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Planarity and genus of the n-total graphs of finite commutative rings

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Abstract

Let R be a finite commutative ring and let n be a positive integer. The n-total graph $\tau_n(R)$ has vertex set R, with distinct vertices x and y adjacent precisely when $x^n + y^n$ is a zero-divisor. We determine the orientable genus of $\tau_n(R)$ for every finite local ring in terms of the order of its maximal ideal, the order q of its residue field, and $\gcd(n, q-1)$. The resulting formula yields a complete planarity criterion and exactly six families of toroidal local ring/exponent pairs. For finite nonlocal rings, we combine a coordinate-zero clique with a residue-fiber adjacency rule to reduce the low-genus problem to a finite collection of products. This gives complete classifications of all nonlocal rings whose n-total graphs are planar or toroidal. We further prove that the least common multiple of the integers $q_i - 1$, taken over the residue fields of the local factors of R, is a period of the labeled graph and hence of its genus. When $n = 1$, our results recover the known genus-at-most-one classification for ordinary total graphs.

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1. Introduction

All rings in this paper are commutative with identity, and all graphs are finite, simple, and undirected. Anderson and Badawi introduced the total graph $T(\Gamma(R))$, whose vertex set is R and in which two distinct vertices x, y are adjacent when $x + y \in Z(R)$; see Ref. [1]. We refer the reader to Ref. [2] for a survey on the topic, Ref. [3] for genus-related results, and Ref. [4] for details on independence and independent domination parameters.

AitElhadi and Badawi replaced $x+y$ by x^n+y^n to define the n-total graph $\tau_n(R)$; see Ref. [5]; hence $\tau_1(R) = T(\Gamma(R))$. For a finite local ring they obtained a component decomposition determined by the maximal-ideal order b, the residue-field order q, and

$$g_n = \gcd(n, q-1), \qquad \delta_n = \frac{q-1}{g_n}.$$

We use it to obtain one genus formula valid for every exponent.

For the ordinary total graph, Maimani, Wickham, and Yassemi classified the finite rings whose total graphs have genus at most one, see Ref. [6]. Power maps change the residue-field fibers, so their classification does not extend formally to $n > 1$: for example, the ordinary total graph of either local ring of order nine is toroidal, whereas its n-total graph is planar for even n; see Example 3.6.

We write $\mathrm{gen}(G)$ for orientable genus, avoiding notation used for domination parameters; "toroidal" means genus exactly one. We determine $\mathrm{gen}(\tau_n(R))$ for every finite local ring and classify all finite nonlocal rings for which $\tau_n(R)$ is planar or

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toroidal. The local decomposition reduces genus to the classical complete and complete-bipartite formulas. For products, a coordinate-zero clique bounds the factor orders and a residue-fiber rule resolves the remaining finite list. We finally record a period of the genus sequence and recover the ordinary case at $n = 1$.

2. Preliminaries

2.1. Finite rings and the n-total graph

Definition 2.1. Let R be a finite commutative ring and let $n \geq 1$. The *n-total graph* $\tau_n(R)$ is the simple graph with vertex set R such that distinct $x, y \in R$ are adjacent if and only if $x^n + y^n \in Z(R)$.

Definition 2.2. Let $\mathbb{F}_q$ be a finite field. The map

$$P_n : \mathbb{F}_q \longrightarrow \mathbb{F}_q, \qquad P_n(a) = a^n,$$

is called the *power map*. A *fiber* of a map $f : X \to Y$ is a set $f^{-1}(y)$, where $y \in Y$. Accordingly, a *power fiber* is a nonempty fiber of P_n. On $\mathbb{F}_q^\times$, every power fiber has order $\gcd(n, q-1)$, because $\mathbb{F}_q^\times$ is cyclic and the restriction of P_n is a group homomorphism.

Every finite commutative ring admits a decomposition

$$R \cong R_1 \times \cdots \times R_t,$$

where each $(R_i, \mathfrak{m}_i)$ is a finite local ring. Moreover,

$$\operatorname{Jac}(R) = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_t, \qquad R/\operatorname{Jac}(R) \cong \prod_{i=1}^{t} \mathbb{F}_{q_i}.$$

See, for example, Ref. [7] for the commutative-ring background used here. An element $x = (x_1, \ldots, x_t)$ of the product R is a zero-divisor if and only if $x_i \in \mathfrak{m}_i$ for some i. Equivalently, its image in at least one residue-field coordinate is zero.

2.2. Orientable genus

The orientable genus $\operatorname{gen}(G)$ of a graph G is the least nonnegative integer h such that G embeds without edge crossings in the orientable surface with h handles. We shall use the following standard facts.

Lemma 2.3 (Classical genus formulas)**.** *For complete and complete bipartite graphs,*

$$\operatorname{gen}(K_r) = \left\lceil \frac{(r-3)(r-4)}{12} \right\rceil, \qquad r \geq 3, \tag{1}$$

$$\operatorname{gen}(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil, \qquad r, s \geq 2. \tag{2}$$

Proof. These are the Ringel–Youngs and Ringel formulas, respectively; see Ref. [8]. □

Lemma 2.4 (Planar obstructions)**.** *A finite graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. In particular, any graph containing K_5 or $K_{3,3}$ as a subgraph is nonplanar.*

Proof. This is Kuratowski's theorem; see Ref. [8]. □

Lemma 2.5 (Euler bound and genus additivity)**.** *For every finite simple graph G with at least three vertices,*

$$|E(G)| \leq 3|V(G)| - 6 + 6\operatorname{gen}(G), \tag{3}$$

and, for pairwise disjoint graphs $G_1, \ldots, G_r$,

$$\operatorname{gen}\left(\dot{\bigcup_{i=1}^{r}} G_i\right) = \sum_{i=1}^{r} \operatorname{gen}(G_i). \tag{4}$$

More generally, genus is additive over the blocks of a graph. In particular, if two graphs meet in exactly one vertex, the genus of their union is the sum of their genera.

Proof. For the inequality, embed G in an orientable surface of genus $\operatorname{gen}(G)$. If necessary, join distinct components by edges drawn inside faces, and then add further edges until a simple cellular triangulation is obtained. Euler's formula applied to the resulting triangulation gives (3), and deleting the added edges preserves the inequality for (G). The additivity assertions are due to Battle, Harary, Kodama, and Youngs; see Ref. [9]. Further background is given in Ref. [8]. □

Motivated by Lemma 2.3, it is convenient to absorb the small exceptional orders in two integer-valued functions. Put

$$\mathcal{K}(u) = \begin{cases} 0, & 1 \leq u \leq 4, \\ \left\lceil \dfrac{(u-3)(u-4)}{12} \right\rceil, & u \geq 5, \end{cases}$$

and

$$\mathcal{B}(u) = \begin{cases} 0, & 1 \leq u \leq 2, \\ \left\lceil \dfrac{(u-2)^2}{4} \right\rceil, & u \geq 3. \end{cases}$$

Thus

$$\mathcal{K}(u) = \operatorname{gen}(K_u), \qquad \mathcal{B}(u) = \operatorname{gen}(K_{u,u}).$$

In particular,

$$\begin{aligned} \mathcal{K}(u) = 0 &\iff u \leq 4, \\ \mathcal{K}(u) = 1 &\iff 5 \leq u \leq 7, \\ \mathcal{B}(u) = 0 &\iff u \leq 2, \\ \mathcal{B}(u) = 1 &\iff u \in \{3, 4\}. \end{aligned}$$

3. The exact genus for finite local rings

Throughout this section, $(R, \mathfrak{m})$ is a finite local ring. We use the notation

$$\begin{aligned} &b = |\mathfrak{m}|, && R/\mathfrak{m} \cong \mathbb{F}_q, \\ &g_n = \gcd(n, q-1), && \delta_n = \frac{q-1}{g_n}, \quad c_n = bg_n. \end{aligned} \tag{5}$$

The case $b = 1$ is precisely the field case.

Lemma 3.1. *If $\mathfrak{m} \neq 0$, then $b = q^e$ for some positive integer e.*

Proof. Since R is finite and local, its maximal ideal is nilpotent. Let s be the least positive integer such that $\mathfrak{m}^s = 0$. Each quotient $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is naturally a vector space over $R/\mathfrak{m} \cong \mathbb{F}_q$: changing a scalar by an element of $\mathfrak{m}$ changes its product with $\mathfrak{m}^i$ by an element of $\mathfrak{m}^{i+1}$. Hence every quotient has order a power of q. Multiplying the orders in the filtration

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \supseteq \mathfrak{m}^s = 0,$$

gives $|\mathfrak{m}| = q^e$ for some $e \geq 1$. □

We recall the local component decomposition in a form suited to the genus calculation.

Theorem 3.2 (AitElhadi–Badawi decomposition)**.** *With the notation in (5), the following hold.*

(i) *If q is even, then*

$$\tau_n(R) \cong K_b \dot{\cup} \delta_n K_{c_n}.$$

(ii) *If q is odd and δ_n is odd, then*

$$\tau_n(R) \cong K_b \dot{\cup} (q-1)bK_1.$$

(iii) *If q is odd and δ_n is even, then*

$$\tau_n(R) \cong K_b \dot{\cup} \frac{\delta_n}{2} K_{c_n,c_n}.$$

Proof. Let $\rho : R \to R/\mathfrak{m} \cong \mathbb{F}_q$ be the residue map. Since $\mathfrak{m} = Z(R)$, for distinct $x, y \in R$ we have

$$x \sim y \iff \rho(x)^n + \rho(y)^n = 0.$$

If $x, y \in \mathfrak{m}$, this equality holds, so $\mathfrak{m}$ induces K_b. If $x \in \mathfrak{m}$ and $y \notin \mathfrak{m}$, then $\rho(x)^n + \rho(y)^n = \rho(y)^n \neq 0$; hence K_b is a component and has no edge to a unit.

Every nonzero residue class has b lifts. By Definition 2.2, the power map on the cyclic group $\mathbb{F}_q^\times$ has kernel of order g_n and image of order δ_n. Thus it has exactly δ_n nonzero power fibers, each containing g_n residue classes and hence $bg_n = c_n$ vertices after lifting to R.

Suppose first that q is even. In this case $-1 = 1$, so two nonzero residues are adjacent precisely when their n-th powers are equal. This remains true for distinct lifts of the same residue, because $2a^n = 0$ in $\mathbb{F}_q$. Consequently, each lifted power fiber is a clique K_{c_n}, different fibers have no edges between them, and there are δ_n such components. This proves (i).

Now suppose that q is odd. There are no edges among two distinct lifts whose residues belong to the same power fiber, because their power sum is $2z \neq 0$, where $z \in \mathbb{F}_q^\times$ is the common power value. Two distinct power fibers are completely joined exactly when their power values are negatives of one another. The image of the power map is cyclic of order δ_n, and it contains -1 if and only if δ_n is even. If δ_n is odd, no two unit vertices are adjacent, which gives (ii). If δ_n is even, negation partitions the δ_n power fibers into $\delta_n/2$ pairs, and each pair lifts to a component K_{c_n,c_n}, proving (iii). These components, together with K_b, account for all $|R| = bq$ vertices. The same decomposition was obtained in Ref. [5]. □

Theorem 3.3 (Local genus formula)**.** *For every finite local ring R and every $n \geq 1$,*

$$\operatorname{gen}(\tau_n(R)) = \mathcal{K}(b) + \begin{cases} \delta_n \mathcal{K}(c_n), & q \text{ even}, \\ 0, & q, \delta_n \text{ odd}, \\ \dfrac{\delta_n}{2}\mathcal{B}(c_n), & q \text{ odd}, \ \delta_n \text{ even}. \end{cases} \tag{6}$$

Proof. Apply Theorem 3.2. By Lemma 2.5, genus is additive over components, and isolated vertices contribute zero. If q is even, the components are K_b and δ_n copies of K_{c_n}, so Lemma 2.3 gives

$$\operatorname{gen}(\tau_n(R)) = \mathcal{K}(b) + \delta_n \mathcal{K}(c_n).$$

If q and δ_n are odd, all units are isolated, leaving only the component K_b, whose genus is $\mathcal{K}(b)$. Finally, if q is odd and δ_n is even, there are $\delta_n/2$ components K_{c_n,c_n}; each has genus $\mathcal{B}(c_n)$. Adding their genera to that of K_b yields the third line of (6). □

Corollary 3.4 (Local planarity criterion)**.** *The graph $\tau_n(R)$ is planar if and only if the applicable line below holds:*

case	*necessary and sufficient conditions*
q even	$b \leq 4$ *and* $c_n = bg_n \leq 4$,
q odd, δ_n odd	$b \leq 4$,
q odd, δ_n even	$b \leq 4$ *and* $c_n = bg_n \leq 2$.

(7)

Proof. The genus in Theorem 3.3 is a sum of nonnegative integers. It is zero precisely when every summand is zero. Now K_u is planar exactly when $u \leq 4$, and $K_{u,u}$ is planar exactly when $u \leq 2$. Substitution gives the three lines of (7). □

The next result translates the equation $\operatorname{gen}(\tau_n(R)) = 1$ into a complete ring-theoretic list.

Theorem 3.5 (Toroidal local n-total graphs)**.** *Let $(R, \mathfrak{m})$ be a finite local ring. Then $\tau_n(R)$ is toroidal if and only if one of the following holds.*

(1) $R \cong \mathbb{F}_8$ *and* $7 \mid n$.
(2) $R \cong \mathbb{F}_7$ *and* $\gcd(n, 6) = 3$.
(3) $R \cong \mathbb{F}_9$ *and* $\gcd(n, 8) = 4$.
(4) $R \cong \mathbb{Z}_9$ *or* $R \cong \mathbb{F}_3[x]/(x^2)$, *and n is odd.*
(5) $R \cong \mathbb{Z}_{25}$ *or* $R \cong \mathbb{F}_5[x]/(x^2)$, *and* $4 \mid n$.
(6) $R \cong \mathbb{Z}_{49}$ *or* $R \cong \mathbb{F}_7[x]/(x^2)$, *and n is even.*

Proof. Assume first that q is even. By Theorem 3.3,

$$\mathcal{K}(b) + \delta_n \mathcal{K}(c_n) = 1.$$

First let $b = 1$, so that $R \cong \mathbb{F}_q$. Then $\mathcal{K}(b) = 0$, and the displayed equation becomes $\delta_n \mathcal{K}(g_n) = 1$. Hence $\delta_n = 1$ and $g_n \in \{5, 6, 7\}$. Since $g_n = q - 1$, the only even prime power among $6, 7, 8$ is $q = 8$. Thus $R \cong \mathbb{F}_8$, and $\delta_n = 1$ is equivalent to $7 \mid n$.

Now suppose $\mathfrak{m} \neq 0$, so $b > 1$. By Lemma 3.1, $b = q^e$ for some $e \geq 1$. If $b \geq 8$, then $\mathcal{K}(b) \geq 2$, contradicting the displayed equation. Hence $b \in \{2, 4\}$ and $\mathcal{K}(b) = 0$. We would then need $\delta_n = 1$ and $c_n = b(q-1) \in \{5, 6, 7\}$. If $q = 2$, then $c_n = b \in \{2, 4\}$; if $q \geq 4$, then $c_n \geq q(q-1) \geq 12$. Both

alternatives are impossible, so no nonfield ring occurs when q is even.

Suppose next that q is odd and δ_n is odd. Then $\operatorname{gen}(\tau_n(R)) = \mathcal{K}(b)$. Hence $b \in \{5, 6, 7\}$. By Lemma 3.1, the only possibilities are $(q, b) = (5, 5)$ and $(q, b) = (7, 7)$. A commutative local ring with $|\mathfrak{m}| = |R/\mathfrak{m}| = p$ has order p^2 and is isomorphic to either $\mathbb{Z}_{p^2}$ or $\mathbb{F}_p[x]/(x^2)$. Indeed, characteristic p^2 gives the first ring, whereas in characteristic p, the ring is a two-dimensional local $\mathbb{F}_p$-algebra whose one-dimensional maximal ideal has square zero. For $q = 5$, the number $4/\gcd(n, 4)$ is odd exactly when $4 \mid n$. For $q = 7$, the number $6/\gcd(n, 6)$ is odd exactly when n is even. This gives items (5) and (6).

It remains to suppose that q is odd and δ_n is even. Equation (6) becomes

$$\mathcal{K}(b) + \frac{\delta_n}{2}\mathcal{B}(c_n) = 1.$$

If $\mathcal{K}(b) \geq 1$, then $b \geq 5$, so $c_n \geq 5$ and $\mathcal{B}(c_n) > 0$; the sum is then larger than one. Consequently $\mathcal{K}(b) = 0$, and the second term must be one. Thus

$$\delta_n = 2, \qquad c_n \in \{3, 4\}.$$

Since q is odd, Lemma 3.1 shows that $b \leq 4$ forces $b = 1$ or $b = 3$.

If $b = 1$, then $g_n = c_n \in \{3, 4\}$ and $q - 1 = \delta_n g_n \in \{6, 8\}$. Hence $R \cong \mathbb{F}_7$ with $\gcd(n, 6) = 3$, or $R \cong \mathbb{F}_9$ with $\gcd(n, 8) = 4$. If $b = 3$, then $q = 3$, $g_n = 1$, and $\delta_n = 2$, which is equivalent to n being odd. The two local rings with $(q, b) = (3, 3)$ are $\mathbb{Z}_9$ and $\mathbb{F}_3[x]/(x^2)$. This gives items (2)–(4), and the proof is complete. □

Example 3.6. Let $R = \mathbb{Z}_9$. Here $b = q = 3$. If n is odd, then $g_n = 1$, $\delta_n = 2$, and

$$\tau_n(\mathbb{Z}_9) \cong K_3 \dot{\cup} K_{3,3}, \qquad \operatorname{gen}(\tau_n(\mathbb{Z}_9)) = 1.$$

If n is even, then $g_n = 2$, $\delta_n = 1$, and

$$\tau_n(\mathbb{Z}_9) \cong K_3 \dot{\cup} 6K_1, \qquad \operatorname{gen}(\tau_n(\mathbb{Z}_9)) = 0.$$

Thus the same ring alternates between toroidal and planar n-total graphs, as illustrated in Figure 1.

4. Products of finite local rings

We now consider

$$R \cong R_1 \times \cdots \times R_t, \qquad t \geq 2, \tag{8}$$

where each R_i is a finite local ring. The following clique is present for every exponent and is the basic topological obstruction in the product case.

Lemma 4.1 (Coordinate-zero clique)**.** *For every $i \in \{1, \ldots, t\}$, the graph $\tau_n(R)$ contains a complete subgraph of order*

$$M_i = \prod_{j \neq i} |R_j| = \frac{|R|}{|R_i|}.$$

Proof. Let

$$C_i = R_1 \times \cdots \times R_{i-1} \times \{0\} \times R_{i+1} \times \cdots \times R_t.$$

Take distinct $x, y \in C_i$. The i-th coordinate of $x^n + y^n$ is $0^n + 0^n = 0$. Hence $x^n + y^n$ is a zero-divisor of the product ring, so x and y are adjacent. Thus C_i induces K_{M_i}. □

The next observation permits an exact analysis of the small product candidates. Write

$$Q = R/\operatorname{Jac}(R) \cong \prod_{i=1}^{t} \mathbb{F}_{q_i}, \qquad s = |\operatorname{Jac}(R)|,$$

and let $\pi : R \longrightarrow Q$ be the quotient map.

Lemma 4.2 (Residue-fiber rule)**.** *Every fiber of π has order s. If $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are distinct elements of Q, then all the edges between $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$ are present if*

$$\alpha_i^n + \beta_i^n = 0 \quad \text{for some } i,$$

and no such edge is present otherwise. Moreover, $\pi^{-1}(\alpha)$ induces K_s if $2\alpha_i^n = 0$ for some i, and it induces sK_1 otherwise.

Proof. The fibers are the additive cosets of $\operatorname{Jac}(R)$, so each has order s. An element of the finite product R is a zero-divisor if and only if its image in at least one residue-field coordinate is zero. Consequently, for $x \in \pi^{-1}(\alpha)$ and $y \in \pi^{-1}(\beta)$,

$$x^n + y^n \in Z(R) \quad \Longleftrightarrow \quad \alpha_i^n + \beta_i^n = 0 \text{ for some } i.$$

This proves the assertion for distinct residue fibers. Taking $\alpha = \beta$ gives the assertion about the subgraph induced by a single fiber; the graph remains simple because only distinct lifts are considered. □

Theorem 4.3 (General product lower bound)**.** *Let $R = \prod_{i=1}^{t} R_i$ be a product of finite local rings, where $t \geq 2$. Then*

$$\operatorname{gen}(\tau_n(R)) \geq \max_{1 \leq i \leq t} \mathcal{K}\left(\frac{|R|}{|R_i|}\right). \tag{9}$$

In particular:

(i) *if $\tau_n(R)$ is planar, then $|R|/|R_i| \leq 4$ for every i;*
(ii) *if $\operatorname{gen}(\tau_n(R)) \leq 1$, then $|R|/|R_i| \leq 7$ for every i.*

Proof. For any $i \in \{1, \ldots, t\}$, Lemma 4.1 shows that K_{M_i} is a subgraph of $\tau_n(R)$. Genus is monotone under taking subgraphs, so $\operatorname{gen}(\tau_n(R)) \geq \operatorname{gen}(K_{M_i}) = \mathcal{K}(M_i)$ for every i. This proves (9). A complete graph is planar only up to order four and has genus at most one only up to order seven, giving (i) and (ii). □

Corollary 4.4 (Order bound at prescribed genus)**.** *Let h be a nonnegative integer. Suppose R is nonlocal and $\operatorname{gen}(\tau_n(R)) \leq h$. Put*

$$B_h = \left\lfloor \frac{7 + \sqrt{1 + 48h}}{2} \right\rfloor.$$

Then

$$|R| \leq B_h^2.$$

Consequently, $|R| \leq 16$ in the planar case and $|R| \leq 49$ when the genus is at most one.



(a) n odd: $\tau_n(\mathbb{Z}_9) \cong K_3 \dot\cup K_{3,3}$. **(b)** n even: $\tau_n(\mathbb{Z}_9) \cong K_3 \dot\cup 6K_1$.

Figure 1: The parity of the exponent changes the n-total graph of $\mathbb{Z}_9$ from a toroidal graph to a planar graph.

Proof. Relabel the factors so that $|R_1| \le |R_i|$ for any $2 \le i \le t$. Since $t \ge 2$, the product of all factors other than R_1 is at least $|R_1|$; hence $M_1 = |R|/|R_1| \ge \sqrt{|R|}$. By Theorem 4.3, $\mathrm{gen}(K_{M_1}) \le h$. If $M_1 \ge 5$, the complete-graph formula implies

$$\frac{(M_1 - 3)(M_1 - 4)}{12} \le h,$$

and solving this quadratic inequality gives

$$M_1 \le \frac{7 + \sqrt{1 + 48h}}{2},$$

so $M_1 \le B_h$. The same conclusion holds when $M_1 \le 4$, because $B_h \ge B_0 = 4$. Therefore $|R| \le M_1^2 \le B_h^2$. Finally, $B_0 = 4$ and $B_1 = 7$. □

The lower bound makes both low-genus classifications finite.

Theorem 4.5 (Complete nonlocal planarity classification)**.** *Let R be a finite nonlocal commutative ring and let $n \ge 1$. Then $\tau_n(R)$ is planar if and only if one of the following holds:*

(1) $R \cong \mathbb{F}_2 \times \mathbb{F}_2$*, with arbitrary n;*
(2) $R \cong \mathbb{F}_2 \times \mathbb{F}_3 \cong \mathbb{Z}_6$*, with arbitrary n;*
(3) $R \cong \mathbb{F}_3 \times \mathbb{F}_3$*, and n is even;*
(4) $R \cong \mathbb{F}_3 \times \mathbb{F}_4$*, and* $\gcd(n, 6) = 2$.

Proof. We first prove sufficiency. On $\mathbb{F}_2 \times \mathbb{F}_2$, taking a positive power does not change either coordinate, and two distinct vertices are adjacent exactly when they agree in one coordinate. Hence $\tau_n(\mathbb{F}_2 \times \mathbb{F}_2) \cong C_4$.

For $\mathbb{F}_2 \times \mathbb{F}_3$, each of the two sets with fixed first coordinate induces a triangle. If n is odd, the edges between the two triangles pair $(0, a)$ with $(1, -a)$, so the graph is a triangular prism. If n is even, the only edge between the two triangles joins $(0, 0)$ to $(1, 0)$. Both graphs are planar.

Suppose $R = \mathbb{F}_3 \times \mathbb{F}_3$ and n is even. In $\mathbb{F}_3$, the n-th power is zero at zero and is one at each nonzero element. Consequently, the graph consists of the triangle whose first coordinate is zero and the triangle whose second coordinate is zero, meeting at $(0, 0)$, together with four isolated vertices. It is planar.

Finally, let $R = \mathbb{F}_3 \times \mathbb{F}_4$ and $\gcd(n, 6) = 2$. The n-th power on $\mathbb{F}_3$ again takes the values zero and one, while the power map on $\mathbb{F}_4^\times$ is a permutation because $3 \nmid n$. Thus the four vertices with first coordinate zero induce K_4. For each $b \in \mathbb{F}_4$, the three vertices with second coordinate b induce K_3, and this triangle meets the K_4 only at $(0, b)$. There are no other edges. The resulting block graph is planar.

The planar structures used in these four sufficiency arguments are displayed schematically in Figure 2. In panel (d), every triangle meets the central K_4 in exactly one cut vertex; hence the drawing also makes the block decomposition used in the proof explicit.

For necessity, Theorem 4.3 gives $1 < t \le 3$. If $t = 2$, both factors belong to

$$\{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2[x]/(x^2)\}.$$

If $t = 3$, all factors are $\mathbb{F}_2$. Let

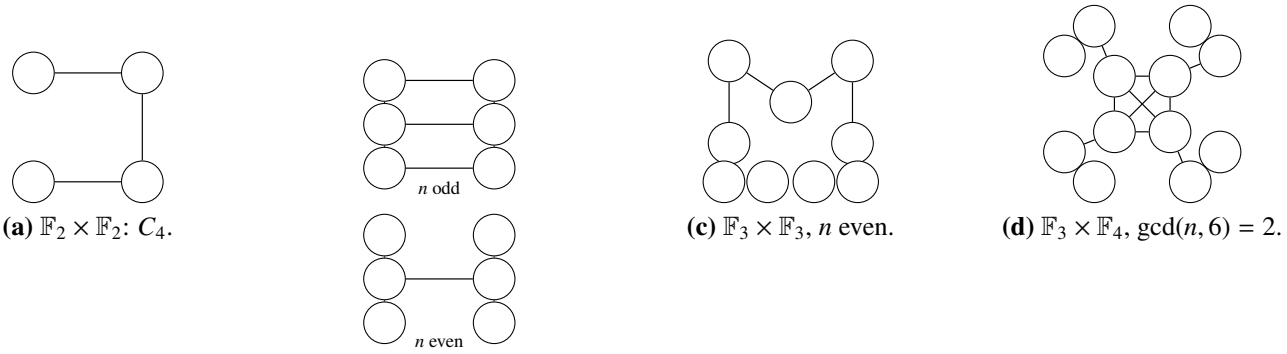
$$T \in \left\{\mathbb{Z}_4, \mathbb{F}_2[x]/(x^2)\right\}.$$

The residue field of T is $\mathbb{F}_2$ and its maximal ideal has order two. Lemma 4.2 gives the following obstructions. If $R = T \times T'$, with both T, T' of this form, then the lifts of the two residue vertices having a fixed first coordinate form K_8. If the other factor is $\mathbb{F}_3$ or $\mathbb{F}_4$, the same construction gives K_6 or K_8, respectively. If the other factor is $\mathbb{F}_2$, the graph has four residue fibers, each inducing K_2, with complete joins between consecutive fibers in a four-cycle. It has eight vertices and $4 + 4 \cdot 4 = 20 > 3 \cdot 8 - 6$ edges, and hence is nonplanar. Therefore no candidate with a nonfield local factor can occur.

It remains to examine products of the fields $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4$. The two positive products involving only $\mathbb{F}_2$ and $\mathbb{F}_3$ were already handled above. For $\mathbb{F}_2 \times \mathbb{F}_4$, if $3 \nmid n$, the graph has two K_4's, say on $u_1, \dots, u_4$ and $v_1, \dots, v_4$, together with the matching $u_i v_i$. It contains a subdivision of K_5: take branch vertices u_1, u_2, u_3, u_4, v_1, use the edge $v_1 u_1$, and use the internally disjoint paths $v_1 v_i u_i$ for $i = 2, 3, 4$. If $3 \mid n$, all three nonzero elements of $\mathbb{F}_4$ have the same n-th power, and the six vertices with nonzero second coordinate form K_6. Thus this product is never planar.

For $\mathbb{F}_3 \times \mathbb{F}_3$, odd n makes the two rows whose first coordinates are 1 and -1 the two parts of a $K_{3,3}$. Hence planarity forces n to be even. For $\mathbb{F}_3 \times \mathbb{F}_4$, if $3 \mid n$, the nine vertices with nonzero second coordinate form K_9. If $3 \nmid n$ and n is odd, the rows with first coordinates 1 and -1 form $K_{4,4}$. The only remaining condition is therefore n even and $3 \nmid n$, equivalently $\gcd(n, 6) = 2$.

For $\mathbb{F}_4 \times \mathbb{F}_4$, divisibility of n by three produces a K_{12} on the vertices with nonzero first coordinate. If $3 \nmid n$, the graph consists of four row cliques K_4 and four column cliques K_4. These edge sets are disjoint, so the graph has 16 vertices and $8\binom{4}{2} = 48 > 3 \cdot 16 - 6$ edges. Finally, in $\mathbb{F}_2^3$ two vertices fail to be adjacent exactly when they are coordinatewise complements.



(a) $\mathbb{F}_2 \times \mathbb{F}_2$: C_4. **(b)** $\mathbb{F}_2 \times \mathbb{F}_3$. **(c)** $\mathbb{F}_3 \times \mathbb{F}_3$, n even. **(d)** $\mathbb{F}_3 \times \mathbb{F}_4$, $\gcd(n, 6) = 2$.

Figure 2: Planar realizations for the four families in Theorem 4.5. Panel (b) shows the two parity-dependent structures; isolated vertices in panel (c) are displayed below the two triangles.

The graph is therefore K_8 with a perfect matching removed and has $28 - 4 = 24 > 3 \cdot 8 - 6$ edges. These obstructions eliminate every remaining candidate and complete the proof. □

Corollary 4.6 (Complete finite-ring planarity criterion)**.** *Let R be any finite commutative ring. Then $\tau_n(R)$ is planar if and only if either R is local and satisfies the applicable line of (7), or R is nonlocal and satisfies one of the four conditions in Theorem 4.5.*

Proof. Every finite commutative ring is either local or has a decomposition into at least two finite local factors. In the local case, Corollary 3.4 gives necessary and sufficient conditions, separated according to the parity of q and δ_n. In the nonlocal case, Theorem 4.5 gives the complete list of planar possibilities. These two cases are mutually exclusive and exhaust all finite commutative rings, proving the equivalence. □

Theorem 4.7 (Complete nonlocal toroidality classification)**.** *Let R be a finite nonlocal commutative ring and let $n \geq 1$. Then* $\mathrm{gen}(\tau_n(R)) = 1$ *if and only if one of the following holds:*

(1) $R \cong \mathbb{F}_2 \times \mathbb{F}_4$, *with arbitrary n;*
(2) $R \cong \mathbb{F}_2 \times \mathbb{Z}_4$ *or* $R \cong \mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2)$, *with arbitrary n;*
(3) $R \cong \mathbb{F}_3 \times \mathbb{F}_3$, *and n is odd;*
(4) $R \cong \mathbb{F}_3 \times \mathbb{F}_5$, *and $4 \mid n$;*
(5) $R \cong \mathbb{F}_3 \times \mathbb{F}_7$, *and n is even;*
(6) $R \cong \mathbb{F}_2^3$, *with arbitrary n.*

Proof. We begin with the positive cases. Let X denote K_8 with a perfect matching removed. Equivalently, $X \cong K_{2,2,2,2}$, and $\mathrm{gen}(X) = 1$; see the standard torus embedding used in Ref. [6]. For $R = \mathbb{F}_2^3$, the exponent has no effect, and two vertices are nonadjacent precisely when they are coordinatewise complements. Thus $\tau_n(R) \cong X$.

Let T be either $\mathbb{Z}_4$ or $\mathbb{F}_2[x]/(x^2)$. The graph of $\mathbb{F}_2 \times T$ consists of four copies of K_2 arranged on a four-cycle, with complete joins between consecutive copies. It has eight vertices and twenty edges, so it is nonplanar by (3). More explicitly, the four copies correspond to the four elements of $\mathbb{F}_2 \times (T/\mathfrak{m}_T) \cong \mathbb{F}_2^2$. The only missing edges between different copies join opposite vertices of the quotient four-cycle. Thus the complement is the disjoint union of two copies of $K_{2,2}$, and in particular it contains a perfect matching. After relabeling the vertices along such a matching, the original graph is a subgraph of $X = K_8 - M$. Therefore its genus is at most one; nonplanarity makes the genus exactly one.

Consider next $\mathbb{F}_2 \times \mathbb{F}_4$. If $3 \nmid n$, the power map on $\mathbb{F}_4$ is a permutation and the graph is the Cartesian product $K_2 \,\square\, K_4$. Here $\square$ denotes the Cartesian product; see Ref. [10] for recent structural relations among standard graph products. Using row labels $u_1, \ldots, u_4$ and $v_1, \ldots, v_4$, the nonedges $u_i v_{i+1}$, with subscripts read modulo four, form a perfect matching in the complement. Hence the graph is a subgraph of X, while the K_5-subdivision exhibited in the proof of Theorem 4.5 shows that it is nonplanar. Hence its genus is one.

Suppose $3 \mid n$. Write the two row cliques as

$$\{u_0, u_1, u_2, u_3\}, \qquad \{v_0, v_1, v_2, v_3\},$$

where the subscript zero corresponds to the zero of $\mathbb{F}_4$. The six nonzero-subscript vertices induce K_6, the vertex u_0 is adjacent to u_1, u_2, u_3, v_0, and v_0 is adjacent to v_1, v_2, v_3, u_0. This graph is obtained from a toroidal embedding of K_7 by splitting one vertex into the adjacent vertices u_0, v_0. Indeed, at the vertex to be split, the six incident edges occur in a cyclic order. Relabel the six neighbors so that the first three consecutive edges are assigned to u_1, u_2, u_3 and the remaining three to v_1, v_2, v_3. Replace the vertex, inside a sufficiently small disk, by the edge $u_0 v_0$, attaching the first three edges to u_0 and the other three to v_0. This local modification creates no crossings and produces exactly the graph described above. Thus the graph is toroidal; since it contains K_6, its genus is exactly one. The two exponent-dependent structures are shown in Figure 3.

If $R = \mathbb{F}_3 \times \mathbb{F}_3$ and n is odd, raising to the n-th power fixes every element of $\mathbb{F}_3$. Hence $\tau_n(R) = \tau_1(R)$, which is toroidal by Ref. [6]; it is nonplanar because the rows with first coordinates 1 and -1 form $K_{3,3}$.

Let $R = \mathbb{F}_3 \times \mathbb{F}_5$ and $4 \mid n$. In both fields, every nonzero element has n-th power one. Thus the graph consists of a K_5 on the row with first coordinate zero and a K_3 on the column with second coordinate zero, meeting at $(0, 0)$, together with isolated vertices. Block additivity gives genus one. Similarly, if $R = \mathbb{F}_3 \times \mathbb{F}_7$ and n is even, then -1 is not an n-th power in $\mathbb{F}_7$.



(a) $3 \nmid n$: $\tau_n(\mathbb{F}_2 \times \mathbb{F}_4) \cong K_2 \square K_4$.

(b) $3 \mid n$: a K_6 core with the split vertices u_0, v_0.

Figure 3: The two structures of $\tau_n(\mathbb{F}_2 \times \mathbb{F}_4)$. The outlined groups in (a) denote complete subgraphs K_4, joined by the displayed matching. Medium-gray edges in (b) form the complete core K_6; the heavier edges record the vertex split used in the toroidal embedding.

The graph consists of a K_7 and a K_3 meeting at $(0, 0)$, together with isolated vertices, and again has genus one.

For necessity, Theorem 4.3 gives $t \leq 3$. For $t = 2$, both factors lie in

$$\{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2[x]/(x^2), \mathbb{F}_5, \mathbb{F}_7\}.$$

Ordering three factor orders leaves only $\mathbb{F}_2^3$ and $\mathbb{F}_2^2 \times \mathbb{F}_3$. First suppose that a factor is one of the two nonfield rings T above. The products $T \times \mathbb{F}_2$ have already been retained. A product of two such nonfield factors contains K_8. If the other factor is $\mathbb{F}_q$, where $q \in \{3, 4, 5, 7\}$, the two sets having a fixed residue in the T-coordinate are disjoint cliques of order $2q$. Their genera have sum at least two. Hence none of these products is toroidal.

We now consider products of fields. If one factor is $\mathbb{F}_2$ or $\mathbb{F}_4$ and the other is $\mathbb{F}_5$ or $\mathbb{F}_7$, fixing a coordinate in the characteristic-two factor gives at least two disjoint copies of K_5 or K_7. If both factors belong to $\{\mathbb{F}_5, \mathbb{F}_7\}$, the zero row and zero column give two complete blocks of genus one meeting at $(0, 0)$. In both situations block additivity forces genus at least two.

For $\mathbb{F}_4 \times \mathbb{F}_4$, if $3 \mid n$, the twelve vertices with nonzero first coordinate form K_{12}. If $3 \nmid n$, split the four first-coordinate values into two pairs. Each pair of rows induces a copy of $K_2 \square K_4$, and the two copies are vertex-disjoint. Each has genus one, so the whole graph has genus at least two.

For $\mathbb{F}_3 \times \mathbb{F}_4$, the case n even and $3 \nmid n$ is planar by Theorem 4.5, while $3 \mid n$ produces a K_9. In the remaining case n is odd and $3 \nmid n$. The rows with first coordinates 1 and -1 give the two parts of a $K_{4,4}$. Choose $b_0 \in \mathbb{F}_4$ and use $(0, b_0)$ as a fifth branch vertex on the first side. Its path to $(-1, b_0)$ is an edge, and for $b \neq b_0$ use the paths

$$(0, b_0) - (0, b) - (-1, b).$$

Together with the direct edges from the row having first coordinate 1, these paths form a subdivision of $K_{5,4}$. Since $\operatorname{gen}(K_{5,4}) = 2$, this graph is not toroidal.

For $\mathbb{F}_3 \times \mathbb{F}_5$, odd n yields a $K_{5,5}$ between the rows with first coordinates 1 and -1. If $n \equiv 2 \pmod 4$, the two opposite power fibers in $\mathbb{F}_5^\times$, each of order two, lift across the three first coordinates to a $K_{6,6}$. These graphs have genus greater than one. The remaining exponents are precisely those divisible by four.

For $\mathbb{F}_3 \times \mathbb{F}_7$, odd n yields $K_{7,7}$, while every even exponent was retained above. The products $\mathbb{F}_2 \times \mathbb{F}_2$ and $\mathbb{F}_2 \times \mathbb{F}_3$ are always planar, and $\mathbb{F}_3 \times \mathbb{F}_3$ is planar for even n. This exhausts all two-factor candidates.

Finally, $\mathbb{F}_2^3$ was retained above. In $\mathbb{F}_2^2 \times \mathbb{F}_3$, the edges forced by the first two coordinates contain four copies of K_3 arranged on a four-cycle with complete joins between consecutive copies. This subgraph has twelve vertices and

$$4\binom{3}{2} + 4 \cdot 3^2 = 48 > 3 \cdot 12,$$

edges, contradicting the toroidal Euler bound. Hence it has genus at least two, and the proof is complete. □

Combining Theorems 3.5 and 4.7, one obtains the following corollary.

Corollary 4.8 (Complete finite-ring toroidality criterion)**.** *Let R be any finite commutative ring. Then* $\operatorname{gen}(\tau_n(R)) = 1$ *if and only if one of the following two cases holds.*

(I) ***Local rings.*** *The ring R is local and satisfies one of the following conditions:*
  (1) $R \cong \mathbb{F}_8$ *and* $7 \mid n$;
  (2) $R \cong \mathbb{F}_7$ *and* $\gcd(n, 6) = 3$;
  (3) $R \cong \mathbb{F}_9$ *and* $\gcd(n, 8) = 4$;
  (4) $R \cong \mathbb{Z}_9$ *or* $R \cong \mathbb{F}_3[x]/(x^2)$, *and n is odd;*
  (5) $R \cong \mathbb{Z}_{25}$ *or* $R \cong \mathbb{F}_5[x]/(x^2)$, *and* $4 \mid n$;
  (6) $R \cong \mathbb{Z}_{49}$ *or* $R \cong \mathbb{F}_7[x]/(x^2)$, *and n is even.*

(II) ***Nonlocal rings.*** *The ring R is nonlocal and satisfies one of the following conditions:*
  (1) $R \cong \mathbb{F}_2 \times \mathbb{F}_4$, *with arbitrary n;*
  (2) $R \cong \mathbb{F}_2 \times \mathbb{Z}_4$ *or* $R \cong \mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2)$, *with arbitrary n;*
  (3) $R \cong \mathbb{F}_3 \times \mathbb{F}_3$, *and n is odd;*
  (4) $R \cong \mathbb{F}_3 \times \mathbb{F}_5$, *and* $4 \mid n$;
  (5) $R \cong \mathbb{F}_3 \times \mathbb{F}_7$, *and n is even;*
  (6) $R \cong \mathbb{F}_2^3$, *with arbitrary n.*

Proof. The local alternatives are exactly those of Theorem 3.5; the nonlocal alternatives are exactly those of Theorem 4.7. Since every finite ring is either local or nonlocal, the two lists give both necessity and sufficiency. □

5. Periodicity of the genus

Theorem 5.1 (A genus period)**.** *Suppose* $R \cong \prod_{i=1}^{t} R_i$*, where* $R_i/\mathfrak{m}_i \cong \mathbb{F}_{q_i}$*, and put* $L(R) = \operatorname{lcm}(q_1 - 1, \ldots, q_t - 1)$*. Then, for every* $n \geq 1$*,*

$$\tau_{n+L(R)}(R) = \tau_n(R),$$
$$\operatorname{gen}(\tau_{n+L(R)}(R)) = \operatorname{gen}(\tau_n(R)).$$

Proof. In $\mathbb{F}_{q_i}$, zero remains zero and every nonzero a satisfies $a^{L(R)} = 1$. Hence $x_i^{n+L(R)} + \mathfrak{m}_i = x_i^n + \mathfrak{m}_i$ for every $x_i \in R_i$. Thus the two power sums vanish in exactly the same residue coordinates, so the labeled graphs, and therefore their genera, coincide. □

Thus planar and toroidal exponents are unions of residue classes modulo $L(R)$. This period need not be least, since nonisomorphic graphs may have the same genus.

Remark 5.2 (The ordinary case $n = 1$). Here $\tau_1(R)$ is the ordinary total graph. Since $g_1 = 1$ and $\delta_1 = q-1$, Theorem 3.2 gives qK_b for even q, and $K_b \dot{\cup} ((q-1)/2)K_{b,b}$ for odd q. Thus our genus formula and classifications recover the results of Ref. [6].

6. Conclusion

We obtained an exact local genus formula and complete planar and toroidal classifications for all finite commutative rings, with periodic dependence on the exponent. Exact higher genus for nonlocal products, the least genus period, and nonorientable genus remain open.

Data availability

This is a theoretical study. No datasets were generated or analysed, and therefore no data repository link is applicable.

Declaration of competing interest

The author declares that there are no competing interests.

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