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Approximate Solutions of the Schrödinger Equation for a Momentum-Dependent potential

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Abstract

The solution of one-dimensional Schrödinger equation for a newly proposed potential called modified shifted Deng-Fan momentum-dependent potential is obtained via supersymmetric approach. The expectation values of momentum and position were calculated using Hellmann Feynman Theorem. The effects of momentum-dependent parameter on the solutions of the system as well as the expectation values were studied. Finally, the special cases of the interacting potential were obtained.

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1. Introduction

In both the relativistic and nonrelativistic mechanics, wave equations play some significant roles. The wave functions embedded information that describes a complete quantum system. It is also noted that the solutions of a complicated physical systems are checked and improved through numerical method by wave equations [1, 2]. However, the major objective of the wave equations is the determination of the eigenvalue and its wave function. The solution of this wave equation can be studied via $H|\Psi\rangle = E|\Psi\rangle$, (in this case, H is Hamiltonian and E is energy). The wave function Ψ is expanded as $|\Psi(r,E)\rangle = \sum_n f_n(E)|\Psi_n(r)\rangle$, the term r stands for set of coordinates for real space.

In most of these cases, several potentials such as hyperbolic potential [3, 4], modified Rosen- Morse [5], Hulthén potential [6, 7], Manning-Rosen potential [8-10], Kratzer potential [11], potential family [12, 13], and others, were used. In recent time, a greater interest has been focused on the wave equation with position dependent-potential [14], constant mass-dependent potential [15], and energy-dependent potential for both relativistic and nonrelativistic wave equations [16]. Motivated by these, we want to study the non-relativistic equation with a modified shifted Deng- Fan momentum-dependent potential. In the present study, the effect of the momentum-dependent parameter on energy of the system will be examined under the proposed potential and its subset potentials. The modified shifted Deng-Fan potential is a combination of Deng-Fan and Hulthén like potentials. The Deng-Fan potential also known as improved Manning-Rosen potential, is an empirical and diatomic molec-

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Table 1. Bound states of the modified shifted Deng-Fan momentum-dependent potential model with $\mu = \hbar = 1$, $\alpha = 0.15$ $Å$, $r_e = 0.4$ $Å$, $D_e = 5$ eV and $D_0 = 0.2$ eV for various n and l.

n	l	$\rho = 1$	$\rho = 0$
0	0	2.4925255	4.0257731
1	0	3.9418902	4.6881046
	1	4.2651097	4.8366869
2	0	4.5080914	4.8958752
	1	4.6571299	4.9527631
	2	4.8134283	4.9944769
3	0	4.7739363	4.9739034
	1	4.8496210	4.9946798
	2	4.9305476	5.0055476
	3	4.9863491	5.0015209

ular potential function proposed in 2012 [17]. This potential was used to examine the molecular vibrations and the energy eigenvalues under relativistic and nonrelativistic wave equation [18]. In ref. [19], it was theoretically used to study HF and information entropy. The Deng-Fan potential model is given by

$$V(r) = D_e\left(1 - \frac{e^{\alpha r_e} - 1}{e^{\alpha r} - 1}\right)^2. \tag{1}$$

where, D_e is the dissociation energy, r_e is the equilibrium bond length, α is the screening parameter and r is the internuclear separation. The scheme of our presentation is as follows: In section 2, we presented the bound state solutions, Results and discussion are given in section 3 while conclusion is given in section 4.

2. Bound State Solutions

In the nonrelativistic quantum mechanics, the radial Schröd–inger equation with an interacting potential $V(r)$ together with a nonrelativistic energy $E_{n\ell}$, is given by

$$-\frac{\hbar^2}{2\mu}\frac{d^2U_{n,\ell}(r)}{dr^2} + \left[V(r) - E_{n,\ell} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]U_{n,\ell}(r) = 0, \tag{2}$$

where, l is the angular quantum number, r is the internuclear distance, n is the quantum number, $\hbar$ is the reduced Planck's constant and $U_{n,\ell}(r)$ is the wave function. In this study, we shall solve the radial Schrödinger equation above for modified shifted Deng-Fan momentum-dependent potential. Thus, the potential function for momentum dependent is given as

$$V(r) = \left[D_e\left(1 - \frac{e^{\alpha r_e} - 1}{e^{\alpha r} - 1}\right)^2 - D_0\left(\frac{1}{e^{\alpha r} - 1}\right)\right](1+\rho). \tag{3}$$

In the equation above, D_e is the dissociation energy with different values for various diatomic molecules, D_0 is the potential strength whose value is arbitrary chosen and r_e is the bond separation. The Schrödinger equation in equation (2) has centrifugal term which needs to be approximated. Several approximation scheme have been used for different potential [20]. In this study, the centrifugal term will be approximated using

$$\frac{1}{r^2} \approx \alpha^2\left(C_0 + \frac{e^{\alpha r}}{(e^{\alpha r} - 1)^2}\right), \tag{4a}$$

where C_0 is a dimensionless constant obtained by using the following power series

$$\alpha^2\left(C_0 + \frac{e^{\alpha r}}{(e^{\alpha r} - 1)^2}\right) = \alpha^2\left[C_0 + \frac{1}{(\alpha r)^2} - \frac{1}{12} + \frac{(\alpha r)^2}{240} - \frac{(\alpha r)^4}{6048} + O((\alpha r)^6)\right]. \tag{4b}$$

This finally give the dimensionless constant as $C_0 = 1/12$. With equation (3) and equation (4), the radial Schrödinger equation in equation (2) becomes

$$\frac{d^2U_{n,\ell}(r)}{dr^2} = \left[\frac{\left(\frac{2\mu D_e\beta_2 b^2 e^{-\alpha r}}{\hbar^2} + \ell(\ell+1)\alpha^2\right)e^{-\alpha r}}{(1 - e^{-\alpha r})^2} - \frac{\frac{2\mu\beta_2(2D_e b + D_0)e^{-\alpha r}}{\hbar^2}}{1 - e^{-\alpha r}} + E_2\right]U_{n,\ell}(r), \tag{5}$$

where,

$$\beta_2 = 1 + \rho, \tag{6}$$

$$E_2 = \frac{2\mu(D_e\beta_2 - E_{n,\ell})}{\hbar^2} + \ell(\ell+1)C_0\alpha^2, \tag{7}$$

$$b = e^{\alpha r_e} - 1. \tag{8}$$

In the present study, the authors adopt supersymmetric approach (SUSYQM) to solve equation (5). To proceed using the basic concept and formalism of SUSYQM, we write the ground state wave function as

$$U_{0,\ell}(r) = exp\left(-\int W(r)dr\right), \tag{9}$$

where $W(r)$ is defined as a superpotential that is propose based on the interacting potential [21]. On the basis of this work, the superpotential is given as

$$W(r) = \rho_0 + \frac{\rho_1}{e^{\alpha r} - 1}, \tag{10}$$

where the constants ρ_0 and ρ_1 will be determine later. To relate the superpotential function in equation (10) to equation (5), we establish a Reccati differential equation of the form

$$W^2(r) - \frac{dW(r)}{dr} = \frac{\left(\frac{2\mu D_e\beta_2 b^2 e^{-\alpha r}}{\hbar^2} + \ell(\ell+1)\alpha^2\right)e^{-\alpha r}}{(1 - e^{-\alpha r})^2} - \frac{\frac{2\mu\beta_2(2D_e b + D_0)e^{-\alpha r}}{\hbar^2}}{1 - e^{-\alpha r}} + E_r. \tag{11}$$

Relating equation (11) with equation (5), we can now determine the two parametric constants in the superpotential function as

$$\rho_0^2 = E_r, \tag{12}$$

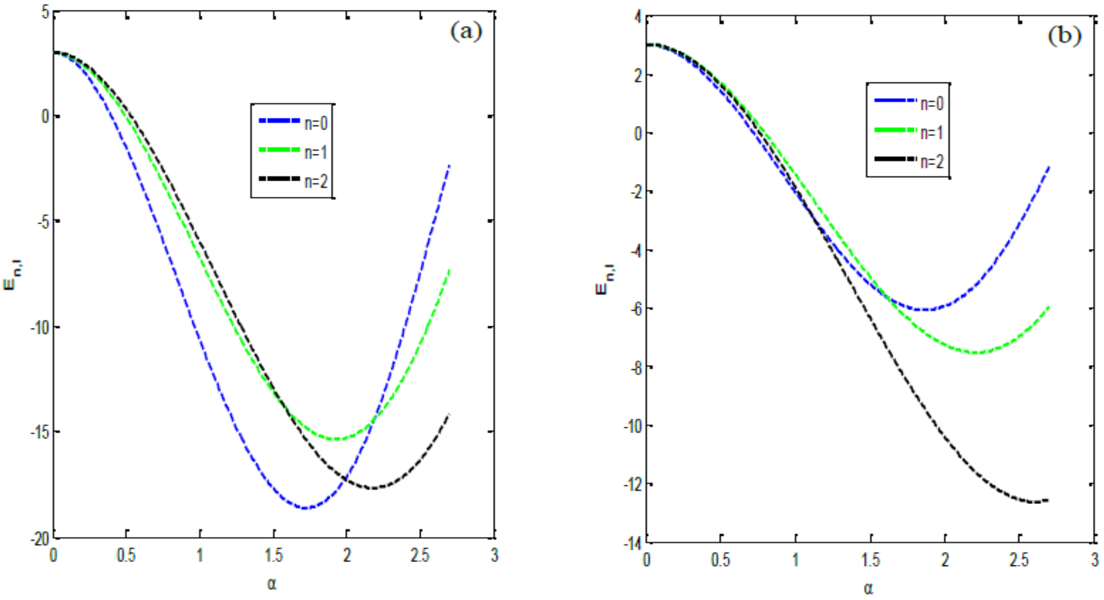

Figure 1. Variation of energy $E_{n,\ell}(eV)$ against the screening parameter $\alpha(\AA)$ with momentum dependent potential ρ_1(a) and without momentum dependent potential ρ_0 (b) for three different quantum states with $\mu = \hbar = l = 1$, $r_e = 0.2\ \AA$, $D_e = 3\ eV$ and $D_0 = 6\ eV$.

Table 2. ro-vibrational energy spectra in (eV) for 2p, 3p, 3d, 4p, 4d, and 4f for Deng-Fan and Hulthèn potential with three different values of the screening parameter for various quantum and angular quantum states.

		$\rho = 0$		$\rho = 1$	
State	α	$D_e = 5, D_0 = 0$	$D_e = 0, D_0 = 5$	$D_e = 5, D_0 = 0$	$D_e = 0, D_0 = 5$
2p	0.05	3.7357639	-1252.5010	1.0959759	-5005.0010
	0.15	3.8435208	-141.39826	1.2510768	-560.56493
	0.25	3.9472499	-52.526042	1.4029712	-205.02604
3p	0.05	4.3380223	-558.05774	2.6082849	-2227.2244
	0.15	4.4738042	-64.248083	2.8289266	-251.93327
	0.25	4.5993040	-24.776910	3.0417528	-93.943576
4p	0.05	4.6792822	-315.00479	3.7798457	-1255.0048
	0.15	4.8154953	-37.265347	4.0535781	-143.93201
	0.25	4.9170844	-15.119792	4.2949652	-55.119792
4d	0.05	4.6480486	-315.00438	3.6133017	-1255.0044
	0.15	4.7878709	-37.261597	3.8809305	-143.92826
	0.25	4.9000869	-15.109375	4.1249749	-55.109375
4f	0.05	4.6313806	-315.00375	3.5085755	-1255.0038
	0.15	4.7760037	-37.255972	3.7735201	-143.92264
	0.25	4.8997892	-15.093750	4.0224272	-55.093750

$$\rho_1 = -\frac{\alpha}{2}\left(1 \pm \sqrt{(1+2l)^2 + \frac{8\mu D_e \beta_2 b^2}{\alpha^2 \hbar^2}}\right), \tag{13}$$

$$\rho_0 = \frac{\frac{2\mu D_e \beta_2 b(b+2)}{\hbar^2}\,\frac{2\mu D_0 \beta_2}{\hbar^2} - \rho_0^2}{2\rho_1}. \tag{14}$$

The bound state solution requires that the wave function satisfies the boundary conditions $U_{n,\ell}(r)/r = 0$ as $r \longrightarrow \infty$ and $U_{n,\ell}(r)/r$ is finite at $r = 0$. The regularity conditions enable us to determine $\rho_0 > 0$ and $\rho_1 < 0$ as can be justify in equation (13) and Eq. (14). By using equation (10), a pair of partner potentials $V_\pm(r) = W^2 \pm dW(r)/dr$ is constructed as:

$$V_+(r) = \rho_0^2 + \frac{\rho_1(2\rho_0 - \rho_1)e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\rho_1(\rho_1 - \alpha)e^{-\alpha r}}{(1 - e^{-\alpha r})^2}, \tag{15}$$

$$V_-(r) = \rho_0^2 + \frac{\rho_1(2\rho_0 - \rho_1)e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\rho_1(\rho_1 + \alpha)e^{-\alpha r}}{(1 - e^{-\alpha r})^2}, \tag{16}$$

The two partner potentials satisfied the relationship

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1), \tag{17}$$

where a_0 is an old set of parameters and a_1 is a new set of parameters uniquely determined from a_0 , the $R(a_1)$ is a reminder term that is independent of the variable r. However,

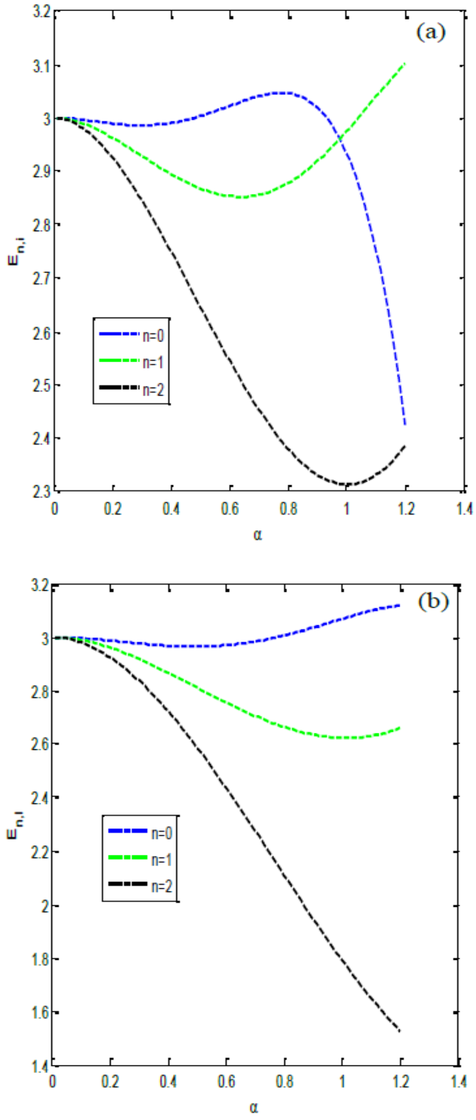

Figure 2. Variation of energy $E_{n,\ell}(eV)$ against the screening parameter $\alpha(\mathring{A})$ with momentum dependent potential ρ_1 (a) and without momentum dependent potential ρ_0 (b) for three different quantum states with $\mu = \hbar = l = 1$, $r_e = 0.2\ \mathring{A}$, $D_e = 3\ eV$ and $D_0 = 6\ eV$.

as $\rho_1 \longrightarrow \rho_1 + \alpha$ for $a_0 = \rho_1$, we can write a recurrent relation of the form $\rho_2 = a_0 + 2\alpha$, $\rho_3 = a_0 + 3\alpha$, $\rho_4 = a_0 + 4\alpha$ and consequently, $\rho_n = a_0 + n\alpha$. For a proper relation, Eq. (17) can be written using the above recurrence relation as

$$R(a_1) = \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_0^2}{2a_0}\right)^2 - \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_1^2}{2a_1}\right)^2, \tag{18}$$

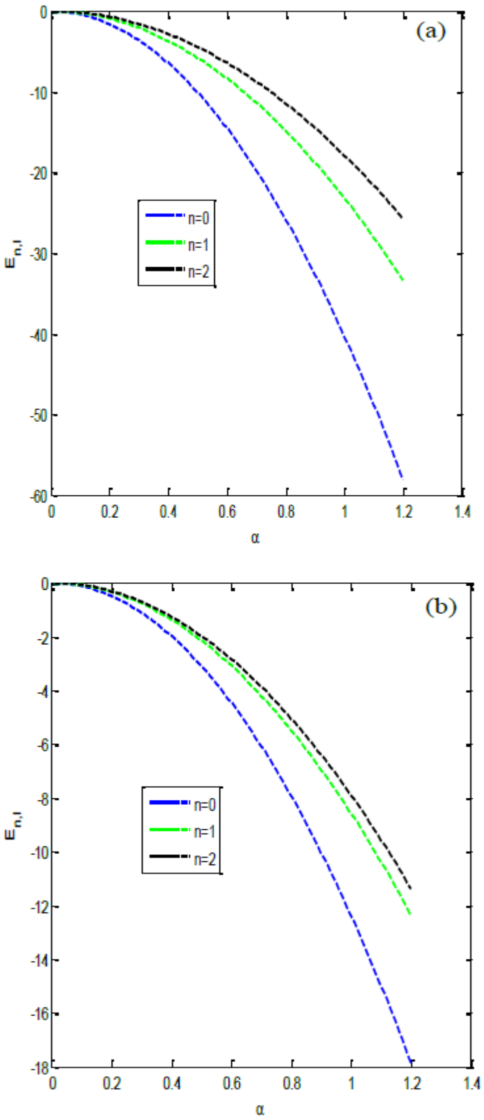

Figure 3. Variation of energy $E_{n,\ell}(eV)$ against the screening parameter α with momentum dependent potential ρ_1 (a) and without momentum dependent potential ρ_0 (b) for three different quantum states with $\mu = \hbar = l = 1$, $r_e = 0.2\ \mathring{A}$, $D_e = 0\ eV$ and $D_0 = 8\ eV$.

$$R(a_2) = \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_1^2}{2a_1}\right)^2 - \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_2^2}{2a_2}\right)^2, \tag{19}$$

$$R(a_3) = \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_2^2}{2a_2}\right)^2 - \left(\frac{\frac{2\mu D_e\beta_2 b(b+2) - 2\mu D_0\beta_2}{\hbar^2} - a_3^2}{2a_3}\right)^2, \tag{20}$$

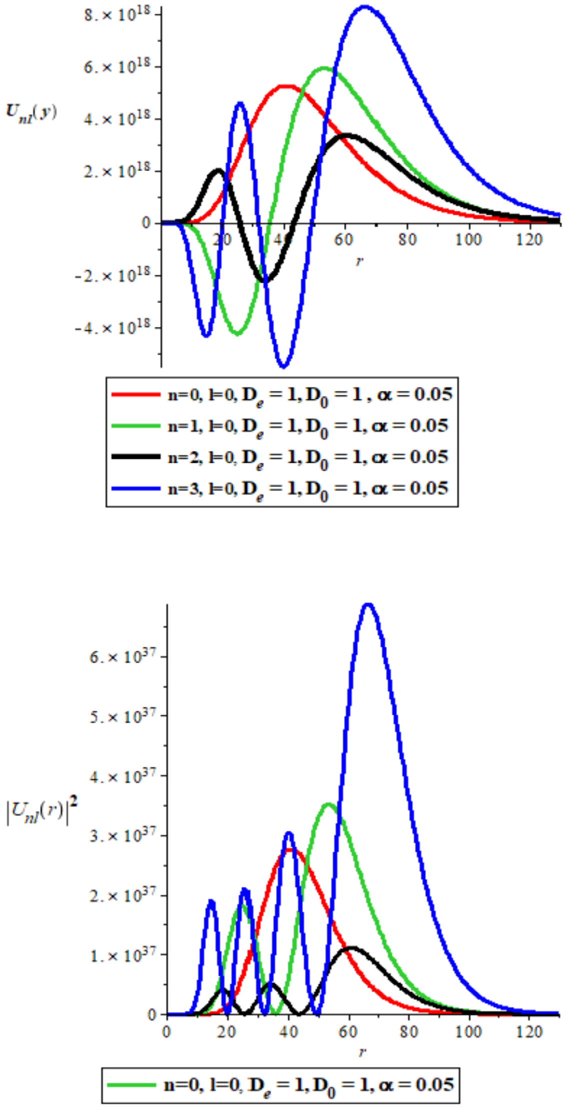

Figure 4. Wave function and probability density for the modified shifted Deng-Fan momentum-dependent potential.

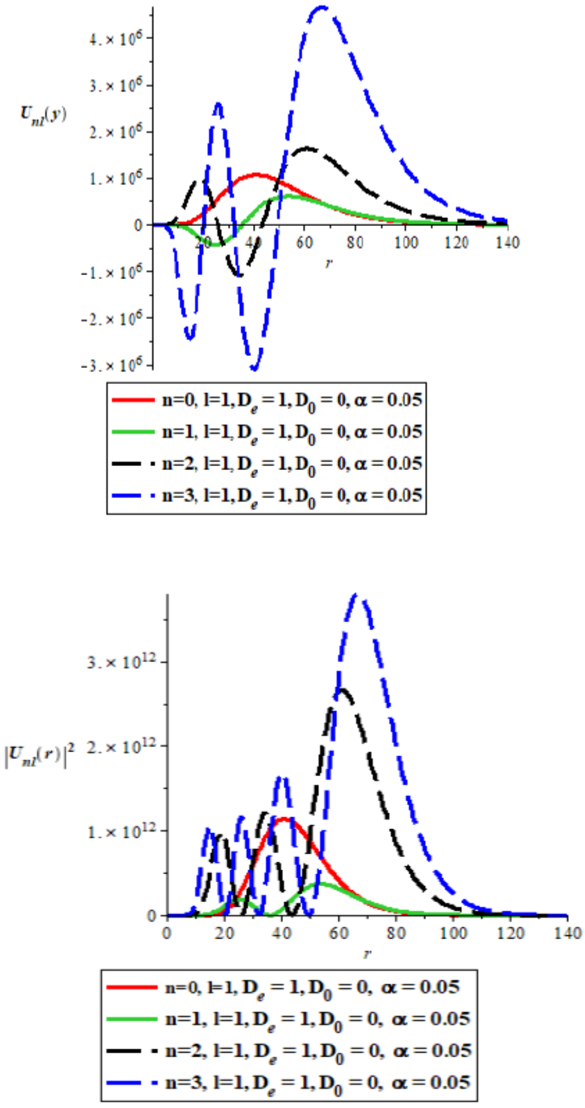

Figure 5. Wave function and probability density for the shifted Deng-Fan momentum-dependent potential.

$$R(a_n)=\left(\frac{\frac{2\mu D_e\beta_2 b(b+2)-2\mu D_0\beta_2}{\hbar^2}-a_{n-1}^2}{2a_{n-1}}\right)^2-\left(\frac{\frac{2\mu D_e\beta_2 b(b+2)-2\mu D_0\beta_2}{\hbar^2}-a_n^2}{2a_n}\right)^2. \tag{21}$$

Using all desirable results inconjuction with the negative partner potential, we finally deduce the energy equation of the non-relativistic equation for modified shifted Deng-Fan momentum dependent potential as

$$E_{n,\ell}=D_e+\frac{\alpha^2\hbar^2}{2\mu}\left[\ell(\ell+1)C_0-\left(\frac{\frac{2\mu\beta_2[D_e b(b+2)-D_0]}{\alpha^2\hbar^2}}{1+2n+\sqrt{(1+2l)^2+\frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}}-\frac{1+2n+\sqrt{(1+2l)^2+\frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}}{4}\right)^2\right]. \tag{22}$$

The radial wave function was obtained using the parametric Nikiforov-Uvarov method. Defining a variable of the form $y=e^{-\alpha r}$ we obtain the wave function in terms of Jacobi polynomial as

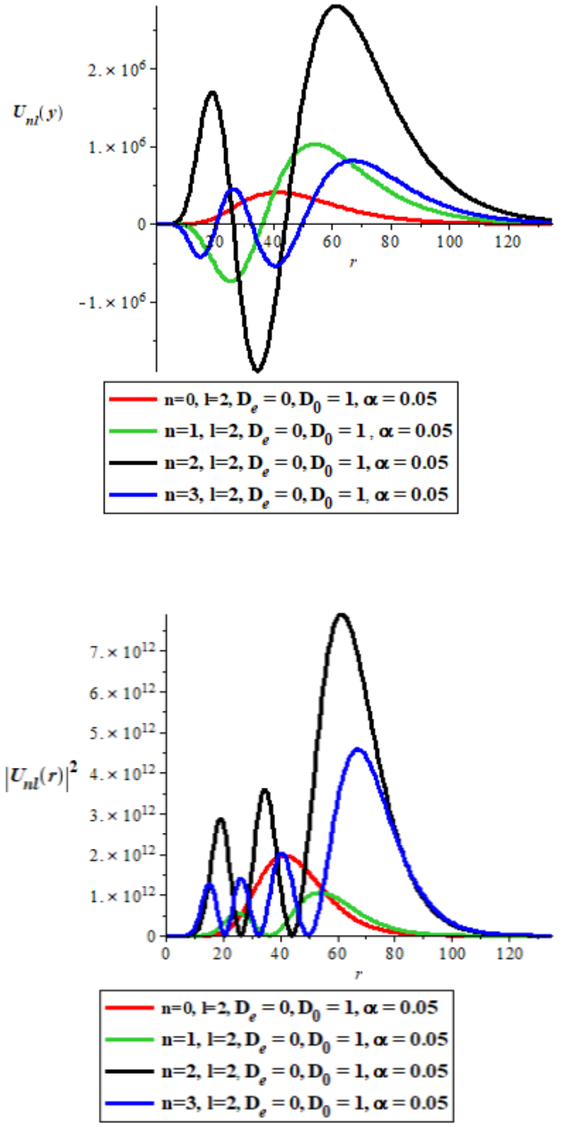

Figure 6. Wave function and probability density for the Hulthèn momentum-dependent potential.

$$U_{n,\ell}(y) = N y^{\sqrt{E_r}}(1-y)^{\frac{1}{2}+\frac{1}{2}\sqrt{(1+2l)^2+\frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}} \times P_n^{\left(2\sqrt{E_r},\sqrt{(1+2l)^2+\frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}\right)}(1-2y). \tag{23}$$

Detail procedure on how to calculate the wave function can be found in ref. [22].

2.1. Expectation Values

In this section, we compute the expectation values using Hellmann-Feynman theorem [23-27]. Given a formula

$$\frac{\partial E(q)}{\partial q} = \left\langle \Psi(q) \left| \frac{\partial H(q)}{\partial q} \right| \Psi(q) \right\rangle. \tag{24}$$

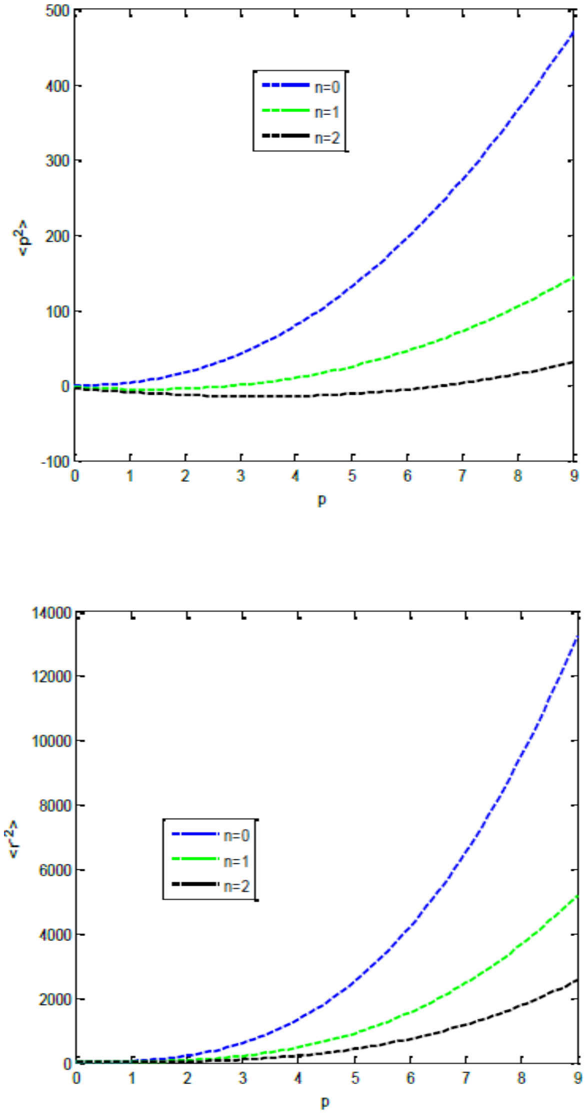

Figure 7. Variation of the expectation values $\left\langle p^2 \right\rangle (eV)$ and $\left\langle r^{-2} \right\rangle (eV)$ against the momentum dependent parameter ρ with $\mu = \hbar = l = 1$, $r_e = 0.2$ Å, $D_e = 3\ eV$, $\alpha = 0.8$ Å and $D_0 = 2\ eV$.

Equation (24) only holds if the normalized eigenfunction is continuous with respect to the parameter. Given the Hamiltonian of potential (1) as

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \left[D_e \left(1 - \frac{e^{\alpha r_e} - 1}{e^{\alpha r} - 1} \right)^2 - D_0 \left(\frac{1}{e^{\alpha r} - 1} \right) \right] (1+\rho), \tag{25}$$

the time and momentum expectation values can now be calculated by transforming the parameter q. To calculate the momentum expectation value $\left\langle p^2 \right\rangle$, we set $q = D_e$ and have

Table 3. Comparison of bound states for Deng-Fan potential with $r_e = 0.4$ Å, $\mu = \hbar = 1$ and $D_e 15\ eV$ for 2p, 3p and 3d at various states.

State		Present	[18]	[28]
2p	0.05	7.86080	7.86080	7.8628
	0.10	7.95330	7.95330	7.95537
	0.15	8.04510	8.04510	8.04724
	0.20	8.13620	8.13620	8.13842
	0.25	8.22663	8.22663	8.22892
3p	0.05	10.99776	10.9978	10.9998
	0.10	11.16256	11.1626	11.1647
	0.15	11.32425	11.3242	11.32647
	0.20	11.48284	11.4828	11.48513
	0.25	11.63834	11.6383	11.64068
3d	0.05	10.21598	10.21598	10.21651
	0.10	10.35354	10.35354	10.35409
	0.15	10.48935	10.48935	10.48992
	0.20	10.62346	10.62346	10.62403
	0.25	10.75591	10.75591	10.75645

Table 4. Comparison of bound states for Hulthèn potential with $\mu = \hbar = 1$ and $D_0 = -\alpha$ for 2p, 3p and 3d at various states.

State		Present	[29]	[30]
2p	0.025	0.1128125	0.1127605	0.1127605
	0.050	0.1012500	0.1010425	0.1010425
	0.075	0.0903125	0.0898478	0.0898478
	0.100	0.0800000	0.0791794	0.0791794
	0.150	0.0612500	0.0594415	0.0594415
3p	0.025	0.0437587	0.0437069	0.0437069
	0.050	0.0333681	0.0331645	0.0331645
	0.075	0.0243837	0.0239397	0.0239397
	0.100	0.0168056	0.0160537	0.0160537
	0.150	0.0058681	0.0044663	0.0044663
3d	0.025	0.0437587	0.0436030	0.0436030
	0.050	0.0333681	0.0327532	0.0327532
	0.075	0.0243837	0.0230307	0.0230307
	0.100	0.0168056	0.0144842	0.0144842
	0.150	0.0018681	0.0013967	0.0013966

$$\left\langle p^2 \right\rangle = 1 - \frac{1}{\mu}\left[\frac{\alpha^2\hbar^2}{2\mu}\left(2\Lambda_0 - n - \frac{1}{2} - \frac{\Lambda_1}{2}\right)\right.$$
$$\left.\times\left(\Lambda_0 - 2\mu D_0\beta_2 - \frac{\mu\beta_2 b^2}{\alpha^2\hbar^2\Lambda_1}\left(\frac{4\Lambda_0}{1+2n+\Lambda_1}\right)\right)\right], \tag{26}$$

$$\left.\begin{aligned} \Lambda_0 &= \frac{2\mu\beta_2(D_e b^2 + 2D_e b - D_0)}{\alpha^2\hbar^2(1+2n+\Lambda_1)}, \\ \Lambda_1 &= \sqrt{(1+2l)^2 + \frac{8\mu\beta_2 D_e b^2}{\alpha^2\hbar^2}} \end{aligned}\right\} \tag{27}$$

To calculate the position expectation value $\left\langle r^{-2} \right\rangle$ we set $q = l$ and obtain

$$\left\langle r^{-2} \right\rangle = \frac{1}{(2l+1)\hbar^2}\left[\alpha^2\hbar^2\left((1+2l)C_0\right.\right.$$
$$-2\left(\frac{2\Lambda_2}{\alpha^2\hbar^2\Lambda_3} - \frac{1}{4} - \frac{n}{2} - \frac{\Lambda_1}{4}\right)$$
$$\left.\left.\times\left(-\frac{4\Lambda_2(1+2l)}{\alpha^2\hbar^2\Lambda_1\Lambda_3^2} - \frac{1+2l}{2\Lambda_1}\right)\right)\right], \tag{28}$$

$$\left.\begin{aligned} \Lambda_2 &= \mu\beta_2(D_e b(b+2) - D_0), \\ \Lambda_3 &= 1 + 2n + \Lambda_1 \end{aligned}\right\} \tag{29}$$

2.2. *Special Cases of the potential*

The two special cases of the potential are the Deng-Fan potential and the Hulthén potential. When we put $D_0 = 0$, the potential becomes the Deng-Fan potential of the form

$$V(r) = D_e\left(1 - \frac{e^{\alpha r_e} - 1}{e^{\alpha r} - 1}\right)^2 (1+\rho), \tag{30}$$

with the energy equation as

$$E_{nl} = D_e + \frac{\alpha^2\hbar^2}{2\mu}\left[\ell(\ell+1)C_0\right.$$
$$-\left(\frac{\frac{2\mu\beta_2 D_e b(b+2)}{\alpha^2\hbar^2}}{1 + 2n + \sqrt{(1+2l)^2 + \frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}}\right.$$
$$\left.\left.-\frac{1 + 2n + \sqrt{(1+2l)^2 + \frac{8\mu D_e\beta_2 b^2}{\alpha^2\hbar^2}}}{4}\right)^2\right]. \tag{31}$$

If we put $D_e = 0$ the interacting potential reduces to Hulthén potential of the form

$$V(r) = -D_0\left(\frac{1}{e^{\alpha r} - 1}\right)(1+\rho), \tag{32}$$

with the energy equation of the form

$$E_{n,\ell} = \frac{\alpha^2\hbar^2}{2\mu}\left[\ell(\ell+1)C_0 - \left(\frac{-\frac{2\mu\beta_2 D_0}{\alpha^2\hbar^2} - \frac{(1+n+l)^2}{2}}{2(1+n+l)}\right)^2\right]. \tag{33}$$

3. Results and Discussion

In Figure 1, we examined the energy of the system with the screening parameter for the modified shifted Deng-Fan potential. The energy of the system goes down while the screening parameter increases. At some point, the energy of the system has a turning point even when the screening parameter continues its linear increase. With momentum-dependent parameter, the energy goes beyond -15 before the turning point while without the momentum-dependent parameter, the energy has the turning point before -13. The energy with momentum dependent parameter for the ground state and first excited has their turning point before the screening parameter equals 2 while without the momentum parameter, all the turning points are obtained above screening parameter equals 2. The turning point physically shows that there is a revised in the direction in the system. In Figure 2, we examined the energy of the system with the screening parameter for the Deng-Fan potential. The variation of energy against the screening parameter in the presence of the momentum dependent parameter at the ground state and at the excited states differs. The energy at the excited states decreases and then have a turning point while the screening parameter increases but for the ground state, the revise case is obtained. This indicates that the effect of momentum-dependent parameter on the energy at the ground state is insignificant but highly significant for the excited states. In the absence of momentum dependent parameter, the energy at different quantum states has the same variation with the screening parameter. For both presence and absence of the momentum dependent parameter, the energy of the system at various state are equal for the screening parameter less than 0.1. In Figure 3, we examined the energy of the system with the screening parameter for Hulthén potential. The energy of the system with and without momentum dependent parameter at various quantum states decreases monotonically as the screening parameter increases. However, with momentum dependent parameter, the energy of the system are lower than their counterpart without momentum dependent parameter. It is noted that the energy of the system at different quantum states without momentum dependent parameter are closer compared to the energy of the system at different quantum states with momentum dependent parameter. In Figures 4, 5 and 6, we plotted the wave function and probability density for modified shifted Deng-Fan momentum-dependent potential, Deng-Fan momentum-dependent potential and Hulthén momentum-dependent potential respectively. The wave function for the modified shifted Deng-Fan momentum-dependent potential has the highest pick both at the negative and positive vertical component, followed by Deng-Fan momentum-dependent potential and lastly, Hulthén momentum-dependent potential. The probability densities are seen to be positive for the three potentials. However, the same trend observed for the wave function are also observed in the probability density for the three potentials. Variation of the expectation values against the momentum dependent parameter ρ with momentum dependent potential are Figure 7. The position expectation value and the momentum expectation value respectively rises as the momentum dependent parameter increases. A clear observation shows that the position expectation value for various quantum states are higher than their counterpart in the momentum expectation value.

In Table 1, we presented the numerical results for modified shifted Deng-Fan momentum-dependent potential and without momentum-dependent potential. The numerical values without momentum-dependent potential are higher than their counterpart with momentum-dependent potential. This simply shows that the present of momentum-dependent potential reduces the energy of the system. The numerical values for Deng-Fan potential ($D_0 = 0$) and Hulthén potential ($D_e = 0$) in the presence and absence of the momentum dependent parameter are presented in Table 2. In both the presence and absence of the momentum dependent parameter, the variation of energy with the screening parameter, quantum and angular quantum numbers are the same. However, the presence of the momentum dependent parameter reduces the energy of the system for both potentials. For Hulthén potential, there are similarities in the numerical values for 3p and 3d as well as 4p, 4d and 4f. The discrepancy for the numerical values rises as the screening parameter increases. In Table 3 and Table 4, we compared the results for Deng-Fan potential and Hulthén potential respectively with existing results. The two results aligned with the previous results.

4. Conclusion

The solutions for modified shifted Deng-Fan potential was obtained with a momentum-dependent potential in the present study. The results showed that the present of the momentum-dependent parameter reduces the energy of the system. This was also noticed for the special cases of the potential. The momentum dependent-parameter showed the same variation against both the position and momentum expectation values.

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