}$ is an open cover of the subspace $O_\tau(\mathcal{I})$, which is Lindelöf by hereditary Lindelöfness of (X, τ), so a countable subfamily $\{G_{x_n}\}$ covers it. Since each $G_{x_n} \subseteq O_\tau(\mathcal{I})$, their union is exactly $O_\tau(\mathcal{I})$, and as $\mathcal{I}$ is a σ-ideal it lies in $\mathcal{I}$. □

Every second countable space, in particular every spacetime with its manifold topology, is hereditarily Lindelöf, so Proposition 9.1 applies to $\mathcal{I}_\mu$, $\mathcal{I}_{\mathrm{mgr}}$ and $\mathcal{I}_{\mathrm{ctb}}$ below.

10. Application: the cost of reconstructing a spacetime topology

We now specialize to a spacetime (M, g), with $\tau = \tau_M$ and σ one of the causally defined coarse topologies of Section 2. The manifold topology is metrizable, hence regular, so $(\tau_M)_\theta = \tau_M$ and the results of Sections 3 and 7 apply in their regular form. The analysis proceeds in two parts. The first is negative: no codense ideal alters the strongly causal set, so deleting a set of zero spacetime volume, or a meager set, from every chronological diamond leaves the topology unrepaired. The second is quantitative, identifying both the regions that may be discarded and the volume this costs.

Throughout, $\mathcal{A}$ is the Alexandrov topology of (M, g), $\mathcal{V}$ its chronology-violating set and μ_g its volume measure.

Definition 10.1. *Let $\mathcal{I}$ be an ideal on M and $\sigma \subseteq \tau_M$ a topology. We say (M, g) is $\mathcal{I}$-strongly causal relative to σ if $\tau_M \subseteq \sigma^*(\mathcal{I})$, and $\mathcal{I}$-strongly causal at p relative to σ if $\sigma^*(\mathcal{I})$ contains a τ_M-neighborhood base at p. When $\sigma = \mathcal{A}$ the qualifier is omitted, and $\mathrm{SC}_{\mathcal{I}}(M)$ denotes the set of points at which (M, g) is $\mathcal{I}$-strongly causal.*

Definition 10.1 formalizes the proposal that strong causality be allowed to fail on a negligible set. It demands that every manifold-open set be reconstructible from chronological diamonds after deletion of negligible sets, this is precisely the information available to an observer who knows the timelike relations and is insensitive to $\mathcal{I}$-small sets. Its relation to the pointwise formulation is determined in Section 12. For $\mathcal{I} = \{\emptyset\}$ it reduces, by Theorem 2.1, to strong causality; for $\mathcal{I} = \mathcal{P}(M)$ it is vacuous, since $\mathcal{A}^*(\mathcal{P}(M))$ is discrete.

Theorem 10.1 (Rigidity of strong causality)**.** *Let (M, g) be a spacetime and $\mathcal{I}$ any τ_M-codense ideal on M. Then*

$$\mathrm{SC}_{\mathcal{I}}(M) = \mathrm{SC}(M).$$

In particular $\tau_M \subseteq \mathcal{A}^*(\mathcal{I})$ if and only if (M, g) is strongly causal, and in that case $\mathcal{A}^*(\mathcal{I}) = \tau^*_M(\mathcal{I})$.

Proof. τ_M is metrizable, hence regular, and $\mathcal{A} \subseteq \tau_M$ by Eq. (2). If $p \in \mathrm{SC}(M)$, Theorem 2.1 makes the diamonds containing p a τ_M-neighborhood base at p, and $\mathcal{A} \subseteq \mathcal{A}^*(\mathcal{I})$ gives $p \in \mathrm{SC}_{\mathcal{I}}(M)$. Conversely, if $p \in \mathrm{SC}_{\mathcal{I}}(M)$ then Theorem 5.1, applied with $\tau = \tau_M$, $\sigma = \mathcal{A}$ and the codense ideal $\mathcal{I}$, shows that $\mathcal{A}$ contains a τ_M-neighborhood base at p, so $p \in \mathrm{SC}(M)$ by Theorem 2.1. The global statement follows, and $\mathcal{A}^*(\mathcal{I}) = \tau^*_M(\mathcal{I})$ from Corollary 4.1. □

Corollary 10.1. *No spacetime failing to be strongly causal at p becomes $\mathcal{I}$-strongly causal at p for $\mathcal{I}$ any of $\mathcal{I}_{\mu_g}$, $\mathcal{I}_{\mathrm{mgr}}$, $\mathcal{I}_{\mathrm{nwd}}$, $\mathcal{I}_{\mathrm{cd}}$, $\mathcal{I}_{\mathrm{ctb}}$ or $\mathcal{I}_{\mathrm{fin}}$; more generally, for any ideal containing no nonempty open set.*

Proof. Each listed ideal is τ_M-codense, since a nonempty open subset of a manifold has positive μ_g-measure, is nonmeager, is somewhere dense, is not discrete, and is uncountable. Apply Theorem 10.1. □

Theorem 5.2 yields a second, logically independent proof of the global statement, using no regularity and only $\mathcal{A}$-codensity.

Theorem 10.2. *Let (M, g) be a spacetime and $\mathcal{I}$ an ideal on M with $\mathcal{I} \cap \mathcal{A} = \{\emptyset\}$. Then $\mathcal{A}^*(\mathcal{I})$ is Hausdorff if and only if (M, g) is strongly causal.*

Proof. By Theorem 5.2, $\mathcal{A}^*(\mathcal{I})$ is Hausdorff if and only if $\mathcal{A}$ is Hausdorff, which by Theorem 2.1 holds if and only if (M, g) is strongly causal. □

Proposition 10.1. *Let (M, g) be any spacetime and $\mathcal{I}$ an ideal containing all singletons, for instance $\mathcal{I}_{\mathrm{fin}}$, $\mathcal{I}_{\mathrm{ctb}}$ or $\mathcal{I}_{\mu_g}$. Then $\mathcal{A}^*(\mathcal{I})$ is T_1. In the Gödel universe [25], where $I^+(p) = I^-(p) = M$ for every p and hence $\mathcal{A} = \{\emptyset, M\}$ is indiscrete, $\mathcal{A}^*(\mathcal{I}_{\mathrm{fin}})$ is the cofinite topology: T_1 and not Hausdorff, while $\mathcal{A}$ is not even T_0.*

Proof. For $x \neq y$ the set $M \setminus \{y\} = M \setminus N$ with $M \in \mathcal{A}$ and $N = \{y\} \in \mathcal{I}$ lies in $\mathcal{A}^*(\mathcal{I})$, contains x and omits y; so $\mathcal{A}^*(\mathcal{I})$ is T_1. In the Gödel universe $\beta(\mathcal{A}, \mathcal{I}_{\mathrm{fin}}) = \{\emptyset\} \cup \{M \setminus N : N \text{ finite}\}$ generates the cofinite topology, which is T_1 and, M being infinite, not Hausdorff, in accordance with Theorem 10.2. □

Corollary 10.2. *For every spacetime (M, g) and every τ_M-codense ideal $\mathcal{I}$,*

$$\tau_M \not\subseteq (\tau^{\uparrow})^*(\mathcal{I}) \qquad \text{and} \qquad \tau_M \not\subseteq (\tau^{\downarrow})^*(\mathcal{I}).$$

Proof. Call $S \subseteq M$ a *future set* if $q \in S$ and $q \leq r$ imply $r \in S$. Each $I^+(p)$ is a future set by Eq. (1), and future sets are closed under arbitrary unions and finite intersections, so every member of $\tau^{\uparrow}$ is a future set.

Suppose $\tau^{\uparrow} = \tau_M$ and fix $p \in M$. Every $U \in \tau_M$ containing p is then a future set, so $J^+(p) \subseteq U$; since τ_M is T_1,

$$J^+(p) \subseteq \bigcap \{U \in \tau_M : p \in U\} = \{p\}.$$

But M carries a global future-directed timelike vector field, whose integral curve γ through p has $\gamma(0) = p$ and $\dot{\gamma} \neq 0$, so $\gamma(s) \neq p$ for some small $s > 0$ while $\gamma(s) \in J^+(p)$. Hence $\tau^{\uparrow} \subsetneq \tau_M$, and Corollary 4.1 with $\sigma = \tau^{\uparrow}$ gives the claim. The past case is dual. □

Remark 11. *Corollary 10.2 is the ideal-theoretic counterpart of the classical requirement that recovery of the manifold topology from causal data use both futures and pasts [1, 3, 4]: one-sided chronological information remains insufficient after arbitrary negligible surgery. By contrast the join $\mathcal{A} = \tau^{\uparrow} \vee \tau^{\downarrow}$ is repairable precisely when it already equals τ_M.*

11.1. Discarded regions and the repair cost

The negative results above raise a quantitative question: if the manifold topology is to be recovered from chronology after discarding something, what must be discarded, and how large must it be? Theorem 9.2 answers the first question completely once its two conditions are read causally.

Definition 11.1. *Let $Y \subseteq M$. We say (M, g) is* strongly causal along Y *if for every $p \in Y$ and every $W \in \tau_M$ with $p \in W$ there are $q \ll p \ll r$ such that*

$$p \in I(q, r) \quad \text{and} \quad I(q, r) \cap Y \subseteq W \cap Y.$$

Equivalently, $\mathcal{A}$ and τ_M induce the same subspace topology on Y.

The equivalence holds because $\mathcal{A} \subseteq \tau_M$ gives $\mathcal{A}|_Y \subseteq \tau_M|_Y$ always, while the displayed condition is exactly $\tau_M|_Y \subseteq \mathcal{A}|_Y$. The condition permits causal curves to leave a neighborhood and return, provided every such excursion is invisible from Y.

Definition 11.2. *A* discarded region *for (M, g) is a set $O \subseteq M$ that is the open core $O_{\tau_M}(\mathcal{I})$ of some ideal $\mathcal{I}$ on M with $\tau_M \subseteq \mathcal{A}^*(\mathcal{I})$.*

Theorem 9.2, applied with $\tau = \tau_M$ and $\sigma = \mathcal{A}$, now translates into a purely causal statement.

Theorem 11.1 (Structure of discarded regions)**.** *Let (M, g) be a spacetime and $O \subseteq M$. Then O is a discarded region if and only if*

(i) O is a union of chronological diamonds, and

(ii) (M, g) is strongly causal along $M \setminus O$.

In that case the ideal $\mathcal{P}(O)$ of all subsets of O satisfies $\tau_M \subseteq \mathcal{A}^*(\mathcal{P}(O))$.

Proof. Since τ_M is regular, Theorem 9.2 applies with $\tau = \tau_M$ and $\sigma = \mathcal{A}$; its condition (b) reads: $O \in \tau_M$, $O \in \mathcal{A}$, and $\mathcal{A}$ and τ_M agree on $M \setminus O$. Now $O \in \mathcal{A}$ means exactly that O is a union of basic $\mathcal{A}$-open sets, that is of chronological diamonds, and this already implies $O \in \tau_M$ by Eq. (2); and the agreement of subspace topologies is Definition 11.1. The final assertion is the last clause of Theorem 9.2. □

Condition (i) is the substantive constraint, and it is the causal reading of the forced σ-openness in Theorem 9.2. A discarded region cannot be an arbitrarily thin set surrounding the pathology: it must be causally saturated, in the sense that each of its points lies in a chronological diamond entirely contained in it. Since chronological diamonds have nonempty interior, this bounds below the size of any repair and makes the following definition meaningful.

Definition 11.3. *The* repair cost *of a spacetime (M, g) is*

$$c(M) = \inf\{\mu_g(O) : O \text{ a discarded region}\},$$

a quantity lying in $[0, \mu_g(M)] \subseteq [0, +\infty]$; the value $\mu_g(M) = +\infty$ is permitted, since (M, g) is not assumed to have finite volume.

Corollary 11.1. *The infimum in Definition 11.3 is over a nonempty family, so $c(M)$ is well defined and $c(M) \leq \mu_g(M)$. Moreover $c(M) = 0$ with the infimum attained if and only if (M, g) is strongly causal; and if (M, g) is not strongly causal, then every discarded region is a nonempty open set and hence has strictly positive volume.*

Proof. $O = M$ is a discarded region, being $\mathcal{A}$-open with $M \setminus O = \emptyset$; alternatively $\mathcal{A}^*(\mathcal{P}(M))$ is discrete. The infimum is attained at 0 exactly when $\emptyset$ is a discarded region, which by Corollary 9.1 happens exactly when $\mathcal{A} = \tau_M$, that is, by Theorem 2.1, exactly when (M, g) is strongly causal. If (M, g) is not strongly causal, Corollary 9.1 gives $O \neq \emptyset$, and a nonempty open subset of a manifold has positive volume. □

Theorem 11.2 (Volume cost of chronology violation)**.** *Let (M, g) be a spacetime and O a discarded region. Then*

$$\mu_g(\mathcal{V} \setminus O) = 0,$$

and consequently

$$\mu_g(O) \geq \mu_g(\mathcal{V}) \qquad \text{and} \qquad c(M) \geq \mu_g(\mathcal{V}).$$

Proof. By Theorem 11.1, $\mathcal{A}$ and τ_M induce the same subspace topology on $Y = M \setminus O$.

Let $p \in \mathcal{V} \cap Y$ and $S \in \mathcal{A}$ with $p \in S$. Choosing a basic diamond with $p \in I(q, r) \subseteq S$, we have $q \ll p$, $p \ll p$ (as $p \in \mathcal{V}$) and $p \ll r$, so every z with $p \ll z \ll p$ satisfies $q \ll z \ll r$ by transitivity of $\ll$; therefore

$$V_p \subseteq I(q, r) \subseteq S, \tag{3}$$

for every $\mathcal{A}$-open S containing p. Fix a countable τ_M-neighborhood base $\{W_n\}$ at p with $\bigcap_n W_n = \{p\}$, available since τ_M is metrizable. By the agreement of subspace topologies there are $S_n \in \mathcal{A}$ with $p \in S_n$ and $S_n \cap Y \subseteq W_n$; with Eq. (3),

$$V_p \cap Y \subseteq \bigcap_n (S_n \cap Y) \subseteq \bigcap_n W_n = \{p\}.$$

Hence each chronology-violating component contains at most one point of $\mathcal{V} \cap Y$: if $p, p' \in \mathcal{V} \cap Y$ lie in a common component then $V_p = V_{p'}$ and $p' \in V_p \cap Y \subseteq \{p\}$, so $p' = p$. The components are a pairwise disjoint family of nonempty open subsets of the second countable space M, hence countable in number, so $\mathcal{V} \setminus O$ is countable and μ_g-null. Therefore $\mu_g(\mathcal{V}) = \mu_g(\mathcal{V} \cap O) \leq \mu_g(O)$, and taking the infimum over discarded regions gives $c(M) \geq \mu_g(\mathcal{V})$. □

Theorem 11.2 admits a reading in terms of the chronology protection conjecture [27], which asserts that the laws of physics prevent the formation of a chronology-violating region. The theorem is independent of that conjecture and of any field equation or energy condition: it states that if such a region is nevertheless present, then recovering the manifold topology from chronology requires discarding a set of volume at least $\mu_g(\mathcal{V})$. Chronology violation is therefore an obstruction to topological reconstruction independent of its dynamical status.

Proposition 11.1 (Inert ideals in the causal setting)**.** *Let (M, g) be a spacetime and $\mathcal{I}$ a τ_M-codense ideal. Then $\mathcal{A}^*(\mathcal{I}) = \tau_M$ if and only if (M, g) is strongly causal and every member of $\mathcal{I}$ is a closed discrete, equivalently locally finite, subset of M.*

Proof. By Proposition 6.1 with $\tau = \tau_M$ and $\sigma = \mathcal{A}$, the equality holds if and only if $\mathcal{A} = \tau_M$ and every member of $\mathcal{I}$ is τ_M-closed, the latter forcing closed discreteness by heredity; Theorem 2.1 identifies $\mathcal{A} = \tau_M$ with strong causality. A subset A of a topological space is *locally finite* if every point has a neighborhood meeting A in a finite set; in a locally compact metrizable space, and in particular in M, this is equivalent to being closed and discrete, since such a set is exactly one meeting every compact set finitely. □

12. A Lorentzian cylinder: vanishing but unattained repair cost

This section computes the repair cost for an explicit spacetime and distinguishes two candidate weakenings of strong causality. The example is a Lorentzian metric of unit determinant on a cylinder whose causal cone degenerates on a single circle. That circle is a closed null geodesic, so causality fails there, while no closed timelike curve exists anywhere. Strong causality fails only on a compact null set, which makes the example discriminating for the two weakenings (Figure 3).

Example 13. *Let $M = \mathbb{R} \times S^1$ with global coordinates (t, ϕ), ϕ periodic of period 2π, and*

$$g = 2\, dt\, d\phi + t^2\, d\phi^2, \qquad \text{that is} \qquad (g_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & t^2 \end{pmatrix}.$$

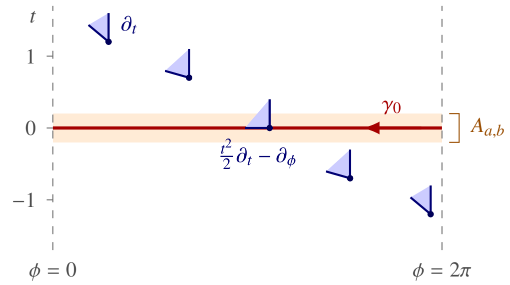

Figure 3: The fundamental domain of the cylinder of Example 13; ϕ is horizontal, t is vertical, and the dashed edges are identified. Future cones are drawn at $t = 1.2, 0.7, 0, -0.7, -1.2$ with arms of equal length, spanned by the null directions ∂_t and $\frac{t^2}{2}\partial_t - \partial_\phi$ of part (c). The second is horizontal exactly at $t = 0$, where it is tangent to the closed null geodesic γ_0, and steepens as $|t|$ grows, so that away from γ_0 the cones force t to increase. The shaded band is an annulus $A_{a,b}$ as in Eq. (5).

(a) Signature and volume. $\det(g_{\alpha\beta}) = -1$ *everywhere, so g is a smooth Lorentzian metric and $\mu_g = dt\,d\phi$ is two-dimensional Lebesgue measure in these coordinates.*

(b) Time orientation. *$X = (\frac{t^2}{2} + 1)\partial_t - \partial_\phi$ satisfies $g(X, X) = -2$ everywhere, so X is globally timelike and (M, g) is time-oriented by declaring X future-directed.*

(c) The causal cone. *For $v = v^t\partial_t + v^\phi\partial_\phi$,*

$$g(v, v) = v^\phi(t^2 v^\phi + 2v^t), \qquad g(v, X) = -v^t + v^\phi\Big(1 - \tfrac{t^2}{2}\Big).$$

Let $v \neq 0$ be causal, so $v^\phi(t^2 v^\phi + 2v^t) \leq 0$. If $v^\phi = 0$ then $g(v, X) = -v^t$, so v is future-directed exactly when $v^t > 0$. If $v^\phi < 0$ then $v^t \geq -t^2 v^\phi/2 > 0$ and $g(v, X) \leq v^\phi < 0$, so v is future-directed. If $v^\phi > 0$ then $v^t \leq -t^2 v^\phi/2 \leq 0$ and $g(v, X) \geq v^\phi > 0$, so v is past-directed. Hence every future-directed causal vector satisfies

$$v^\phi \leq 0 \qquad \text{and} \qquad v^t \;\geq\; \tfrac{t^2}{2}\,|v^\phi| \;\geq\; 0, \tag{4}$$

and ∂_t is null and future-directed.

(d) Chronology holds; causality fails exactly on $\{t = 0\}$. *By Eq. (4) the coordinate t is nondecreasing along every future-directed causal curve. A closed such curve has $\Delta t = 0$, hence $\dot t \equiv 0$, and then Eq. (4) forces $t^2|\dot\phi| \equiv 0$; since $\dot t = \dot\phi = 0$ is impossible for a curve, $t \equiv 0$. The circle $\gamma_0 = \{t = 0\}$, traversed in the direction $\dot\phi < 0$, satisfies $g(\dot\gamma_0, \dot\gamma_0) = 0$, so it is a closed null curve. It is moreover an affinely parametrized geodesic: from $(g^{\alpha\beta}) = \left(\begin{smallmatrix} -t^2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ one computes*

$$\Gamma^t_{\phi\phi} = t^3, \qquad \Gamma^\phi_{\phi\phi} = -t,$$

all remaining Christoffel symbols being $\Gamma^t_{t\phi} = \Gamma^t_{tt} = \Gamma^\phi_{t\phi} = \Gamma^\phi_{tt} = 0$. Along $s \mapsto (0, \phi_0 - s)$ the velocity is $\dot\gamma_0 = -\partial_\phi$ with $\ddot\gamma_0 = 0$, and both $\Gamma^t_{\phi\phi}$ and $\Gamma^\phi_{\phi\phi}$ vanish on $\{t = 0\}$, so the geodesic equation $\ddot x^\alpha + \Gamma^\alpha_{\beta\gamma}\dot x^\beta \dot x^\gamma = 0$ holds. Related geodesic computations by differential-form methods appear in [30]. Hence $\mathcal{V} = \emptyset$, and the causality-violating set is exactly γ_0.

(e) Strong causality fails exactly on $\{t = 0\}$. *Let $p_0 = (t_0, \phi_0)$ with $t_0 \neq 0$, fix $\eta \in (0, \pi]$ and*

$$0 < \delta < \min\Big\{\tfrac{|t_0|}{2},\ \tfrac{\eta t_0^2}{32}\Big\},$$

and set

$$U = \{(t, \phi) : |t - t_0| < \delta,\ d_{S^1}(\phi, \phi_0) < \eta/2\},$$

where d_{S^1} denotes arclength distance on S^1; since $\eta \leq \pi$, the ϕ-extent of U is a single arc of length $\eta < 2\pi$. Below, $\Delta\phi$ denotes the increment of a continuous lift of ϕ to $\mathbb{R}$ along the curve, so that $|\Delta\phi|$ is the total angular variation, wrap-around included. Suppose a future-directed causal curve starts in U, leaves U and returns. Since t is nondecreasing and its values at the initial and final points lie in $(t_0 - \delta, t_0 + \delta)$, monotonicity confines t to that interval throughout the intervening segment, so

$$|t| \;\geq\; |t_0| - \delta \;\geq\; \tfrac{|t_0|}{2}, \qquad \Delta t < 2\delta.$$

Hence $\dot t \geq \frac{t_0^2}{8}|\dot\phi|$ by Eq. (4), and the total ϕ-variation satisfies

$$|\Delta\phi| \;\leq\; \frac{8}{t_0^2}\,\Delta t \;<\; \frac{16\delta}{t_0^2} \;<\; \frac{\eta}{2} \;\leq\; \eta.$$

But ϕ is nonincreasing by Eq. (4), so exiting an arc of length η and re-entering it requires $|\Delta\phi| \geq 2\pi - \eta \geq \eta$, a contradiction. As η, δ may be taken arbitrarily small, strong causality holds at p_0. With (d) this gives

$$\mathrm{SC}(M) = \{t \neq 0\}, \qquad M \setminus \mathrm{SC}(M) = \gamma_0,$$

a compact, μ_g-null, nowhere dense, closed subset of M, in accordance with the structure result recalled after Theorem 2.1.

(f) Every diamond through a point of γ_0 contains an annulus. *Let $p = (0, \phi_0)$ and $q \ll p \ll r$; by Eq. (4), $q = (t_q, \phi_q)$ with $t_q < 0$ and $r = (t_r, \phi_r)$ with $t_r > 0$. For $\varepsilon > 0$ let γ_ε be the curve through q with*

$$\dot\phi = -1, \qquad \dot t = \tfrac{t^2}{2} + \varepsilon,$$

future-directed timelike since $g(\dot\gamma_\varepsilon, \dot\gamma_\varepsilon) = (-1)(-t^2 + t^2 + 2\varepsilon) = -2\varepsilon < 0$ and $\dot\phi < 0$. Fix $a \in (t_q, 0)$. Along γ_ε, $d\phi/dt = -(\frac{t^2}{2} + \varepsilon)^{-1}$, so the ϕ-variation accumulated while t runs from t_q to a is

$$\Phi(\varepsilon, a) = \int_{t_q}^{a} \frac{dt}{\frac{t^2}{2} + \varepsilon} \;\xrightarrow[\varepsilon\downarrow 0]{}\; \int_{t_q}^{a} \frac{2\,dt}{t^2} = \frac{2}{|a|} - \frac{2}{|t_q|},$$

by monotone convergence. Choosing $|a|$ small enough that $2/|a| - 2/|t_q| > 2\pi$, and then ε small enough that $\Phi(\varepsilon, a) > 2\pi$, the curve γ_ε meets every $\phi^* \in S^1$ at some parameter with $t \leq a$. Fix ϕ^* and let $z = (t_z, \phi^*)$ be such a point, so $q \ll z$ and $t_z \leq a$. The curve $s \mapsto (t_z + s, \phi^*)$ has tangent ∂_t, future-directed null by (c), so $z \leq (t, \phi^*)$ for all $t \geq t_z$, and Eq. (1) gives $q \ll (t, \phi^*)$. As ϕ^* was arbitrary, $I^+(q) \supseteq \{(t, \phi) : t \geq a\}$; dually there is $b \in (0, t_r)$ with $I^-(r) \supseteq \{(t, \phi) : t \leq b\}$. Hence

$$A_{a,b} := \{(t, \phi) : a < t < b\} \subseteq I(q, r), \qquad a < 0 < b. \tag{5}$$

Proposition 13.1. *For the spacetime of Example 13 and every τ_M-codense ideal $\mathcal{I}$ on M:*

(i) strong causality holds at every point outside the compact, μ_g-null, nowhere dense set γ_0;

(ii) $\mathrm{SC}_{\mathcal{I}}(M) = M \setminus \gamma_0 \neq M$, so (M, g) is not $\mathcal{I}$-strongly causal for any such $\mathcal{I}$;

(iii) $\mathcal{A}^*(\mathcal{I})$ is not Hausdorff.

Proof. (i) is Example 13(e). For (ii), Theorem 10.1 gives $\mathrm{SC}_{\mathcal{I}}(M) = \mathrm{SC}(M) = M \setminus \gamma_0$. For (iii), (M, g) is not strongly causal, so $\mathcal{A}$ is not Hausdorff by Theorem 2.1; a τ_M-codense ideal is a fortiori $\mathcal{A}$-codense, so Theorem 10.2 applies. □

Proposition 13.2 (Vanishing but unattained repair cost)**.** *For the spacetime of Example 13, the sets*

$$O_\varepsilon = \{(t, \phi) : |t| < \varepsilon\}, \qquad \varepsilon > 0,$$

are discarded regions with $\mu_g(O_\varepsilon) = 4\pi\varepsilon$. Consequently

$$c(M) = 0,$$

while every discarded region has strictly positive volume; the infimum in Definition 11.3 is therefore not attained.

Proof. We verify the two conditions of Theorem 11.1 for O_ε.

(i) O_ε is a union of chronological diamonds. Let $p = (t_p, \phi_p) \in O_\varepsilon$, so $|t_p| < \varepsilon$. Choose points q and r on a past-directed and a future-directed timelike curve through p, respectively, close enough to p that their t-coordinates satisfy $-\varepsilon < t_q < t_p < t_r < \varepsilon$; this is possible since t varies continuously along such curves. Then $p \in I(q, r)$, and since t is nondecreasing along future-directed causal curves,

$$I(q, r) \subseteq \{(t, \phi) : t_q < t < t_r\} \subseteq O_\varepsilon.$$

(ii) (M, g) is strongly causal along $Y_\varepsilon = M \setminus O_\varepsilon = \{|t| \geq \varepsilon\}$. Every $p \in Y_\varepsilon$ has $t_p \neq 0$, so (M, g) is strongly causal at p by Example 13(e); by Theorem 2.1 the diamonds containing p form a τ_M-neighborhood base at p, so for $W \in \tau_M$ with $p \in W$ there are $q \ll p \ll r$ with $I(q, r) \subseteq W$ and a fortiori $I(q, r) \cap Y_\varepsilon \subseteq W$.

By Theorem 11.1, O_ε is a discarded region, and $\mu_g(O_\varepsilon) = \int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} dt\, d\phi = 4\pi\varepsilon$. Letting $\varepsilon \downarrow 0$ gives $c(M) = 0$. Finally (M, g) is not strongly causal, so by Corollary 11.1 every discarded region is nonempty and open, hence of positive volume; in particular no discarded region attains the value 0. □

Corollary 13.1 (Dichotomy)**.** *Let (M, g) be a spacetime that is not strongly causal.*

(i) If $\mathcal{V} \neq \emptyset$, then $c(M) \geq \mu_g(\mathcal{V}) > 0$: chronology violation carries an irreducible volume cost.

(ii) If $\mathcal{V} = \emptyset$, the cost may vanish: the cylinder of Example 13 is chronological, violates causality and strong causality, and has $c(M) = 0$.

In both cases the manifold topology cannot be recovered after discarding a set of zero volume, since by Corollary 11.1 every discarded region has positive volume.

Proof. (i) is Theorem 11.2 together with the openness of $\mathcal{V}$; (ii) is Proposition 13.2. The final statement is Corollary 11.1. □

Remark 14. *The obstruction in Proposition 13.1 is already visible in Eq. (5), without invoking Theorem 10.1. Let $p = (0, \phi_0)$ and let U be a coordinate ball about p of Euclidean radius $\rho < 1$. For any $q \ll p \ll r$ the excess $I(q, r) \setminus U$ contains the nonempty open set $A_{a,b} \setminus \overline{U}$, of measure at least $2\pi \min\{|a|, b\} - \pi\rho^2$, positive for small ρ. An open set is negligible for no codense ideal, so the criterion of Lemma 4.1 fails at p. The obstruction is not the size of the set on which strong causality fails, which is null, but the size of the region into which every diamond through p necessarily extends. Proposition 13.2 shows that this region can nonetheless be made to have arbitrarily small volume, which is why the cost vanishes without being attained.*

Remark 15. *The two candidate weakenings of strong causality are therefore strictly ordered. Since $\tau_M \subseteq \mathcal{A}^*(\mathcal{I})$ forces $\mathrm{SC}(M) = M$, the topological condition of Definition 10.1 implies the pointwise one, $M \setminus \mathrm{SC}(M) \in \mathcal{I}$, for every codense $\mathcal{I}$, and Proposition 13.1 shows the converse to fail. By Theorem 10.1 the topological condition is no weakening at all, being equivalent to strong causality itself.*

16. Concluding remarks

The results above isolate a structural feature of the relationship between a topology and any coarsening of it: the deficit of a coarse topology is an open-set-level phenomenon, invisible to and unrepairable by any notion of negligibility that assigns no weight to open sets. Theorem 4.1 shows that a codense ideal refinement can recover at most the non-θ part of the fine topology, so that regularity — the condition $\tau_\theta = \tau$ — is not an incidental hypothesis but exactly the circumstance in which the theorem yields total rigidity, and Proposition 8.1 shows the bound to be attained. Regularity cannot be relaxed to T_2; which non-regular topologies admit a repairable strict coarsening is not settled here. Theorem 5.2 operates on a different principle, that two basic sets with negligible intersection have empty intersection, and Example 6 shows that this principle does not extend to T_0 or T_1.

Theorem 9.2 converts these negative results into a quantitative theory by identifying the possible open cores exactly, the essential point being that the core is necessarily open in the coarse topology as well. In the relativistic setting this states that a discarded region must be a union of chronological diamonds, so that a causal pathology can be excised only along a causally saturated set. The resulting repair cost behaves asymmetrically in the two cases, though the two statements are not of the same logical strength. For chronology violation the bound $c(M) \geq \mu_g(\mathcal{V}) > 0$ of Theorem 11.2 is universal, holding in every spacetime with $\mathcal{V} \neq \emptyset$. In the complementary case the paper establishes only an existence statement: the cylinder of Example 13 is chronological, violates causality, and has $c(M) = 0$ unattained, so that causality violation without chronology violation can cost arbitrarily little while never costing nothing. Whether every such spacetime has vanishing repair cost is not determined here.

The two are treated differently in the physical literature: closed timelike curves are the object of chronology protection [26, 27], whereas closed null curves without them attract no comparable prohibition. The results here are consistent with that asymmetry, with the caveat just noted that only the chronology-violating side carries a universal lower bound. This bears on several current lines of work.

In causal set theory [31] one recovers a continuum geometry from an order together with a counting measure, and the results here indicate that the order-theoretic and measure-theoretic ingredients play strictly complementary roles: no amount of measure-theoretic discarding compensates for an order that fails to generate the topology. In the theory of spacetimes with metrics of low regularity [32, 33], where constructions frequently yield statements valid only up to sets of measure zero, Theorem 10.1 shows that a causal condition asserted almost everywhere does not upgrade to the corresponding topological statement, and Example 13 realizes the gap already with a smooth metric on a cylinder.

One limitation should be stated plainly. The topology recovered here is throughout the Alexandrov topology, the coarsest of the causal topologies, and the results say nothing about the finer causal topologies that encode differential or conformal structure, such as those of Zeeman [1] and of Hawking *et al.* [2]. Malament's theorem [3] shows that the chronological order determines the manifold topology in a distinguishing spacetime; the present results quantify, for the Alexandrov topology specifically, the extent to which that determination survives the deletion of negligible sets. The topology considered here is point-set rather than analytic: the finer topologies of function spaces on Lie groups, such as the Schwartz-space topology on the Euclidean motion group studied in [34, 35], lie outside the present framework.

Several questions remain. First, is the repair cost attained whenever it is positive? Theorem 11.2 bounds it below by $\mu_g(\mathcal{V})$ when $\mathcal{V} \neq \emptyset$, but neither attainment nor an exact value is known to us in that case. Second, Theorem 4.1 gives $\tau_\theta \subseteq \sigma \subseteq \tau$, and Proposition 8.1 realizes the lower end; which intermediate topologies are realizable, for a given non-regular τ, as repairable coarsenings? Third, the pointwise invariant B_σ of Corollary 5.1 is defined for any pair $\sigma \subseteq \tau$, and the characterization of the pairs for which it is open or dense remains open; for $\sigma = \mathcal{A}$, Theorem 2.1 gives openness. Fourth, the analysis concerns coarsenings of τ_M, whereas the Zeeman [1] and path [2] topologies are refinements; the dual question, whether a fine causal topology can be coarsened to τ_M by a construction based on a filter of conegligible sets, lies beyond the present methods and remains open. Finally, the rigidity phenomena established here are stated for ideals; the corresponding questions for grills and for general Kuratowski closure perturbations remain open.

Data availability

No data were generated or analyzed in the course of this work. All results are established by proof within the article, and the Lorentzian metric used in Example 13 is stated explicitly in closed form, so that every computation reported here can be reproduced directly from the text.

Declaration of competing interest

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