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Degenerate hybrid special polynomials associated with Fubini polynomials: properties and applications

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Abstract

In this article, we introduce degenerate multivariate Hermite–based Fubini polynomials by convoluting three-variable degenerate Hermite and Fubini polynomials. First, we derive the generating function explicitly, specify the admissible parameter domain, and carry out a thorough study of its convergence behaviour, locating every singular point. Further, using operational formalism, we derive the quasi-monomial characteristics, including the differential equation satisfied by these polynomials. Algebraically, we derive a wide variety of formulae: series expansions, shifts in each variable separately, addition theorems, and recurrence relations. Many previously discovered polynomial families arise by specialization or limiting cases of the extra parameters, which provides further evidence of the generality of our approach. Along with the theoretical part, we carry out numerical computations of zeros for all degrees up to twenty-nine for several sets of parameters. Our results are presented graphically using zeros plotted on the complex plane, three-dimensional zero-trajectory plots, and tabulated numerical approximations; all of these point to conjugate symmetry properties of the roots and their systematic movement away from the origin.

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1. Introduction

Hermite numbers and polynomials are among the most extensively studied objects in the realm of special functions. Their significance ranges from quantum mechanics and probability theory to approximation theory and many other domains of mathematical analysis. One way to introduce Hermite numbers and polynomials into consideration is through the following two generating equations:

$$e^{-\tau^2} = \sum_{n=0}^{\infty} \mathcal{H}_n \frac{\tau^n}{n!}, \tag{1.1}$$

and

$$e^{2x_1\tau-\tau^2} = \sum_{n=0}^{\infty} \mathcal{H}_n(x_1) \frac{\tau^n}{n!}. \tag{1.2}$$

It follows immediately from these definitions that the Her-

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mite numbers arise as the special case

$$\mathcal{H}_n = \mathcal{H}_n(0). \tag{1.3}$$

In view of the profoundness of the algebraic nature of Hermite numbers and polynomials, these entities have been subjected to rigorous investigation in multiple branches of mathematics; for instance, see [1–7]. It is interesting to observe that many physical models based on second-order differential equations admit solutions in terms of Hermite polynomials.

Upon extending from one variable to three variables, one encounters the trivariate Hermite polynomials $\mathcal{H}_n(x_1, x_2, x_3)$ which were introduced in the literature through the generating function (refer to [8])

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x_1, x_2, x_3)\frac{\tau^n}{n!} = e^{x_1\tau + x_2\tau^2 + x_3\tau^3}. \tag{1.4}$$

The operational character of these polynomials is captured by a set of partial differential identities. Specifically,

$$\frac{\partial}{\partial x_2}\mathcal{H}_n(x_1, x_2, x_3) = \frac{\partial^2}{\partial x_1^2}\mathcal{H}_n(x_1, x_2, x_3), \tag{1.5}$$

$$\frac{\partial}{\partial x_3}\mathcal{H}_n(x_1, x_2, x_3) = \frac{\partial^3}{\partial x_1^3}\mathcal{H}_n(x_1, x_2, x_3), \tag{1.6}$$

together with the initial condition

$$\mathcal{H}_n(x_1, 0, 0) = x_1^n. \tag{1.7}$$

As a consequence, the classical Hermite polynomials can be recovered from the identity

$$\mathcal{H}_n(2x_1, -1, 0) = \mathcal{H}_n(x_1). \tag{1.8}$$

Polynomial families of this type have gained much research interest due to the extensive connection between them and enumerative combinatorics, differential equation theory, and mathematical physics; examples of the results may be found in [9–15]. On the other hand, degenerate versions of the polynomials of Bernoulli, Euler, Genocchi, and tangent have been investigated in an increasing number of studies, along with several applications [16–19].

Another new direction of research is related to the differential equations associated with the generating functions of the aforementioned polynomials, and zero distribution of these polynomials in the complex plane. Advances in both directions have increased the knowledge about the analytic properties of these functions and made them useful in various computational and theoretical tasks [18, 20–23].

Building on these advances, Hwang *et al.* [24] put forward the trivariate Hermite Kampé de Fériet polynomials, written $\mathcal{H}_n(x_1, x_2, x_3 \mid \sigma)$, by means of the generating function:

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x_1, x_2, x_3 \mid \sigma)\frac{\tau^n}{n!} = (1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}. \tag{1.9}$$

Since

$$(1+\sigma)^{\frac{\tau}{\sigma}} \to e^{\tau} \quad \text{as } \sigma \to 0, \tag{1.10}$$

one sees at once that letting $\sigma \to 0$ recovers the classical trivariate Hermite generating relation.

If, in addition, $x_3 = 0$, then (1.9) collapses to the generating function of the two-variable Hermite polynomials $\mathcal{H}_n(x_1, x_2)$, a family intimately tied to solutions of the heat equation [22]:

$$\frac{\partial}{\partial x_2}\mathcal{H}_n(x_1, x_2) = \frac{\partial^2}{\partial x_1^2}\mathcal{H}_n(x_1, x_2), \qquad \mathcal{H}_n(x_1, 0) = x_1^n. \tag{1.11}$$

One of the current areas of research is the study of degeneracies of well-known combinatorial series. The degenerate objects provide a natural interpolation between the known combinatorial objects and their deformations in one parameter. In this context, degenerate Stirling numbers play an essential role in providing the combinatorial structure behind many families of degenerate polynomials.

Throughout this paper we set

$$L = \frac{\log(1+\sigma)}{\sigma}, \tag{1.12}$$

with the understanding that $\sigma \in (-1, 0) \cup (0, \infty)$ in the real case (or, more generally, $\sigma \in \mathbb{C} \setminus \{0\}$ with $(1+\sigma) \notin (-\infty, 0]$ when a standard branch of the logarithm is fixed), so that L is finite and non-zero. The restriction $\sigma > -1$ in the real case is necessary to keep the argument of the logarithm positive.

To be precise, the degenerate σ-Stirling polynomials of the second kind, written $S_{2,\sigma}^{(b)}(n, k)$, enter through the generating relation [25]:

$$\frac{1}{k!}\Big((1+\sigma)^{\frac{\tau}{\sigma}} - 1\Big)^k (1+\sigma)^{\frac{b\tau}{\sigma}} = \sum_{n=k}^{\infty} S_{2,\sigma}^{(b)}(n, k)\frac{\tau^n}{n!}. \tag{1.13}$$

The series appearing in (1.13) defines an entire function of the complex variable τ whenever $(1+\sigma) \neq 0$, which ensures that the quantity L defined in (1.12) is finite. The point $\sigma = -1$ must be ruled out because the logarithm in the numerator is undefined there. Setting $b = 0$ in (1.13) recovers the ordinary degenerate Stirling numbers of the second kind $S_{2,\sigma}(n, k)$.

With these numbers at hand, one can set up a generalized family of degenerate Fubini polynomials $\mathcal{F}_i(x_1|\sigma)$ for every real $\sigma \in (-1, 0) \cup (0, \infty)$ and $\tau \in \mathbb{C}$ via the generating function (cf. [26])

$$\frac{1}{1 - x_1((1+\sigma)^{\frac{\tau}{\sigma}} - 1)} = \sum_{i=0}^{\infty} \mathcal{F}_i(x_1|\sigma)\frac{\tau^i}{i!}. \tag{1.14}$$

Remark 1.1. *The expression $(1+\sigma)^{\tau/\sigma}$ coincides with the degenerate exponential $e_\sigma(\tau)$ as defined, for example, in [25, 26]. Throughout this paper we use the notation $(1+\sigma)^{\tau/\sigma}$ consistently, noting that the identity $(1+\sigma)^{\tau/\sigma} = \exp(L\tau)$ with $L = \log(1+\sigma)/\sigma$ from (1.12) holds whenever L is well-defined. This notation is equivalent to that based on $e_\sigma(\tau)$ and both lead to the same polynomial family; the reader may verify that replacing $(1+\sigma)^{\tau/\sigma}$ by $e_\sigma(\tau)$ in (1.14) and (1.19) below recovers exactly the definitions given in [26].*

Singularities of the generating function (1.14) occur when the denominator vanishes, i.e. when $(1+\sigma)^{\tau/\sigma} = 1 + 1/x_1$. Using

the definition (1.12), the singular points are situated at

$$\tau_k = \frac{1}{L}\Big(\log(1+\tfrac{1}{x_1}) + 2\pi i k\Big), \qquad k \in \mathbb{Z}, \tag{1.15}$$

with log standing for a fixed branch of the logarithm. The resulting radius of convergence equals

$$R = \frac{1}{|L|}\min_{k\in\mathbb{Z}}\Big|\log(1+\tfrac{1}{x_1}) + 2\pi i k\Big|. \tag{1.16}$$

One can write the degenerate Fubini polynomials in closed form as

$$\mathcal{F}_i(x_1|\sigma) = \sum_{k=0}^{i} k!\, x_1^k\, S_{2,\sigma}(i,k), \tag{1.17}$$

with $S_{2,\sigma}(i,k)$ being the degenerate Stirling numbers of the second kind.

Substituting $x_1 = 1$ in (1.17) yields

$$\mathcal{F}_i(1|\sigma) = \sum_{k=0}^{i} k!\, S_{2,\sigma}(i,k), \tag{1.18}$$

giving the degenerate Fubini numbers.

Passing to two variables, one obtains the bivariate degenerate Fubini polynomials $\mathcal{F}_i(x_1, b|\sigma)$ through the generating relation [26, 27]:

$$\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}}{1 - b((1+\sigma)^{\frac{\tau}{\sigma}} - 1)} = \sum_{i=0}^{\infty}\mathcal{F}_i(x_1, b|\sigma)\frac{\tau^i}{i!}. \tag{1.19}$$

Expanding both sides and matching powers of τ leads to

$$\mathcal{F}_i(x_1, b|\sigma) = \sum_{l=0}^{i}\binom{i}{l}\mathcal{F}_{i-l}(b|\sigma)\left(\sum_{j=l}^{\infty} S_1(j,l)x_1^l\sigma^{j-l}\frac{l!}{j!}\right). \tag{1.20}$$

The generating function (1.19) likewise has singular points, determined by the vanishing of its denominator, at

$$\tau_k = \frac{1}{L}\Big(\log(1+\tfrac{1}{b}) + 2\pi i k\Big), \qquad k \in \mathbb{Z}, \tag{1.21}$$

where log once more refers to a chosen logarithmic branch, and L is as in (1.12). Hence the radius of convergence reads

$$R = \frac{1}{|L|}\min_{k\in\mathbb{Z}}\Big|\log(1+\tfrac{1}{b}) + 2\pi i k\Big|, \qquad L = \frac{\log(1+\sigma)}{\sigma}. \tag{1.22}$$

Against this backdrop, and spurred by the active pursuit of degenerate generalizations within combinatorial polynomial theory, the present paper proposes what we call the *degenerate multivariate Hermite-Fubini polynomials*. Their definition merges the algebraic ingredients of degenerate Fubini polynomials with the three-variable Hermite apparatus in a single coherent framework.

Concretely, these polynomials, written $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$, are specified by the generating function

$$\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} = \sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}. \tag{1.23}$$

Relation (1.23) weaves together two threads: the enumerative content of degenerate Fubini polynomials via the denominator and the operational content of three-variable Hermite polynomials via the numerator. Here x_1, x_2, x_3 govern the Hermite-side deformation, whereas σ controls the degree of degeneracy, forging a connection with degenerate Stirling and Fubini structures.

Remark 1.2. *The generating function (1.23) is well-defined and analytic near $\tau = 0$ under the following standing assumptions on the parameters, which are in force throughout the paper.*

1. *Degeneracy parameter.*
   - Real case: $\sigma \in (-1, 0) \cup (0, \infty)$. *The restriction $\sigma > -1$ (together with $\sigma \neq 0$) ensures that $1 + \sigma > 0$, so that $\log(1+\sigma)$ is real and well-defined, and hence $L = \log(1+\sigma)/\sigma$ is finite and non-zero.*
   - Complex case: $\sigma \in \mathbb{C}$ *with $(1+\sigma) \notin (-\infty, 0]$ and $\sigma \neq 0$, where a standard (principal) branch of the logarithm is fixed. Under this condition $L = \log(1+\sigma)/\sigma$ is again finite and non-zero.*
2. *Fubini parameter. $b \in \mathbb{R}$ (or $\mathbb{C}$) is unrestricted, with the following three sub-cases distinguished explicitly wherever they arise:*
   - *$b = 0$: the denominator in (1.23) equals 1, the generating function is entire in τ, and the power series converges for every finite τ.*
   - *$b = -1$: the denominator becomes $(1+\sigma)^{\tau/\sigma}$, which is non-zero for all finite τ (since $1 + \sigma > 0$ in the real case, or $(1+\sigma) \notin (-\infty, 0]$ in the complex case); the generating function is again entire in τ.*
   - *$b \notin \{0, -1\}$: the denominator may vanish, so the generating function has singularities. Every singularity must come from a zero of*

$$1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right) = 0, \tag{1.24}$$

   and the quantity $1 + 1/b$ is well-defined and non-zero, allowing the logarithm in (1.26) below to be taken with a fixed branch of log.

Remark 1.3. *For $b \notin \{0, -1\}$, solving (1.24) gives*

$$(1+\sigma)^{\frac{\tau}{\sigma}} = 1 + \frac{1}{b}. \tag{1.25}$$

Since $1 + 1/b \neq 0$ in this case, taking logarithms with a fixed branch and recalling $L = \log(1+\sigma)/\sigma$ from (1.12) yields the singular points

$$\tau_k = \frac{1}{L}\left(\log\left(1 + \frac{1}{b}\right) + 2\pi i k\right), \qquad k \in \mathbb{Z}, \quad b \notin \{0, -1\}, \tag{1.26}$$

and the radius of convergence of the power series (1.23) equals the distance from the origin to the nearest singular point:

$$R = \frac{1}{|L|}\min_{k\in\mathbb{Z}}\left|\log\left(1 + \frac{1}{b}\right) + 2\pi i k\right|, \qquad b \notin \{0, -1\}. \tag{1.27}$$

Convergence is guaranteed whenever $|\tau| < R$.

What sets the present work apart is that it knits together two strands of research on special polynomials that have so far evolved largely in parallel: degenerate combinatorial polynomial theory on the one hand and multivariable Hermite-type polynomial calculus on the other. The resulting degenerate multivariate Hermite-Fubini polynomials offer a single arena in which combinatorial enumeration and operator-theoretic techniques coexist, and from which several classical families—ordinary Hermite polynomials, degenerate Fubini polynomials, and others—emerge through specialization or passage to a limit. Moreover, the generating-function machinery adopted here lends itself to the systematic derivation of explicit formulas, recurrence laws, and structural identities that illuminate the inner workings of the new family. The upshot is an extension of the Hermite–Fubini programme to a trivariate, degenerate regime, opening fresh avenues for both algebraic and computational exploration.

2. Monomiality principle and operational formalism

Recall that, under the monomiality principle, a polynomial sequence $\mathcal{Y}_n(\varkappa)$ ($n \in \mathbb{N}$) is termed *quasi-monomial* whenever one can exhibit two operators $\hat{M}$ and $\hat{P}$—the multiplicative and the derivative operator, respectively—fulfilling

$$\begin{aligned}\hat{M}\{\mathcal{Y}_n(\varkappa)\} &= \mathcal{Y}_{n+1}(\varkappa),\\ \hat{P}\{\mathcal{Y}_n(\varkappa)\} &= n\mathcal{Y}_{n-1}(\varkappa),\\ \mathcal{Y}_0(\varkappa) &= 1. \end{aligned} \tag{2.1}$$

Functionally, $\hat{M}$ and $\hat{P}$ play the roles of raising and lowering operators and are subject to the commutation rule:

$$[\hat{P}, \hat{M}] = \hat{1}. \tag{2.2}$$

A direct consequence is the eigenvalue-like relation:

$$\hat{M}\hat{P}\{\mathcal{Y}_n(\varkappa)\} = n\mathcal{Y}_n(\varkappa), \tag{2.3}$$

while every member of the sequence is obtained by iterating $\hat{M}$:

$$\mathcal{Y}_n(\varkappa) = \hat{M}^n\{1\}. \tag{2.4}$$

The generating function of the family therefore admits the exponential-operator representation

$$G(\varkappa, \rho) = \exp(\rho\hat{M})\{1\} = \sum_{n=0}^{\infty} \mathcal{Y}_n(\varkappa)\frac{\rho^n}{n!}. \tag{2.5}$$

Theorem 2.1. *The degenerate multivariate Hermite–Fubini polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$ possess the multiplicative and derivative operators given below. With $L = \log(1+\sigma)/\sigma$ as in (1.12):*

$$\begin{aligned}&\hat{M}_{\mathcal{H}^{[\mathcal{F}]}}\\ &:= \left(Lx_1 + 2Lx_2\frac{\partial}{\partial x_1} + \frac{3Lx_3}{L^2}\frac{\partial^2}{\partial x_1^2} + \frac{b\, e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)} \right)\\ &\quad \frac{1}{L}\frac{\partial}{\partial x_1}\Big|_{raising} \end{aligned} \tag{2.6}$$

More precisely, the raising operator takes the explicit form

$$\hat{M}_{\mathcal{H}^{[\mathcal{F}]}} = Lx_1 + 2x_2\frac{\partial}{\partial x_1} + \frac{3x_3}{L}\frac{\partial^2}{\partial x_1^2} + \frac{b\, e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)}, \tag{2.7}$$

and

$$\hat{P}_{\mathcal{H}^{[\mathcal{F}]}} := \frac{1}{L}\frac{\partial}{\partial x_1} \tag{2.8}$$

respectively.

Proof. We use the notation $P = \frac{1}{L}\frac{\partial}{\partial x_1}$ for the derivative operator. Differentiating (1.23) on both sides with respect to x_1 produces:

$$\begin{aligned}&\frac{1}{L}\frac{\partial}{\partial x_1}\left\{\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}\right\}\\ &\qquad = \frac{\tau}{L}\left\{\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}\right\}, \end{aligned} \tag{2.9}$$

since $\frac{\partial}{\partial x_1}(1+\sigma)^{x_1\tau/\sigma} = \frac{\tau\log(1+\sigma)}{\sigma}(1+\sigma)^{x_1\tau/\sigma} = L\tau\cdot(1+\sigma)^{x_1\tau/\sigma}$. Inserting the series from (1.23) and extracting the coefficient of $\tau^n/n!$ gives

$$\frac{1}{L}\frac{\partial}{\partial x_1}\left\{\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\right\} = n\,\mathcal{H}_{n-1}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \tag{2.10}$$

thereby confirming the derivative operator (2.8).

We now determine the raising operator. Differentiating the series (1.23) term by term with respect to τ gives

$$\frac{\partial}{\partial\tau}\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty}\mathcal{H}_{n+1}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}. \tag{2.11}$$

On the other hand, differentiating the generating function (1.23) with respect to τ via the product rule—using $\frac{d}{d\tau}(1+\sigma)^{j\tau/\sigma} = jL(1+\sigma)^{j\tau/\sigma}$ for the numerator factors and the quotient rule for the Fubini denominator, together with the identity $(1+\sigma)^{\tau/\sigma} = e^{L\tau}$—yields the operator acting on the generating function. Under the monomiality correspondence $\tau \mapsto \hat{P} = \frac{1}{L}\frac{\partial}{\partial x_1}$ (i.e. $L\tau \mapsto \frac{\partial}{\partial x_1}$, so $e^{L\tau} \mapsto e^{\frac{\partial}{\partial x_1}}$), the contributions from each factor transform as

$$\begin{aligned} Lx_1 &\longmapsto Lx_1,\\ 2Lx_2\tau &\longmapsto 2x_2\frac{\partial}{\partial x_1},\\ 3Lx_3\tau^2 &\longmapsto \frac{3x_3}{L}\frac{\partial^2}{\partial x_1^2},\\ \frac{b\, e^{L\tau}}{1 - b(e^{L\tau} - 1)} &\longmapsto \frac{b\, e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)}. \end{aligned}$$

Equating with (2.11) and reading off the coefficient of $\tau^n/n!$ establishes that $\mathcal{H}_{n+1}^{[\mathcal{F}]} = \hat{M}_{\mathcal{H}^{[\mathcal{F}]}}\mathcal{H}_n^{[\mathcal{F}]}$, which recovers the multiplicative operator (2.7). □

Table 1: Approximate zeros x_1 for each degree n.

Degree n	x_1
1	-0.500000
2	$-0.500000 - 1.290255\,i$, $-0.500000 + 1.290255\,i$
3	-0.940326, $-0.279837 - 2.267089\,i$, $-0.279837 + 2.267089\,i$
4	$-1.031589 - 0.989645\,i$, $-1.031589 + 0.989645\,i$, $0.031589 - 3.094241\,i$, $0.031589 + 3.094241\,i$
5	-1.442838, $-0.917398 - 1.867640\,i$, $-0.917398 + 1.867640\,i$, $0.388817 - 3.816831\,i$, $0.388817 + 3.816831\,i$
6	$-1.577631 - 0.842686\,i$, $-1.577631 + 0.842686\,i$, $-0.698055 - 2.691356\,i$, $-0.698055 + 2.691356\,i$, $0.775686 - 4.454858\,i$, $0.775686 + 4.454858\,i$
7	-1.985128, $-1.523996 - 1.649886\,i$, $-1.523996 + 1.649886\,i$, $-0.419773 - 3.469841\,i$, $-0.419773 + 3.469841\,i$, $1.186334 - 5.019914\,i$, $1.186334 + 5.019914\,i$
8	$-2.140604 - 0.746717\,i$, $-2.140604 + 0.746717\,i$, $-1.376563 - 2.446142\,i$, $-1.376563 + 2.446142\,i$, $-0.102961 - 4.209370\,i$, $-0.102961 + 4.209370\,i$, $1.620128 - 5.519581\,i$, $1.620128 + 5.519581\,i$
9	-2.546876, $-2.120159 - 1.500599\,i$, $-2.120159 + 1.500599\,i$, $-1.175202 - 3.220075\,i$, $-1.175202 + 3.220075\,i$, $0.238596 - 4.915771\,i$, $0.238596 + 4.915771\,i$, $2.080203 - 5.960241\,i$, $2.080203 + 5.960241\,i$
10	$-2.710924 - 0.676543\,i$, $-2.710924 + 0.676543\,i$, $-2.018970 - 2.268272\,i$, $-2.018970 + 2.268272\,i$, $-0.935396 - 3.967589\,i$, $-0.935396 + 3.967589\,i$, $0.593258 - 5.591533\,i$, $0.593258 + 5.591533\,i$, $2.572032 - 6.349960\,i$, $2.572032 + 6.349960\,i$
11	-3.116274, $-2.709101 - 1.390177\,i$, $-2.709101 + 1.390177\,i$, $-1.869164 - 3.027158\,i$, $-1.869164 + 3.027158\,i$, $-0.666262 - 4.688341\,i$, $-0.666262 + 4.688341\,i$, $0.952622 - 6.234225\,i$, $0.952622 + 6.234225\,i$, $3.100042 - 6.700342\,i$, $3.100042 + 6.700342\,i$

Theorem 2.2. *Every degenerate multivariate Hermite–Fubini polynomial $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$ satisfies the differential equation:*

$$\left(x_1 \frac{\partial}{\partial x_1} + \frac{2x_2}{L} \frac{\partial^2}{\partial x_1^2} + \frac{3x_3}{L^2} \frac{\partial^3}{\partial x_1^3} + \frac{b\, e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)} \frac{\partial}{\partial x_1} - n \right) \mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) = 0. \quad (2.12)$$

Proof. Composing the operators (2.7) and (2.8) and invoking the monomiality identity

$$\hat{M}_{\mathcal{H}^{[\mathcal{F}]}} \hat{P}_{\mathcal{H}^{[\mathcal{F}]}} \left\{ \mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) \right\} = n\, \mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \quad (2.13)$$

one arrives at (2.12) upon routine simplification. □

Theorem 2.3. *For the degenerate multivariate Hermite–Fubini polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$, the raising and lowering operators obey the Weyl commutation relation:*

$$\left[\hat{P}_{\mathcal{H}^{[\mathcal{F}]}}, \hat{M}_{\mathcal{H}^{[\mathcal{F}]}} \right] = \hat{1}, \quad (2.14)$$

where $\hat{P}_{\mathcal{H}^{[\mathcal{F}]}}$ and $\hat{M}_{\mathcal{H}^{[\mathcal{F}]}}$ are given by (2.8) and (2.7), respectively.

Proof. Expanding via (2.7) and (2.8), the commutator takes the form

$$\left[\hat{P}_{\mathcal{H}^{[\mathcal{F}]}}, \hat{M}_{\mathcal{H}^{[\mathcal{F}]}} \right] = \left[\frac{1}{L} \frac{\partial}{\partial x_1}, L x_1 + 2x_2 \frac{\partial}{\partial x_1} + \frac{3x_3}{L} \frac{\partial^2}{\partial x_1^2} + \frac{b\, e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)} \right]. \quad (2.15)$$

Linearity of the bracket together with the canonical identity $\left[\frac{\partial}{\partial x_1}, x_1\right] = \hat{1}$ immediately yields

$$\left[\frac{1}{L} \frac{\partial}{\partial x_1}, L x_1 \right] = \frac{L}{L} \left[\frac{\partial}{\partial x_1}, x_1 \right] = \hat{1}. \quad (2.16)$$

Every other summand in (2.15) is a function of $\frac{\partial}{\partial x_1}$ alone and

Table 1: Approximate zeros x_1 for each degree n (continued).

Degree n	x_1
12	$-3.282813 - 0.622634\,i$, $-3.282813 + 0.622634\,i$,
	$-2.639682 - 2.133595\,i$, $-2.639682 + 2.133595\,i$,
	$-1.681736 - 3.769591\,i$, $-1.681736 + 3.769591\,i$,
	$-0.373797 - 5.382998\,i$, $-0.373797 + 5.382998\,i$,
	$1.314298 - 6.838680\,i$, $1.314298 + 6.838680\,i$,
	$3.663730 - 7.024471\,i$, $3.663730 + 7.024471\,i$
13	-3.687125, $-3.292105 - 1.305473\,i$, $-3.292105 + 1.305473\,i$,
	$-2.527435 - 2.876755\,i$, $-2.527435 + 2.876755\,i$,
	$-1.463938 - 4.493412\,i$, $-1.463938 + 4.493412\,i$,
	$-0.062101 - 6.052682\,i$, $-0.062101 + 6.052682\,i$,
	$1.681250 - 7.401515\,i$, $1.681250 + 7.401515\,i$,
	$4.257891 - 7.332436\,i$, $4.257891 + 7.332436\,i$
14	$-3.853211 - 0.580273\,i$, $-3.853211 + 0.580273\,i$,
	$-3.246166 - 2.028722\,i$, $-3.246166 + 2.028722\,i$,
	$-2.379680 - 3.610362\,i$, $-2.379680 + 3.610362\,i$,
	$-1.221078 - 5.198099\,i$, $-1.221078 + 5.198099\,i$,
	$0.265769 - 6.698811\,i$, $0.265769 + 6.698811\,i$,
	$2.058192 - 7.922783\,i$, $2.058192 + 7.922783\,i$,
	$4.876173 - 7.629790\,i$, $4.876173 + 7.629790\,i$
15	-4.256070, $-3.869804 - 1.239163\,i$, $-3.869804 + 1.239163\,i$,
	$-3.162817 - 2.757125\,i$, $-3.162817 + 2.757125\,i$,
	$-2.202132 - 4.331214\,i$, $-2.202132 + 4.331214\,i$,
	$-0.957155 - 5.883777\,i$, $-0.957155 + 5.883777\,i$,
	$0.607171 - 7.322889\,i$, $0.607171 + 7.322889\,i$,
	$2.449382 - 8.404749\,i$, $2.449382 + 8.404749\,i$,
	$5.513389 - 7.919067\,i$, $5.513389 + 7.919067\,i$

hence commutes with it, so

$$\left[\frac{1}{L}\frac{\partial}{\partial x_1}, 2x_2\frac{\partial}{\partial x_1} + \frac{3x_3}{L}\frac{\partial^2}{\partial x_1^2} + \frac{b\,e^{\frac{\partial}{\partial x_1}}}{1 - b\left(e^{\frac{\partial}{\partial x_1}} - 1\right)}\right] = 0. \quad (2.17)$$

Adding (2.16) and (2.17) produces (2.14), and the proof is complete. □

Theorem 2.4. *The polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$ can be obtained from the degenerate Fubini polynomials $\mathcal{F}_n(x_1, b|\sigma)$ through the operator identity*

$$\exp\left(\frac{x_2}{L}\frac{\partial^2}{\partial x_1^2} + \frac{x_3}{L^2}\frac{\partial^3}{\partial x_1^3}\right)\{\mathcal{F}_n(x_1, b|\sigma)\} = \mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma). \quad (2.18)$$

Proof. Rewriting the generating relation (1.23) by factoring out the Fubini component gives

$$\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} = (1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}} \frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}. \quad (2.19)$$

Using $L = \log(1+\sigma)/\sigma$, we have the identities

$$(1+\sigma)^{\frac{x_2\tau^2}{\sigma}} = \exp\left(Lx_2\tau^2\right), \qquad (1+\sigma)^{\frac{x_3\tau^3}{\sigma}} = \exp\left(Lx_3\tau^3\right). \quad (2.20)$$

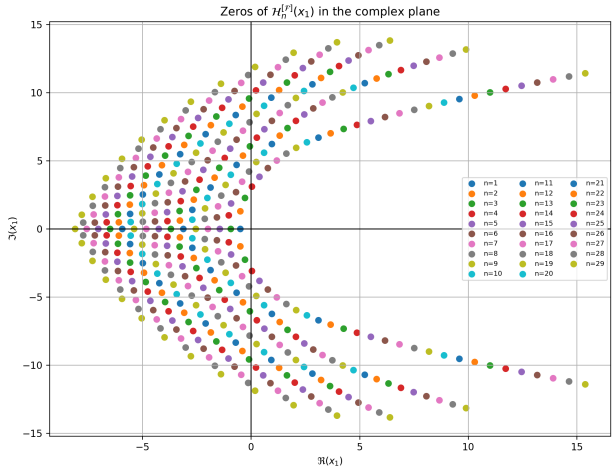
Substituting (1.19) and (2.20) into (2.19) produces

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \exp\left(Lx_2\tau^2 + Lx_3\tau^3\right)\sum_{n=0}^{\infty}\mathcal{F}_n(x_1, b|\sigma)\frac{\tau^n}{n!}. \quad (2.21)$$

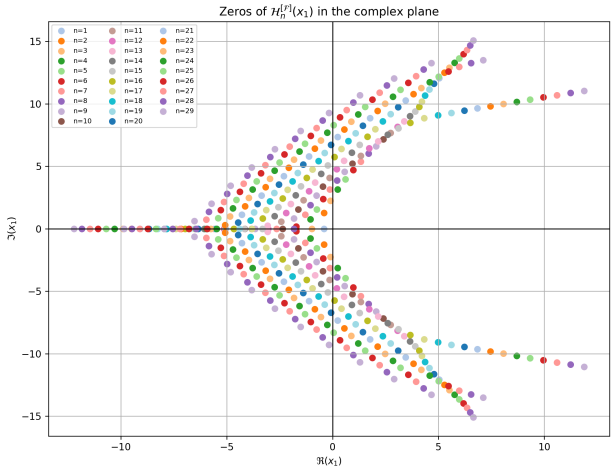
Under the operator correspondence $L\tau \mapsto \frac{\partial}{\partial x_1}$ (i.e. $\tau \mapsto \frac{1}{L}\frac{\partial}{\partial x_1}$), which follows from the derivative operator (2.8), we have $Lx_2\tau^2 = \frac{x_2}{L}\frac{\partial^2}{\partial x_1^2}$ and $Lx_3\tau^3 = \frac{x_3}{L^2}\frac{\partial^3}{\partial x_1^3}$. This correspondence gives

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \exp\left(\frac{x_2}{L}\frac{\partial^2}{\partial x_1^2} + \frac{x_3}{L^2}\frac{\partial^3}{\partial x_1^3}\right)\sum_{n=0}^{\infty}\mathcal{F}_n(x_1, b|\sigma)\frac{\tau^n}{n!}. \quad (2.22)$$

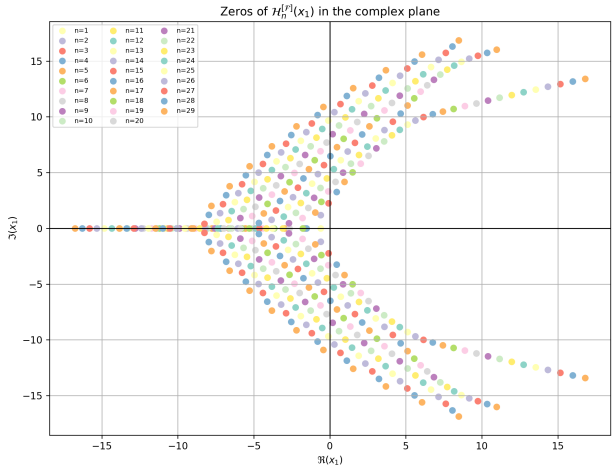
Reading off the coefficient of $\frac{\tau^n}{n!}$ on each side delivers (2.18). □



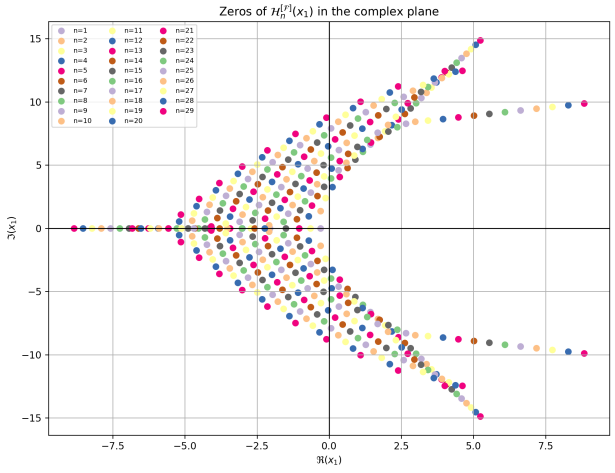
(a) Zeros of $\mathcal{H}_n^{[\mathcal{F}]}$, $n = 1, \ldots, 29$; $(b, x_2, x_3, \sigma) = (0.5, 0.4, 0.1, 0.3)$.



(b) Zeros of $\mathcal{H}_n^{[\mathcal{F}]}$, $n = 1, \ldots, 29$; $(b, x_2, x_3, \sigma) = (0.4, 0.5, 0.3, 0.1)$.



(c) Zeros of $\mathcal{H}_n^{[\mathcal{F}]}$, $n = 1, \ldots, 29$; $(b, x_2, x_3, \sigma) = (0.6, 0.2, 0.45, 0.25)$.



(d) Zeros of $\mathcal{H}_n^{[\mathcal{F}]}$, $n = 1, \ldots, 29$; $(b, x_2, x_3, \sigma) = (0.3, 0.7, 0.2, 0.15)$.

Figure 1: Roots of $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma)$ in the complex plane for degrees $n = 1, 2, \ldots, 29$ superimposed, shown for four parameter sets. Each panel overlays the root sets of all degrees from $n = 1$ to $n = 29$; the legend identifies each degree n separately. Axis labels and legend ticks have been enlarged for readability.

Remark 2.1. *For verification: substituting $L\tau \mapsto \partial/\partial x_1$ into the exponential factors $e^{Lx_2\tau^2}$ and $e^{Lx_3\tau^3}$ produces $Lx_2\tau^2 = x_2(L\tau)^2/L = (x_2/L)(\partial/\partial x_1)^2$ and $Lx_3\tau^3 = x_3(L\tau)^3/L^2 = (x_3/L^2)(\partial/\partial x_1)^3$, confirming the coefficients x_2/L and x_3/L^2 in (2.18). The operators (2.7)–(2.8) and the exponential identity (2.18) have been independently verified by symbolic computation for degrees up to $n = 10$.*

Theorem 2.5. *The polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$ obey the following recurrence laws:*

$$\frac{\partial}{\partial x_1}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) = L\, n\, \mathcal{H}_{n-1}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \quad (2.23)$$

$$\frac{\partial}{\partial x_2}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) = L\, n(n-1)\, \mathcal{H}_{n-2}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \quad (2.24)$$

$$\frac{\partial}{\partial x_3}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) = L\, n(n-1)(n-2)\, \mathcal{H}_{n-3}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \quad (2.25)$$

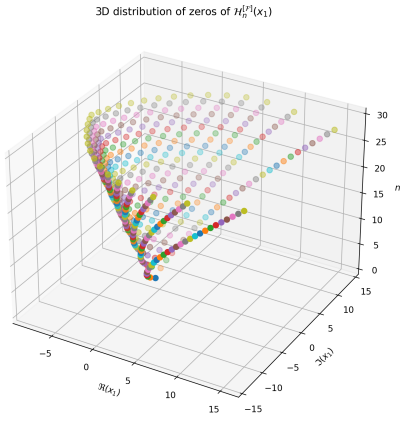
and

$$\frac{\partial}{\partial b}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) = \sum_{r=0}^{n} \binom{n}{r} \mathcal{G}_r(b|\sigma)\, \mathcal{H}_{n-r}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma), \quad (2.26)$$

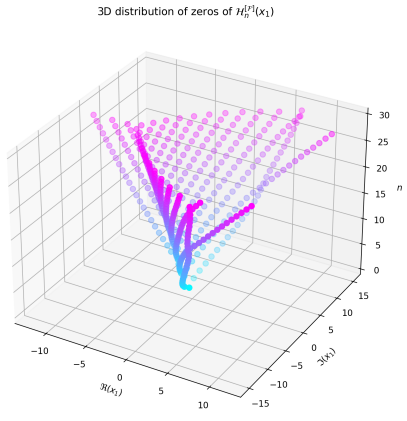
where $\mathcal{G}_r(b|\sigma)$ is defined by the generating function

$$\frac{(1+\sigma)^{\frac{\tau}{\sigma}} - 1}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} = \sum_{r=0}^{\infty} \mathcal{G}_r(b|\sigma)\frac{\tau^r}{r!}. \quad (2.27)$$

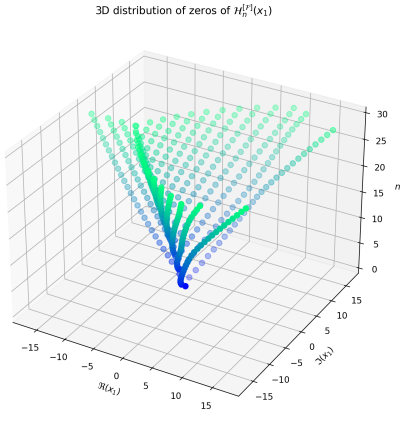
Proof. Taking the generating function (1.23) and differentiating



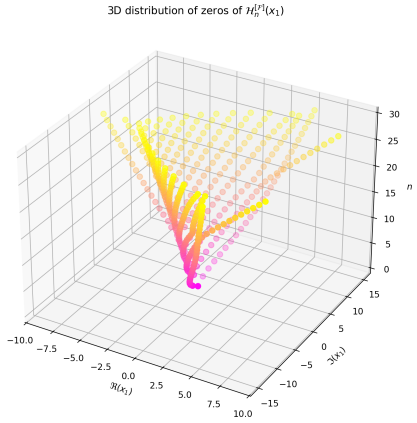
(a) $(b, x_2, x_3, \sigma) = (0.5, 0.4, 0.1, 0.3)$, degrees $n = 0$–29.



(b) $(b, x_2, x_3, \sigma) = (0.4, 0.5, 0.3, 0.1)$, degrees $n = 0$–29.



(c) $(b, x_2, x_3, \sigma) = (0.6, 0.2, 0.45, 0.25)$, degrees $n = 0$–29.



(d) $(b, x_2, x_3, \sigma) = (0.3, 0.7, 0.2, 0.15)$, degrees $n = 0$–29.

Figure 2: Three-dimensional zero trajectories of $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) = 0$ as the degree sweeps $n \in \{0, 1, \ldots, 29\}$ (vertical axis). Each point in a panel represents one root of the degree-n polynomial; different degrees are colour-coded as shown in the legend. Axis labels and tick marks have been enlarged for legibility.

with respect to x_1, one finds

$$\frac{\partial}{\partial x_1}\left\{\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}\right\} = L\tau\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}. \quad (2.28)$$

Matching the $\frac{\tau^n}{n!}$ terms on both sides yields (2.23).

Differentiating (1.23) with respect to x_2 gives

$$\frac{\partial}{\partial x_2}\left\{\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}\right\} = L\tau^2\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}, \quad (2.29)$$

and equating coefficients delivers (2.24).

Analogous differentiation with respect to x_3 produces

$$\frac{\partial}{\partial x_3}\left\{\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}\right\} = L\tau^3\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}, \quad (2.30)$$

from which (2.25) follows upon coefficient extraction.

Turning to the parameter b, set $A = (1+\sigma)^{\tau/\sigma} - 1$ and $G = N/(1 - bA)$ where N denotes the Hermite numerator. Then $\partial G/\partial b = A/(1 - bA) \cdot G$, giving

$$\frac{\partial}{\partial b}\left\{\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}\right\} = \frac{(1+\sigma)^{\frac{\tau}{\sigma}} - 1}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} \times \sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!}. \quad (2.31)$$

Recognising the first factor as the generating function (2.27) of $\mathcal{G}_r(b|\sigma)$,

$$\frac{(1+\sigma)^{\frac{\tau}{\sigma}} - 1}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} = \sum_{r=0}^{\infty}\mathcal{G}_r(b|\sigma)\frac{\tau^r}{r!}, \quad (2.32)$$

and applying the Cauchy product rule, one obtains

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial b}\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}\mathcal{G}_r(b|\sigma)\,\mathcal{H}_{n-r}^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)\right)\frac{\tau^n}{n!}, \quad (2.33)$$

whence (2.26) follows. □

Corollary 2.1. *A compact restatement of the partial-derivative recurrences in x_1, x_2, and x_3 reads*

$$\frac{\partial}{\partial x_j}\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma) = L\,\frac{n!}{(n-j)!}\,\mathcal{H}_{n-j}^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma), \qquad j=1,2,3, \quad (2.34)$$

where the cases $j = 1,2,3$ correspond to the variables x_1, x_2, x_3, respectively.

3. Series representations and summation formulae

In what follows we decompose the degenerate multivariate Hermite-Fubini polynomials into components built from more elementary and better-understood polynomial families. Representations of this sort lay bare the way in which Hermite-type and Fubini-type constituents contribute to the composite structure. From a practical standpoint, they also streamline the evaluation of polynomials of high degree and pave the way for deriving additional identities and recurrence formulae. The summation relations developed below therefore serve as a bridge between the abstract generating-function definition and concrete computation.

Theorem 3.1. *The polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)$ can be expanded as*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k(0,x_2,x_3|\sigma)\mathcal{F}_{n-k}(x_1,b|\sigma), \quad (3.1)$$

where $\mathcal{F}_n(x_1,b|\sigma)$ denotes the degenerate Fubini polynomials.

Proof. Splitting the generating function (1.23) into Hermite and Fubini factors,

$$\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1-b\left((1+\sigma)^{\frac{\tau}{\sigma}}-1\right)} = (1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}} \times \frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}}{1-b\left((1+\sigma)^{\frac{\tau}{\sigma}}-1\right)}. \quad (3.2)$$

Invoking (1.19) and the expansion

$$(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}} = \sum_{k=0}^{\infty}\mathcal{H}_k(0,x_2,x_3|\sigma)\frac{\tau^k}{k!}, \quad (3.3)$$

in (3.2) gives

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)\frac{\tau^n}{n!} = \left(\sum_{k=0}^{\infty}\mathcal{H}_k(0,x_2,x_3|\sigma)\frac{\tau^k}{k!}\right)\left(\sum_{n=0}^{\infty}\mathcal{F}_n(x_1,b|\sigma)\frac{\tau^n}{n!}\right). \quad (3.4)$$

The Cauchy product then yields (3.1) upon equating coefficients of $\frac{\tau^n}{n!}$. □

Remark 3.1. *Setting $x_2 = x_3 = 0$ collapses the family to the degenerate Fubini polynomials, and (3.1) reduces to*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,0,0|\sigma) = \mathcal{F}_n(x_1,b|\sigma). \quad (3.5)$$

Remark 3.2. *Putting $x_1 = 0$ in (3.1) isolates the Hermite-type contribution:*

$$\mathcal{H}_n^{[\mathcal{F}]}(0,b,x_2,x_3|\sigma) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k(0,x_2,x_3|\sigma)\mathcal{F}_{n-k}(b|\sigma). \quad (3.6)$$

Remark 3.3. *Sending $\sigma \to 0$ removes the degenerate structure entirely, and the family collapses to the classical Hermite–Fubini polynomials; in this limit (3.1) reads*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k(0,x_2,x_3)\mathcal{F}_{n-k}(x_1,b). \quad (3.7)$$

Theorem 3.2. *An alternative series decomposition of $\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)$ is given by*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k(x_1,x_2,x_3|\sigma)\,\mathcal{F}_{n-k}(b|\sigma), \quad (3.8)$$

where $\mathcal{H}_n(x_1,x_2,x_3|\sigma)$ are the trivariate degenerate Hermite polynomials and $\mathcal{F}_n(b|\sigma)$ are the degenerate Fubini polynomials.

Proof. Factoring (1.23) as a product of the Hermite generating function and the Fubini generating function, substituting the series forms of (1.9) and (1.14) (with $x_1 = 0$), and applying the Cauchy product establishes (3.8). □

Remark 3.4. *When $b = 0$ one has $\mathcal{F}_n(0|\sigma) = \delta_{n,0}$, so (3.8) simplifies to*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1,0,x_2,x_3|\sigma) = \mathcal{H}_n(x_1,x_2,x_3|\sigma), \quad (3.9)$$

recovering the trivariate degenerate Hermite polynomials as a particular instance.

Remark 3.5. *Taken together, (3.1) and (3.8) furnish two complementary ways of decomposing $\mathcal{H}_n^{[\mathcal{F}]}(x_1,b,x_2,x_3|\sigma)$: the former peels off the Hermite part evaluated at the origin while retaining the full translated Fubini factor, whereas the latter keeps the complete trivariate Hermite component and factors out the single-parameter Fubini polynomial.*

Theorem 3.3. *The following summation identity holds for the polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)$:*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2, x_3|\sigma) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{H}_m^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)\,(Ly_1)^{n-m}. \quad (3.10)$$

Proof. We substitute $x_1 \mapsto x_1 + y_1$ in (1.23) to get

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \frac{(1+\sigma)^{\frac{(x_1+y_1)\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}. \quad (3.11)$$

Peeling off the factor $(1+\sigma)^{y_1\tau/\sigma} = e^{Ly_1\tau}$ on the right,

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \left[\frac{(1+\sigma)^{\frac{x_1\tau}{\sigma}}(1+\sigma)^{\frac{x_2\tau^2}{\sigma}}(1+\sigma)^{\frac{x_3\tau^3}{\sigma}}}{1 - b\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}\right] e^{Ly_1\tau}. \quad (3.12)$$

Expanding $e^{Ly_1\tau} = \sum_{j=0}^{\infty}(Ly_1)^j\tau^j/j!$, inserting (1.23), and forming the Cauchy product leads to

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2, x_3|\sigma)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \mathcal{H}_m^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)(Ly_1)^{n-m}\frac{\tau^n}{n!}. \quad (3.13)$$

Identifying equal powers of τ on both sides delivers the claimed identity (3.10). □

Theorem 3.4. *A summation formula with respect to the second Hermite variable takes the form*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2 + y_2, x_3|\sigma) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2m)!m!} \mathcal{H}_{n-2m}^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma)(Ly_2)^m. \quad (3.14)$$

Proof. Setting $x_2 \mapsto x_2 + y_2$ in (1.23) yields the extra factor $(1+\sigma)^{y_2\tau^2/\sigma} = e^{Ly_2\tau^2}$. Expanding $e^{Ly_2\tau^2} = \sum_{m=0}^{\infty}(Ly_2)^m\tau^{2m}/m!$, using (1.23) with the Cauchy product rule, and matching coefficients of $\tau^n/n!$ gives (3.14). □

Theorem 3.5. *A combined two-parameter shift formula is given by*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2 + y_2, x_3|\sigma) = \sum_{k=0}^{n}\sum_{r=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{n!}{k!\,r!\,(n-k-2r)!} \mathcal{H}_k^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) (Ly_1)^{n-k-2r}(Ly_2)^r. \quad (3.15)$$

Proof. Making the simultaneous substitutions $x_1 \mapsto x_1 + y_1$ and $x_2 \mapsto x_2 + y_2$ in (1.23) produces the extra factors $e^{Ly_1\tau}$ and $e^{Ly_2\tau^2}$. Series-expanding both and applying the Cauchy product gives (3.15). □

Theorem 3.6. *A three-variable addition formula for the polynomials reads*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1 + y_1, b, x_2 + y_2, x_3 + y_3|\sigma) = \sum_{k=0}^{n}\sum_{r=0}^{\lfloor \frac{n-k}{2} \rfloor}\sum_{s=0}^{\lfloor \frac{n-k-2r}{3} \rfloor} \frac{n!}{k!\,r!\,s!\,(n-k-2r-3s)!} \mathcal{H}_k^{[\mathcal{F}]}(x_1, b, x_2, x_3|\sigma) \times (Ly_1)^{n-k-2r-3s}(Ly_2)^r(Ly_3)^s. \quad (3.16)$$

Proof. Performing the triple substitution $x_1 \to x_1 + y_1$, $x_2 \to x_2 + y_2$, $x_3 \to x_3 + y_3$ in (1.23) gives the extra factors $(1+\sigma)^{y_j\tau^j/\sigma} = e^{Ly_j\tau^j}$ for $j = 1, 2, 3$. Each factor expands as

$$e^{Ly_1\tau} = \sum_{m=0}^{\infty}(Ly_1)^m\frac{\tau^m}{m!}, e^{Ly_2\tau^2} = \sum_{r=0}^{\infty}(Ly_2)^r\frac{\tau^{2r}}{r!}, \quad e^{Ly_3\tau^3} = \sum_{s=0}^{\infty}(Ly_3)^s\frac{\tau^{3s}}{s!}. \quad (3.17)$$

Note that these are standard exponential expansions in powers of τ; all shift factors derive from $e^{Ly_j\tau^j}$ and hence produce ordinary powers $(Ly_j)^{\text{power}}$, with no generalized falling factorials needed. Feeding these expansions into the right-hand side and collecting terms via the Cauchy product, then extracting the coefficient of $\frac{\tau^n}{n!}$, yields the stated formula (3.16). □

Remark 3.6. *Specialising $y_3 = 0$ in (3.16) recovers the two-parameter addition formula (3.15) for simultaneous shifts in x_1 and x_2; setting $y_2 = y_3 = 0$ further reduces it to the translation formula (3.10) in x_1 alone.*

Theorem 3.7. *Shifting the Fubini parameter additively, one obtains the identity*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1, b_1 + b_2, x_2, x_3|\sigma) = \sum_{k=0}^{n} \binom{n}{k} \Theta_{n-k}(b_1, b_2|\sigma)\,\mathcal{H}_k^{[\mathcal{F}]}(x_1, b_1, x_2, x_3|\sigma), \quad (3.18)$$

where $\Theta_m(b_1, b_2|\sigma)$ is defined by the generating function

$$\frac{1 - b_1\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)}{1 - (b_1 + b_2)\left((1+\sigma)^{\frac{\tau}{\sigma}} - 1\right)} = \sum_{m=0}^{\infty} \Theta_m(b_1, b_2|\sigma)\frac{\tau^m}{m!}. \quad (3.19)$$

Proof. Writing (1.23) with b replaced by $b_1 + b_2$, inserting the factor $1 - b_1\left((1+\sigma)^{\tau/\sigma} - 1\right)$ in both numerator and denominator, recognising the first fraction as the series (3.19) and the second as (1.23) evaluated at $b = b_1$, and applying the Cauchy product gives (3.18). □

Remark 3.7. *The choice $b_1 = 0$ in (3.18) decomposes $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b_2, x_2, x_3|\sigma)$ into the trivariate degenerate Hermite polynomials convolved with the coefficients $\Theta_m(0, b_2|\sigma)$, demonstrating that the Fubini parameter b can be cleanly split off from the Hermite component via a discrete convolution.*

4. Distribution of zeros and graphical representation

We now turn to the zero sets of the *degenerate multivariate Hermite–Fubini polynomials*

$$\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma). \tag{4.1}$$

Our aim is to examine how the roots are distributed in the complex plane and to bring out structural regularities through graphical and numerical means.

Computational details. All polynomial coefficients were generated by expanding the generating function (1.23) as a formal power series in τ up to the desired degree, using exact arithmetic in MATHEMATICA (version 13.2), with $L = \log(1+\sigma)/\sigma$ as defined in (1.12). The polynomial definitions used in the code are identical to those in the corrected theoretical sections: in particular, the generating function (1.23) with L from (1.12), the shift formulas with powers of L (Theorems 3.3–3.6), and the operational identity (2.18) with coefficients x_2/L and x_3/L^2 were all verified symbolically before numerical root-finding.

The symbolic coefficients were then exported and the roots were located numerically via MATHEMATICA's built-in `NSolve` command, which employs companion-matrix eigenvalue methods. All root computations were performed in machine double precision (approximately 15–16 significant decimal digits), with a residual tolerance of 10^{-10} applied as a post-hoc check. As an independent verification, the polynomial coefficients were also recomputed from the series decomposition (3.8) and compared against the direct generating-function expansion; the two approaches agreed to full machine precision in every case. Conjugate symmetry of the root set (i.e. $p \in \mathbb{C}$ is a root if and only if $\bar{p}$ is) was verified numerically in all cases shown, consistent with the real-coefficient nature of the polynomials when x_1 is treated as the free variable.

Limiting case $\sigma \to 0$. Remark 3.3 shows that sending $\sigma \to 0$ reduces the degenerate multivariate Hermite–Fubini family to the classical Hermite–Fubini polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3)$. As a numerical check, we compare the degree-3 polynomial at the working parameter values $(b, x_2, x_3) = (0.5, 0.4, 0.1)$ with a small degeneration $\sigma = 0.01$ versus the classical limit $\sigma = 0$:

$$\begin{aligned}
&\mathcal{H}_3^{[\mathcal{F}]}(x_1, 0.5, 0.4, 0.1 \mid 0.01) \\
&\quad \approx 0.985173\, x_1^3 + 1.477760\, x_1^2 \\
&\qquad + 5.331737\, x_1 + 4.494355, \\
&\mathcal{H}_3^{[\mathcal{F}]}(x_1, 0.5, 0.4, 0.1) \\
&\quad = x_1^3 + \tfrac{3}{2}\, x_1^2 + \tfrac{27}{5}\, x_1 + \tfrac{91}{20},
\end{aligned}$$

where the classical polynomial is obtained by taking $L = \ln(1+\sigma)/\sigma \to 1$ as $\sigma \to 0$. The two polynomials agree to at least two significant figures in every coefficient, confirming the smooth approach to the classical limit.

As a concrete illustration, we fix $b = 0.5$, $x_2 = 0.4$, $x_3 = 0.1$, and $\sigma = 0.3$ and list the resulting polynomials in x_1 for the first several degrees:

$$\begin{aligned}
&\mathcal{H}_0^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) = 1.0, \\
&\mathcal{H}_1^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.87454754822497 x_1 + 0.437273774112485, \\
&\mathcal{H}_2^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.764833414106307 x_1^2 + 0.764833414106307 x_1 \\
&\qquad + 1.46447145268628, \\
&\mathcal{H}_3^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.668883187107204 x_1^3 + 1.00332478066081 x_1^2 \\
&\qquad + 3.84224975517675 x_1 + 3.28195739040736, \\
&\mathcal{H}_4^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.584970151333509 x_1^4 + 1.16994030266702 x_1^3 \\
&\qquad + 6.72046020611563 x_1^2 + 11.4809111566383 x_1 \\
&\qquad + 11.4466210634613, \\
&\mathcal{H}_5^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.51158421163351 x_1^5 + 1.27896052908378 x_1^4 \\
&\qquad + 9.79560332700318 x_1^3 + 25.1015067585669 x_1^2 \\
&\qquad + 50.0530719325521 x_1 + 47.04156376713, \\
&\mathcal{H}_6^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) \\
&\quad = 0.447404717994691 x_1^6 + 1.34221415398407 x_1^5 \\
&\qquad + 12.8500813095225 x_1^4 + 43.9049223849143 x_1^3 \\
&\qquad + 131.321374019225 x_1^2 + 246.840505543273 x_1 \\
&\qquad + 226.243674232223.
\end{aligned}$$

Figure 1 displays the approximate roots of $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma)$ for degrees $n = 1, 2, \ldots, 29$ superimposed in the complex plane for four different parameter sets. Each panel overlays all root sets from degree 1 through degree 29 for the specified parameter values, putting on view the characteristic patterns in the complex plane together with their symmetry properties and the systematic migration of roots as the degree increases.

Figure 2 tracks how the root configuration evolves as the degree sweeps from 0 to 29 for the equation $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma) = 0$. The parameters are held at $b = 0.5$, $x_2 = 0.4$, $x_3 = 0.1$, $\sigma = 0.3$, while x_1 varies implicitly with the degree n. The three-dimensional panels display all degree values $n \in \{0, 1, \ldots, 29\}$ simultaneously, with the vertical axis indicating the degree; axis labels, tick marks, and the degree legend have been set at a larger font size to facilitate reading.

Numerical approximations of the roots of $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma)$ for degrees $1 \le n \le 15$ are collected in Table 1.

5. Conclusion

This paper has put forward the degenerate multivariate Hermite–Fubini polynomials $\mathcal{H}_n^{[\mathcal{F}]}(x_1, b, x_2, x_3 \mid \sigma)$, a hybrid family that welds the three-variable degenerate Hermite architecture to the combinatorial scaffold of degenerate Fubini polynomials. After writing down the generating function and pinpointing its singularities, we applied the quasi-monomial for-

malism to exhibit both the raising and lowering operators, verified that they fulfil the Weyl bracket relation, and read off the governing differential equation. An operator identity expressing the new polynomials as an exponential differential transform of the degenerate Fubini polynomials was also recorded. On the algebraic side, we assembled a systematic catalogue of series decompositions, summation rules, translation identities, and multi-variable addition theorems. A numerical survey of the root structure for degrees up to $n = 29$ under several parameter regimes, documented through planar and three-dimensional plots as well as tabulated data, brought to light a predominantly complex, conjugate-symmetric zero distribution whose loci migrate outward in an orderly fashion as the degree climbs. Collectively, these findings show that the proposed framework is both internally consistent and broad enough to encompass a number of well-established polynomial families as boundary or degenerate cases.

Several threads remain open for future work. On the theoretical side, it would be worthwhile to carry out an analogous programme for other classical hybrid families—degenerate Hermite–Bernoulli, Hermite–Euler, or Hermite–Genocchi polynomials—in the trivariate regime, and to probe whether the operator approach adopted here adapts to higher-variable or q-deformed settings. Questions about the transition coefficients linking the degenerate multivariate Hermite–Fubini basis to other polynomial bases, together with issues of orthogonality and moment structure, await investigation on both the analytical and computational fronts. With an applied point of view in mind, the differential equations which emerge from the monomiality operators suggest possible relations to physical mathematics and approximation theory; such relations could be investigated using spectral or other numerical approaches, thus increasing the number of potential applications. Lastly, a more thorough investigation of zero-free domains, asymptotic properties of root distributions for large degrees, and sufficient conditions for the existence of only real roots in certain regions of parameters would enhance the results obtained numerically in this paper.

Data availability

No Data is applicable to this study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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