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On spectral geometry of metric graphs: Diameter bounds, cycles, and rigidity

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Abstract

We study how the spectral gap $\lambda_1(G)$ of a compact metric graph equipped with the Kirchhoff Laplacian depends on its geometric and topological features. First, we examine $\lambda_1(G) \geq \pi^2/D^2$ and demonstrate that equality is not unique to interval graphs: a single cycle graph C of length l (with diameter $D = l/2$) satisfies $\lambda_1(C) = 4\pi^2/l^2 = \pi^2/D^2$, providing a non-interval equality case. Furthermore, we analyze basic topological operations, proving that edge subdivision preserves the spectrum, pendant edge contraction strictly increases $\lambda_1(G)$ unless a first eigenfunction vanishes at the attachment vertex (in which case $\lambda_1(G)$ is invariant), and function addition yields a weighted average of Rayleigh quotients rather than an additive sum even under mutual H^1-orthogonality.

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1. Introduction

A metric graph is obtained by taking a finite graph and assigning to each edge a positive length, so that the edge becomes a genuine interval and the whole graph becomes a one-dimensional metric space. Functions on this space can be differentiated along each edge, and at every vertex one imposes the usual Kirchhoff-Neumann matching conditions: continuity of the function together with a balance of outgoing derivatives. This simple setup already puts metric graphs at the crossing point of discrete graph theory and analysis on singular spaces, and it is why they show up so often in quantum graph theory, spectral geometry, and the modelling of networks.

For a compact connected metric graph G, the natural operator to study is the Laplacian, and its spectrum carries information about both the local structure near each vertex and the graph's global shape. We are mainly interested in the first nonzero eigenvalue $\lambda_1(G)$, usually called the spectral gap, which controls how fast heat diffuses on G, how waves propagate, how quickly a random walk mixes, and how stable the graph is under perturbation. From here on, $\lambda_1(G)$ always refers to the first nonzero eigenvalue of the Kirchhoff-Neumann Laplacian, and every bound or stability statement in the paper is stated with respect to this quantity.

The spectral theory of metric graphs, also called quantum graphs, is by now a mature subject; see the books of Berkolaiko and Kuchment [1] and Post [2], and the survey of Kuchment [3], for background. Nicaise [4] and Friedlander [5] were among the first to look at extremal eigenvalue problems under a length

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constraint, and their examples already show that a cycle can push λ_1 either up or down depending on how it sits inside the graph. Borthwick *et al.* [6] later obtained a diameter bound that holds for every compact connected metric graph, not just trees, and pinned down the equality case: the interval is, up to isometry, the only minimizer.

On the maximization side, Band *et al.* [7] found the graphs that maximize λ_1 for a fixed total length, so-called pumpkin and flower graphs. What these examples make clear is that cycles can move the higher part of the spectrum quite a lot, while their effect on the spectral gap itself is much more sensitive to where, exactly, the cycle is placed.

Discrete spectral graph theory gives some intuition here: raising the cyclomatic number tends to push up the higher eigenvalues, though its effect on the very first one is far less clear [8]. In the metric setting, a cycle introduces extra oscillatory modes, and these can interact with the geodesics that realize the diameter in ways that are hard to predict. The folklore expectation is that cycles increase λ_1, yet there is no systematic account of how the number, size, or arrangement of cycles actually governs the spectral gap.

There has also been a good deal of recent interest in curvature notions for graphs. Ollivier's discrete Ricci curvature [9], later developed further by Lin *et al.* [10], gives eigenvalue estimates and functional inequalities in a purely combinatorial setting, and Gozlan *et al.* [11] have pushed related ideas through Bochner-type identities and entropy convexity. These curvature-based results, however, are almost always stated for discrete Laplacians and do not directly see metric data such as diameter or edge length.

Stability of eigenvalues under geometric perturbation is well understood on smooth spaces, largely through variational and optimal transport techniques [12, 13]. For metric graphs the picture is much more fragmentary: what is known concerns mainly limits from graph-like manifolds or edge contraction procedures [2], and a quantitative stability theory for the spectral gap under changes to edge lengths or to the underlying topology is still largely missing.

In particular, Borthwick *et al.* [6] established the bound $\lambda_1(G) \geq \frac{\pi^2}{D^2}$ for compact connected metric graphs and characterized the equality case, showing that equality occurs only for an interval graph (up to isometry). We therefore take this result as the starting point rather than as a new contribution. The purpose of the present work is to investigate the additional spectral and geometric structure surrounding this extremal situation.

More precisely, the main contributions of this paper are as follows. Section 1 is the introduction related to this research work. Section 2 deals with background material required for proving results. Section 3 contains all the main findings of this research work. Section 4 concludes all the findings.

2. Preliminaries

We collect here the basic definitions and background results on metric graphs that will be used repeatedly in the rest of the paper.

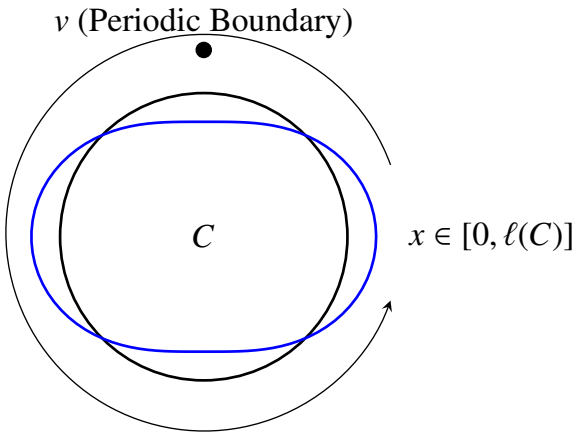


Figure 1: A single cycle C of length $\ell(C)$, showing its periodic geometry.

Table 1: Geometric and spectral characteristics of representative metric graphs.

Graph	Diameter	Total Length	First Nonzero Eigenvalue
Interval I_D	D	D	$\frac{\pi^2}{D^2}$
Cycle C_ℓ	$\frac{\ell}{2}$	ℓ	$\frac{4\pi^2}{\ell^2} = \frac{\pi^2}{D^2}$
Star Graph S_n	$2L$	nL	$\frac{\pi^2}{4L^2} = \frac{\pi^2}{D^2}$

Definition 2.1. A metric graph $G = (V, E, \ell)$ consists of a finite connected graph with vertex set V and edge set E, where each edge $e \in E$ is identified with a compact interval $[0, \ell_e]$, $\ell_e > 0$. The graph is endowed with the natural path metric.

Theorem 2.1. (Min-Max Principle): Let G be a compact metric graph equipped with the standard Kirchhoff Laplacian $-\Delta$. The spectrum of $-\Delta$ is discrete and satisfies

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

The first nonzero eigenvalue $\lambda_1(G)$ admits the variational characterization

$$\lambda_1(G) = \inf_{\substack{u \in H^1(G) \\ \int_G u = 0 \\ u \neq 0}} \frac{\int_G |u'(x)|^2 \, dx}{\int_G |u(x)|^2 \, dx}.$$

More generally, the k-th eigenvalue is given by the Courant–Fischer min–max formula

$$\lambda_k(G) = \inf_{\substack{V \subset H^1(G) \\ \dim V = k+1}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\int_G |u'(x)|^2 \, dx}{\int_G |u(x)|^2 \, dx}.$$

3. Results on geometric and spectral bounds for metric graphs

Theorem 3.1. Let G be a metric graph consisting of a single cycle C of total length $\ell(C)$, equipped with the standard Kirchhoff Laplacian. Then, the first nonzero eigenvalue $\lambda_1(G)$ satisfies

$$\lambda_1(G) = \frac{4\pi^2}{\ell(C)^2}.$$

In particular, for any function $u \in H^1(C)$ satisfying the zero-mean condition, the sharp Poincaré-type inequality holds:

$$\int_C |u'(x)|^2\,dx \;\geq\; \frac{4\pi^2}{\ell(C)^2}\int_C |u(x)|^2\,dx.$$

Proof. A graph consisting of a single cycle C is nothing more than a circle of circumference $\ell(C)$, so it is convenient to parametrize C by $[0,\ell(C)]$ with the two endpoints glued together. Under this identification, functions on C become exactly the periodic functions on $[0,\ell(C)]$, and the Kirchhoff condition at the (single) vertex is satisfied automatically by any such periodic function. The Kirchhoff Laplacian on G is just the usual operator $-d^2/dx^2$, acting on H^2-functions on $[0,\ell(C)]$ subject to periodic boundary conditions. The eigenvalue problem $-u'' = \lambda u$ on $[0,\ell(C)]$ with periodic boundary conditions is solved explicitly by the complex exponentials

$$u_k(x) = e^{2\pi i k x/\ell(C)}, \quad k \in \mathbb{Z},$$

with

$$\lambda_k = \left(\frac{2\pi k}{\ell(C)}\right)^2.$$

Here, $k = 0$ gives the constant functions, i.e., $\lambda_0 = 0$, and the smallest nonzero eigenvalues come from $k = \pm 1$, so

$$\lambda_1(G) = \frac{4\pi^2}{\ell(C)^2}.$$

Theorem 2.1 identifies $\lambda_1(G)$ with the infimum

$$\lambda_1(G) = \inf_{\substack{u\in H^1(C)\\ \int_C u=0\\ u\neq 0}} \frac{\int_C |u'(x)|^2\,dx}{\int_C |u(x)|^2\,dx},$$

and combining this with the value of $\lambda_1(G)$ computed above gives the stated Poincaré-type inequality

$$\int_C |u'(x)|^2\,dx \;\geq\; \frac{4\pi^2}{\ell(C)^2}\int_C |u(x)|^2\,dx$$

for every $u \in H^1(C)$ with zero mean. □

To illustrate the geometric setting of Theorem 3.1, see Figure 1, which presents a metric graph consisting of a single cycle of total length $\ell(C)$. Since a cycle graph is isometric to a circle with periodic boundary conditions, its spectral analysis reduces to the classical eigenvalue problem on a closed interval with identified endpoints. The figure highlights the cyclic structure and serves as a visual representation of the geometry responsible for the explicit spectral gap formula $\lambda_1(C) = 4\pi^2/\ell(C)^2$.

Proposition 3.1. Let G be a compact connected metric graph of diameter D, and let $\gamma \subset G$ be a diameter-realizing path of length D. Assume that $\lambda_1(G) = \frac{\pi^2}{D^2}$, and let u be a first eigenfunction normalized by $\int_G |u|^2\,dx = 1$ and $\int_G u\,dx = 0$. Suppose, in addition, that $\int_\gamma u\,dx = 0$ and $\int_\gamma |u|^2\,dx = 1$. Then

$$u|_\gamma(x) = c\cos\left(\frac{\pi x}{D}\right),$$

for some $c \neq 0$, and $u = 0$ a.e. on $G \setminus \gamma$.

Proof. Parametrize the diameter-realizing path γ by arc length $x \in [0,D]$. Assume that $\int_\gamma u(x)\,dx = 0$. The sharp one-dimensional Neumann–Poincaré inequality gives

$$\int_\gamma |u'(x)|^2\,dx \geq \frac{\pi^2}{D^2}\int_\gamma |u(x)|^2\,dx.$$

Since $\int_\gamma |u|^2\,dx = 1$ and $\int_G |u|^2\,dx = 1$, we have

$$\int_{G\setminus\gamma} |u|^2\,dx = 0.$$

Due to this, $u = 0$ almost everywhere on $G \setminus \gamma$. Moreover,

$$\int_G |u'|^2\,dx = \lambda_1(G) = \frac{\pi^2}{D^2}.$$

Since $u = 0$ almost everywhere on $G\setminus\gamma$, the energy is supported on γ, yielding

$$\int_\gamma |u'|^2\,dx = \frac{\pi^2}{D^2}\int_\gamma |u|^2\,dx.$$

Therefore, equality holds in the one-dimensional Neumann–Poincaré inequality. Its equality characterization gives $u|_\gamma(x) = c\cos\left(\frac{\pi x}{D}\right)$ for some $c \neq 0$. □

Theorem 3.2. Let G be a compact connected metric graph of diameter D, and let $\gamma \subset G$ be a diameter-realizing path of length D. Assume that

$$\lambda_1(G) = \frac{\pi^2}{D^2}.$$

Let u be a first eigenfunction normalized by $\int_G |u|^2\,dx = 1$, $\int_G u\,dx = 0$, and suppose that $\int_\gamma u\,dx = 0$, $\int_\gamma |u|^2\,dx = 1$. Then

$$u|_\gamma(x) = c\cos\left(\frac{\pi x}{D}\right), \qquad x \in [0,D],$$

for some $c \neq 0$, and $u = 0$ a.e. on $G \setminus \gamma$. Moreover, every connected component of $G \setminus \gamma$ can be attached to γ only at the midpoint of γ, namely at the point corresponding to $x = D/2$. In particular, if $G \setminus \gamma = \varnothing$, then G is isometric to the interval $[0,D]$.

Proof. We begin by parametrizing the diameter-realizing path γ by arc length, so that $\gamma \simeq [0,D]$. Through hypothesis, we get $\int_\gamma u(x)\,dx = 0$.

Now, the one-dimensional Neumann–Poincaré inequality on $[0, D]$ is applicable to $u|_\gamma$ and gives

$$\int_\gamma |u'(x)|^2\,dx \geq \frac{\pi^2}{D^2}\int_\gamma |u(x)|^2\,dx.$$

Since $\int_\gamma |u(x)|^2\,dx = 1$ and $\int_G |u(x)|^2\,dx = 1$, we obtain $\int_{G\setminus\gamma} |u(x)|^2\,dx = 0$, which implies $u = 0$ a.e. on $G \setminus \gamma$. Because u is a first eigenfunction,

$$\int_G |u'(x)|^2\,dx = \lambda_1(G)\int_G |u(x)|^2\,dx = \frac{\pi^2}{D^2}.$$

Since u vanishes almost everywhere on $G \setminus \gamma$, the energy is concentrated on γ, and consequently

$$\int_\gamma |u'(x)|^2\,dx = \frac{\pi^2}{D^2}\int_\gamma |u(x)|^2\,dx.$$

Equality holds in the one-dimensional Neumann–Poincaré inequality, yielding

$$u|_\gamma(x) = c\cos\left(\frac{\pi x}{D}\right)$$

for some nonzero constant c. We now determine where a component of $G \setminus \gamma$ can meet γ. Let H be a connected component of $G \setminus \gamma$ and suppose that H is attached to γ at a vertex corresponding to the point $x_0 \in [0, D]$. Since $u = 0$ on H, continuity of H^1-functions on metric graphs implies $u(x_0) = 0$. On γ we have $u(x_0) = c\cos\left(\frac{\pi x_0}{D}\right)$.

Since $c \neq 0$, it follows that $\cos\left(\frac{\pi x_0}{D}\right) = 0$. For $x_0 \in [0, D]$, the only solution is $x_0 = \frac{D}{2}$. Every connected component of $G \setminus \gamma$ can meet γ only at its midpoint, where the restriction of u satisfies $u\left(\frac{D}{2}\right) = 0$.

Moreover,

$$u'(x) = -\frac{c\pi}{D}\sin\left(\frac{\pi x}{D}\right),$$

and

$$u'\left(\frac{D}{2}^-\right) = -\frac{c\pi}{D}, \qquad u'\left(\frac{D}{2}^+\right) = -\frac{c\pi}{D}.$$

When expressed as outward derivatives from the midpoint vertex, the two derivatives along the two directions of γ have opposite signs and therefore cancel. Since u is identically zero on every component attached at the midpoint, the outward derivatives along those complementary edges are also zero. The Kirchhoff condition is compatible with the localization described above. Finally, if $G \setminus \gamma = \varnothing$, every point and every edge of G belongs to γ, meaning G is precisely the path γ, which is isometric to the interval $[0, D]$. □

Corollary 3.1 (Cycle equality case)**.** Let C_ℓ be a compact metric graph consisting of a single cycle of total length ℓ. Then

$$D(C_\ell) = \frac{\ell}{2}$$

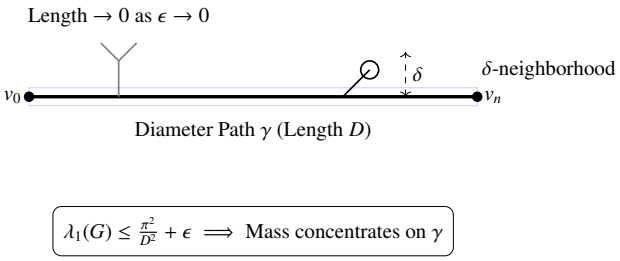


Figure 2: Near-extremal graph, diameter path γ, its δ-neighborhood, and concentration of eigenfunction mass.

and

$$\lambda_1(C_\ell) = \frac{4\pi^2}{\ell^2} = \frac{\pi^2}{D(C_\ell)^2}.$$

Consequently, the value π^2/D^2 is attained by a non-interval metric graph. In particular, the presence of a cycle does not, in general, imply strict inequality in the diameter expression.

Proof. By Theorem 3.1, a metric graph consisting of a single cycle C_ℓ of total length ℓ satisfies $\lambda_1(C_\ell) = \frac{4\pi^2}{\ell^2}$. The diameter of a cycle of circumference ℓ is the length of a shortest semicircle, so $D(C_\ell) = \frac{\ell}{2}$. Now,

$$\frac{\pi^2}{D(C_\ell)^2} = \frac{\pi^2}{(\ell/2)^2} = \frac{4\pi^2}{\ell^2}.$$

Combining the two identities gives $\lambda_1(C_\ell) = \frac{\pi^2}{D(C_\ell)^2}$. Since a cycle is not isometric to an interval, this provides a non-interval equality example. Thus, the mere presence of a cycle does not imply $\lambda_1(G) > \frac{\pi^2}{D^2}$. □

Theorem 3.3. Let G be a compact connected metric graph with Kirchhoff Laplacian, and let $u, v \in H^1(G) \setminus \{0\}$ satisfy $\int_G u\,dx = \int_G v\,dx = 0$ with $u + v \not\equiv 0$. For $w \neq 0$, let $R(w) = \frac{\|w'\|_{L^2}^2}{\|w\|_{L^2}^2}$. In general,

$$R(u+v) = \frac{\|u'\|_{L^2}^2 + \|v'\|_{L^2}^2 + 2\int_G u'v'\,dx}{\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + 2\int_G uv\,dx}.$$

If $\int_G uv\,dx = \int_G u'v'\,dx = 0$, then $R(u + v)$ reduces to the weighted average

$$R(u+v) = \frac{R(u)\|u\|_{L^2}^2 + R(v)\|v\|_{L^2}^2}{\|u\|_{L^2}^2 + \|v\|_{L^2}^2} \in [\min\{R(u), R(v)\},\ \max\{R(u), R(v)\}].$$

Orthogonality yields a weighted average rather than an additive sum, precluding additive combinations of Rayleigh quotients for variational bounds on $\lambda_1(G)$.

Proof. By definition of the Rayleigh quotient,

$$R(u+v) = \frac{\int_G |(u+v)'|^2\,dx}{\int_G |u+v|^2\,dx}.$$

Expanding the Dirichlet energy and L^2-norm yields

$$\int_G |(u+v)'|^2\,dx = \int_G |u'|^2\,dx + \int_G |v'|^2\,dx + 2\int_G u'v'\,dx,$$
$$\int_G |u+v|^2\,dx = \int_G |u|^2\,dx + \int_G |v|^2\,dx + 2\int_G uv\,dx.$$

Substituting these expansions gives the general identity for $R(u+v)$. When both cross terms vanish ($\int_G uv\,dx = 0$ and $\int_G u'v'\,dx = 0$), the expression simplifies to

$$R(u+v) = \frac{\int_G |u'|^2\,dx + \int_G |v'|^2\,dx}{\|u\|^2_{L^2(G)} + \|v\|^2_{L^2(G)}}.$$

Substituting $\int_G |u'|^2\,dx = R(u)\|u\|^2_{L^2(G)}$ and $\int_G |v'|^2\,dx = R(v)\|v\|^2_{L^2(G)}$ gives

$$R(u+v) = \frac{R(u)\|u\|^2_{L^2(G)} + R(v)\|v\|^2_{L^2(G)}}{\|u\|^2_{L^2(G)} + \|v\|^2_{L^2(G)}},$$

which is precisely a convex combination (weighted average) of $R(u)$ and $R(v)$. This completes the proof. □

The Rayleigh quotient is not additive with respect to sums of functions. Its value for $u+v$ depends on the energy and L^2-norm cross terms. When both cross terms vanish, the resulting quotient is a weighted average of $R(u)$ and $R(v)$, rather than their sum. Consequently, variational estimates for sums of admissible functions must be derived from the quadratic form identity together with the relevant normalization and orthogonality conditions.

Theorem 3.4. Let G be a compact metric graph equipped with the standard Kirchhoff Laplacian, and let $\widetilde{G}$ be obtained from G by subdividing an edge into finitely many edges whose total length equals the original edge length. Then, the first nonzero eigenvalue satisfies $\lambda_1(\widetilde{G}) = \lambda_1(G)$. More generally, the entire spectrum of the Kirchhoff Laplacian is invariant under edge subdivision.

Proof. Subdividing an edge without changing its total length leaves the underlying metric space, and hence the global topology of the graph, completely untouched while only changing the combinatorial bookkeeping. Each new subdivision vertex has degree exactly 2, and the Kirchhoff condition there simply says that the function is continuous and that the two outgoing derivatives sum to zero. This is nothing but the usual requirement of C^1-smoothness at an interior point of an interval, so a degree 2 vertex imposes no genuine constraint beyond ordinary differentiability.

It follows that the identity map gives an isometry between $H^1(G)$ and $H^1(\widetilde{G})$, under which the Dirichlet forms agree exactly:

$$\int_G |u'(x)|^2\,dx = \int_{\widetilde{G}} |u'(x)|^2\,dx.$$

The L^2-norm and the zero-mean condition are likewise unchanged by this identification, so the Rayleigh quotient is identical before and after subdivision. The min-max principle then gives

$$\lambda_1(\widetilde{G}) = \lambda_1(G),$$

and since the isometry respects every finite-dimensional subspace appearing in the min-max formula, the same conclusion holds for every eigenvalue, not just λ_1. □

Corollary 3.2. Let G be a compact metric graph, and let $\{G_n\}_{n\geq 1}$ be a sequence of graphs obtained from G by iterative edge subdivision such that the total length of the graph remains invariant. Then, the k-th eigenvalue satisfies

$$\lambda_k(G_n) = \lambda_k(G)$$

for all $k \geq 0$ and all $n \geq 1$. In particular, repeated edge subdivision preserves the entire spectrum of the Kirchhoff Laplacian.

Proof. Every subdivision step in the sequence preserves the metric geometry and the topology of the graph, and only introduces degree 2 dummy vertices. Theorem 3.4 shows that one such step leaves the spectrum unchanged, and applying this repeatedly along $\{G_n\}_{n\geq 1}$ gives

$$\lambda_k(G_n) = \lambda_k(G)$$

for every $n \geq 1$ and $k \geq 0$, as claimed. □

Lemma 3.1. Let G be a compact metric graph equipped with the standard Kirchhoff Laplacian, and let $\widetilde{G}$ be the metric graph obtained from G by subdividing a single edge e into two edges whose total length equals the length of e. Then, every function $f \in H^1(G)$ naturally defines a function $\widetilde{f} \in H^1(\widetilde{G})$ such that

$$\int_{\widetilde{G}} |\widetilde{f}'(x)|^2\,dx = \int_G |f'(x)|^2\,dx,$$
$$\int_{\widetilde{G}} |\widetilde{f}(x)|^2\,dx = \int_G |f(x)|^2\,dx.$$

Proof. Identify e with $[0, L]$. The subdivided graph $\widetilde{G}$ is formed by placing a new vertex at some interior point $x_0 \in (0, L)$, so that e is replaced by two edges $e_1 = [0, x_0]$ and $e_2 = [x_0, L]$ with $\ell(e_1) + \ell(e_2) = L$.

Given $f \in H^1(G)$, define $\widetilde{f}$ on $\widetilde{G}$ by restricting f to e_1 and e_2 and leaving it unchanged on every other edge of G. Since $f \in H^1(G)$ is continuous on $[0, L]$, the resulting $\widetilde{f}$ is automatically continuous at the new vertex x_0.

By construction $\widetilde{f} \in H^1(\widetilde{G})$, and its weak derivative agrees almost everywhere with that of f on the corresponding pieces of the original edge, so

$$\int_{\widetilde{G}} |\widetilde{f}'(x)|^2\,dx = \int_G |f'(x)|^2\,dx,$$
$$\int_{\widetilde{G}} |\widetilde{f}(x)|^2\,dx = \int_G |f(x)|^2\,dx.$$

If, moreover, f is an eigenfunction of the Kirchhoff Laplacian on G, then $\widetilde{f}$ satisfies the Kirchhoff condition at the new vertex

x_0 as well: the one-sided derivatives from e_1 and e_2 at x_0 both equal $f'(x_0)$, so their contributions to the flux balance cancel,

$$\left.\frac{d\widetilde{f}}{dx}\right|_{e_1}(x_0) - \left.\frac{d\widetilde{f}}{dx}\right|_{e_2}(x_0) = f'(x_0) - f'(x_0) = 0.$$

So edge subdivision carries eigenfunctions to eigenfunctions with the same eigenvalue, and preserves the underlying variational structure of the Kirchhoff Laplacian. □

Proposition 3.2. Let $G = [0, L]$ be an interval equipped with the standard Kirchhoff (Neumann) Laplacian. Let $b \in (0, L)$ with $b \neq L/2$, and let G_a be obtained from G by attaching a pendant edge of length $a > 0$ at the point b. Then

$$\lambda_1(G_a) < \lambda_1(G) = \frac{\pi^2}{L^2}.$$

Consequently, the first nonzero eigenvalue is, in general, strictly decreased by the addition of a pendant edge.

Proof. The first nonzero eigenvalue of the interval $G = [0, L]$ with Neumann boundary conditions is $\lambda_1(G) = \frac{\pi^2}{L^2}$, with corresponding eigenfunction

$$u(x) = \cos\left(\frac{\pi x}{L}\right).$$

We normalize u for convenience; in particular, $\int_0^L u(x)\,dx = 0$. Let e be the pendant edge of length a attached at $x = b$, where $b \in (0, L)$ and $b \neq L/2$. Define

$$c = u(b) = \cos\left(\frac{\pi b}{L}\right).$$

Since $b \neq L/2$, we have $c \neq 0$. Extend u to a continuous function $\widetilde{u}$ on G_a by setting

$$\widetilde{u}(x) = \begin{cases} u(x), & x \in G, \\ c, & x \in e. \end{cases}$$

Moreover, $\widetilde{u}$ is constant on the pendant edge, so $\int_e |\widetilde{u}'(x)|^2\,dx = 0$. Now,

$$\int_{G_a} |\widetilde{u}'(x)|^2\,dx = \int_0^L |u'(x)|^2\,dx = \frac{\pi^2}{L^2}\int_0^L |u(x)|^2\,dx.$$

The only issue is that $\widetilde{u}$ does not have zero mean on G_a. Indeed, $\int_{G_a} \widetilde{u}\,dx = \int_0^L u(x)\,dx + ac = ac$. We subtract the constant $m = \frac{ac}{L+a}$ and define $v = \widetilde{u} - m$. Due to this, we get $\int_{G_a} v\,dx = \int_{G_a} \widetilde{u}\,dx - (L + a)m = 0$, so v is an admissible test function for the variational characterization of $\lambda_1(G_a)$. Since subtracting a constant does not change the derivative,

$$\int_{G_a} |v'|^2\,dx = \int_{G_a} |\widetilde{u}'|^2\,dx = \int_0^L |u'|^2\,dx.$$

For the denominator, we use

$$\int_{G_a} |v|^2\,dx = \int_{G_a} |\widetilde{u} - m|^2\,dx = \int_{G_a} |\widetilde{u}|^2\,dx - \frac{\left(\int_{G_a} \widetilde{u}\,dx\right)^2}{L + a}.$$

Now, since $\int_{G_a} |\widetilde{u}|^2\,dx = \int_0^L |u(x)|^2\,dx + ac^2$ and $\int_{G_a} \widetilde{u}\,dx = ac$, we obtain

$$\begin{aligned}\int_{G_a} |v|^2\,dx &= \int_0^L |u(x)|^2\,dx + ac^2 - \frac{a^2c^2}{L+a} \\ &= \int_0^L |u(x)|^2\,dx + \frac{aL}{L+a}c^2.\end{aligned}$$

Because $a > 0$, $L > 0$, and $c \neq 0$, $\frac{aL}{L+a}c^2 > 0$. Consequently, $\int_{G_a} |v|^2\,dx > \int_0^L |u(x)|^2\,dx$. So, the Rayleigh quotient of v satisfies

$$\mathcal{R}_{G_a}(v) < \frac{\int_0^L |u'(x)|^2\,dx}{\int_0^L |u(x)|^2\,dx} = \lambda_1(G) = \frac{\pi^2}{L^2}.$$

By the variational characterization of the first nonzero eigenvalue,

$$\lambda_1(G_a) \leq \mathcal{R}_{G_a}(v) < \frac{\pi^2}{L^2}.$$

Thus, $\lambda_1(G_a) < \lambda_1(G)$. This completes the proof. □

Theorem 3.5 (Spectral Gap Increase Under Contraction of a Pendant Edge)**.** Let G be a compact connected metric graph equipped with the standard Kirchhoff Laplacian. Let e be a pendant edge of length $a > 0$, attached to the remaining graph G_c at a vertex v. Let G_c be the metric graph obtained by contracting e to v. Then

$$\lambda_1(G) \leq \lambda_1(G_c).$$

Moreover, if G_c has a first eigenfunction f satisfying $f(v) \neq 0$, then

$$\lambda_1(G) < \lambda_1(G_c).$$

In particular, equality can occur only if every first eigenfunction of G_c vanishes at the attachment vertex v.

Proof. Let f be a first eigenfunction of G_c, normalized so that

$$\int_{G_c} f(x)\,dx = 0, \quad \int_{G_c} |f(x)|^2\,dx = 1.$$

Write $\lambda_c = \lambda_1(G_c)$, $c = f(v)$, and let $\ell(G_c)$ denote the total length of G_c. We first prove that $\lambda_1(G) \leq \lambda_c$. Define a function $\widetilde{f}$ on G by setting $\widetilde{f}(x) = f(x)$ for $x \in G_c$ and 0 for $x \in e$. This function is continuous at the attachment vertex only when $f(v) = 0$. Thus, this zero extension is admissible in that case, and its mean and energy are unchanged, giving $\lambda_1(G) \leq \lambda_c$ whenever $f(v) = 0$.

For the general case, extend f constantly along the pendant edge:

$$f_e(x) = \begin{cases} f(x), & x \in G_c, \\ c, & x \in e. \end{cases}$$

This extension belongs to $H^1(G)$. Since it is constant on e, we have $\int_G |f_e'|^2\,dx = \int_{G_c} |f'|^2\,dx = \lambda_c$, and its integral over G is ac.

Defining the mean-corrected function $g = f_e - ac/(\ell(G_c) + a)$, its energy is unchanged, and its L^2-norm satisfies:

$$\int_G |g|^2\,dx = \int_G |f_e|^2\,dx - \frac{a^2c^2}{\ell(G_c)+a} = 1 + \frac{a\,\ell(G_c)}{\ell(G_c)+a}c^2.$$

Therefore, the Rayleigh quotient is bounded by:

$$R_G(g) = \frac{\lambda_c}{1 + \dfrac{a\,\ell(G_c)}{\ell(G_c)+a}c^2} \le \lambda_c.$$

By the variational characterization, $\lambda_1(G) \le R_G(g) \le \lambda_c$. If $c = f(v) \neq 0$, the denominator is strictly greater than 1, yielding $R_G(g) < \lambda_c$ and hence $\lambda_1(G) < \lambda_1(G_c)$. Thus, equality can occur only if every first eigenfunction of G_c vanishes at v. □

Theorem 3.6. [Continuity under the addition of a shrinking pendant edge] Let G be a compact connected metric graph equipped with the standard Kirchhoff Laplacian. Let $v \in G$ be a vertex, and let G_ε be the metric graph obtained from G by attaching a pendant edge e_ε of length $\varepsilon > 0$ at v. Then

$$\lim_{\varepsilon\to 0^+} \lambda_1(G_\varepsilon) = \lambda_1(G).$$

Proof. Let u be a first eigenfunction of G, normalized by $\int_G u(x)\,dx = 0$ and $\int_G |u(x)|^2\,dx = 1$, so that $\int_G |u'(x)|^2\,dx = \lambda_1(G)$. By extending u constantly to e_ε via $\widetilde{u}_\varepsilon(x) = u(x)$ for $x \in G$ and $u(v)$ for $x \in e_\varepsilon$, we have $\int_{G_\varepsilon} |\widetilde{u}'_\varepsilon(x)|^2\,dx = \lambda_1(G)$ and $\int_{G_\varepsilon} \widetilde{u}_\varepsilon\,dx = \varepsilon u(v)$.

Setting $m_\varepsilon = \frac{\varepsilon u(v)}{|G|+\varepsilon}$ and $w_\varepsilon = \widetilde{u}_\varepsilon - m_\varepsilon$, we obtain $\int_{G_\varepsilon} w_\varepsilon\,dx = 0$ and $\int_{G_\varepsilon} |w'_\varepsilon|^2\,dx = \lambda_1(G)$. Using the variance identity under mean subtraction,

$$\int_{G_\varepsilon} |w_\varepsilon|^2\,dx = 1 + \frac{\varepsilon|G|}{|G|+\varepsilon}|u(v)|^2,$$

which yields the bound

$$\lambda_1(G_\varepsilon) \le \frac{\lambda_1(G)}{1 + \dfrac{\varepsilon|G|}{|G|+\varepsilon}|u(v)|^2},$$

giving $\limsup_{\varepsilon\to 0^+} \lambda_1(G_\varepsilon) \le \lambda_1(G)$.

For the reverse inequality, let f_ε be a first eigenfunction of G_ε satisfying $\int_{G_\varepsilon} |f_\varepsilon|^2\,dx = 1$ and $\int_{G_\varepsilon} f_\varepsilon\,dx = 0$. Boundedness ensures f_ε is uniformly bounded in H^1. Restricting f_ε to G, defining $c_\varepsilon = \frac{1}{|G|}\int_G f_\varepsilon(x)\,dx$, and setting $h_\varepsilon = f_\varepsilon|_G - c_\varepsilon$, Cauchy–Schwarz gives $|c_\varepsilon| \le \frac{\sqrt{\varepsilon}}{|G|} \to 0$.

Furthermore, writing $e_\varepsilon = [0,\varepsilon]$ and applying the fundamental theorem of calculus, $\int_{e_\varepsilon} |f_\varepsilon|^2\,dx \to 0$, leading to $\int_G |h_\varepsilon|^2\,dx \to 1$. By the variational characterization,

$$\lambda_1(G) \le \frac{\displaystyle\int_G |h'_\varepsilon|^2\,dx}{\displaystyle\int_G |h_\varepsilon|^2\,dx} \le \int_{G_\varepsilon} |f'_\varepsilon|^2\,dx = \lambda_1(G_\varepsilon).$$

Taking the infimum limit gives $\lambda_1(G) \le \liminf_{\varepsilon\to 0^+} \lambda_1(G_\varepsilon)$, concluding that $\lim_{\varepsilon\to 0^+} \lambda_1(G_\varepsilon) = \lambda_1(G)$. □

Table 2: Effect of graph operations on the first nonzero eigenvalue λ_1.

Graph Operation	Effect on λ_1
Subdivision of an edge	No change
Addition of a pendant edge	May decrease
Contraction of a pendant edge	May increase
Interval graph	Attains the minimum value $\frac{\pi^2}{D^2}$

4. Conclusion

In this paper, we investigated the spectral gap of compact metric graphs with the standard Kirchhoff Laplacian, focusing on the interaction between diameter, metric structure, topology, and variational properties. We obtained the exact first nonzero eigenvalue for a single cycle and examined the behavior of the spectral gap in relation to the diameter bound and its near-extremal regime. We also investigated the variational behavior of sums of admissible functions and demonstrated that Rayleigh quotients are not generally additive.

Overall, the results demonstrate that the spectral gap of a metric graph is governed not only by its combinatorial structure but also by its metric geometry and global variational properties. These observations suggest several directions for future work, including the spectral response to simultaneous or iterated graph perturbations and the investigation of diameter-based rigidity and stability under other self-adjoint vertex conditions, such as Robin, Dirichlet, and more general vertex couplings.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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