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Jackknife Kibria-Lukman M-Estimator: Simulation and Application

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Abstract

The ordinary least square (OLS) method is very efficient in estimating the regression parameters in a linear regression model under classical assumptions. If the model contains outliers, the performance of the OLS estimator becomes imprecise. Multicollinearity is another issue that can reduce the performance of the OLS estimator. This study proposed the Robust Jackknife Kibria-Lukman (RJKL) estimator based on the M-estimator to deal with multicollinearity and outliers. We examine the superiority of the estimator over existing estimators using theoretical proofs and Monte Carlo simulations. We put the estimator to the test once more using real-world data. We observed that the estimator performs better than the existing estimators.

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1. Introduction

The regression model is commonly used in many disciplines to analyze data. The model's analysis is a form of predictive modelling technique which statistically examines the relationship between two different sets of variables. The first variable is called the dependent, also known as the target variable. The second variable is called the independent, also known as the predictors because they are usually more than one. The model is frequently used for forecasting. Mathematically, the general regression model includes; an n × 1 vector of observations referred to as dependent variable labelled y, a known full column rank of n × p standardize and centered independent variables labelled X, a p × 1 vector of unknown parameters labelled β and an n × 1 vector of disturbances labelled ε. ε is assumed to be normally distributed with $E(\varepsilon) = 0$ and dispersion matrix $Cov(\varepsilon) = \sigma^2 I$. The model is mathematically written as

$$y = X\beta + \varepsilon \tag{1}$$

The Ordinary Least Square (OLS) method is very efficient in estimating the regression parameters in a linear regression model under classical assumptions. The Gauss Markov theorem establishes this fact. The theorem stated the OLS estimator has the best linear unbiased estimator (BLUE) with minimum vari-

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ance in the class of all unbiased linear estimators. However, if the dataset for the regression analysis contains outliers, the performance of the OLS estimator becomes imprecise [1–3]. The OLS estimator of β is given by

$$\widehat{\beta} = S^{-1}X'y \tag{2}$$

where $S = X'X$.
When outliers are present, robust regression is used, which gives better results than the OLS method[4–7]. The M-estimation approach is the most common robust regression method which is used to handle outlier in the y-direction [8]. It is a generalization to maximum likelihood estimation in context of location models. The means that it is nearly as efficient as the OLS. The approach involves minimizing residual function rather than minimizing the number of squared errors as the objective. Generally, the approach considers the likelihood function of β and

$$L(\beta,\sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} f\left(\frac{y_i - x_i{}'\beta}{\sigma}\right) = \frac{1}{\sigma^n}\prod_{i=1}^{n} f\left(\frac{y_i - x_i{}'\beta}{\sigma}\right), \tag{3}$$

and by replacing the OLS criterion with a robust criterion, M-estimator of β is

$$\widehat{\beta}_M = min_\beta \sum_{i=1}^{n} \rho\left(\frac{y_i - x_i{}'\beta}{\widehat{\sigma}_M}\right) \tag{4}$$

The purpose of this study is to propose an estimator that solves the problem of multicollinearity and outlier in linear regression model. We investigate the superiority of the estimator through theoretical comparison, simulations and practical application.

2. Existing Shrinkage Estimators

A popular shrinkage estimator is the ridge estimator (RE), developed by [9] and expressed as:

$$\widehat{\beta}_K = (S + kI)^{-1}X'y \quad = W(k)\widehat{\beta} \tag{5}$$

where $W(k) = (S + kI)^{-1}S$ and $k > 0$. However, RE can be sensitive to outliers in the y-direction. Silvapulle [10] combined the advantage of the ridge estimator and the M-estimator to form the Ridge M-estimator (RME), expressed as follows:

$$\widehat{\beta}_{M\_K} = W(k)\widehat{\beta}_M, \tag{6}$$

Kibria and Lukman [11] recently proposed an estimator called the Kibria-Lukman (KL) estimator, the estimator is expressed as:

$$\widehat{\beta}_{KL} = (S + kI)^{-1}(S - kI)\widehat{\beta} \quad = M(k)\widehat{\beta} \tag{7}$$

where $M(k) = (S + kI)^{-1}(S - kI) = \left(I - 2k(S + kI)^{-1}\right)$.
Grafting the M-estimator into the KL estimator produced the robust KL estimator as follows:

$$\widehat{\beta}_{M\_KL} = M(k)\widehat{\beta}_{M,} \tag{8}$$

The use of Jackknife is also a popular approach for reducing the biasness in bias estimator proposed for handling multicollinearity [12–14]. Ugwuowo et al. [15] proposed a jackknife KL (JKL) estimator under the proposition that the use of jackknife procedure reduces the bias of the ridge estimator [16]. The JKL estimator is expressed as:

$$\widehat{\beta}_{JKL} = \left(I - 2k(S + kI)^{-1}\right)^2\left(I + 2k(S + kI)^{-1}\right)\widehat{\beta} = (M(k))^2N(k)\widehat{\beta} \tag{9}$$

where $N(k) = (S + kI)^{-1}(S + 3kI) = \left(I + 2k(S + kI)^{-1}\right)$.

2.1. A New Robust Estimator

Using the same approach as [10,17,3], we combined the JKL estimator in (9) and the M-estimator to form the Robust Jackknife Kibria-Lukman (RJKL) estimator. It is obvious that the presence of outliers in the y-direction will reduced the efficiency of the JKL estimator. Thus, we defined the RJKL estimator as follows

$$\widehat{\beta}_{\text{RJKL}} = (M(k))^2N(k)\widehat{\beta}_M \tag{10}$$

where $k > 0$.
The canonical form of model (1) is written as

$$Y = Z\alpha + \varepsilon, \tag{11}$$

where $Z = XT$, $\alpha = T'\beta$ and T is the ortogonal matrix whose columns contains the eigenvectors of $X'X$. Then

$$Z'Z = T'X'XT = \Lambda = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_p\right), \tag{12}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p > 0$ are the ordered eigenvalues of $X'X$. Let $\widehat{\alpha}_M$ be an M-defined by the solution of the M-estimating equations $\sum\varphi(e_i/s)z_i = 0$ where $e_i = y_i - z_i\widehat{\alpha}_M$, s is an estimator of scale for the errors and $\varphi(.)$ is some suitably chosen function [18]. Thus, the estimators presented in (2-9) can be written in canonical form as follows:

$$\widehat{\alpha} = \Lambda^{-1}Z'y \tag{13}$$

$$\widehat{\alpha}_K = (\Lambda\text{+kI})^{-1}Z'y \; = W^*(k)\widehat{\alpha} \tag{14}$$

$$\widehat{\alpha}_{M\_K} = W^*(k)\widehat{\alpha}_M \tag{15}$$

$$\widehat{\alpha}_{KL} = \left(I - 2k(\Lambda + kI)^{-1}\right)\widehat{\alpha} = M^*(k)\widehat{\alpha} \tag{16}$$

$$\widehat{\alpha}_{M\_KL} = M^*(k)\widehat{\alpha}_M \tag{17}$$

$$\widehat{\alpha}_{JKL} = \left(I - 2k(\Lambda + kI)^{-1}\right)^2\left(I + 2k(\Lambda + kI)^{-1}\right)\widehat{\alpha} = (M^*(k))^2N^*(k)\widehat{\alpha} \tag{18}$$

$$\widehat{\alpha}_{RJKL} = (M^*(k))^2N^*(k)\widehat{\alpha}_M \tag{19}$$

where $W^*(k) = \Lambda\,(\Lambda\text{+kI})^{-1}$, $M^*(k) = \left(I - 2k(\Lambda + kI)^{-1}\right)$, $N^*(k) = \left(I + 2k(\Lambda + kI)^{-1}\right)$ for k>0.
The organization of this article is as follows. The theoretical comparison among estimators is given in section 2.2. Robust choice of the biasing parameters are discussed in section 3, a simulation study conducted to evaluate the performance of the proposed estimator in section 4 and a real life data was analyzed in Section 6 to illustrate the finding of the paper. Section 7 ends with some concluding remarks.

2.2. Superiority of the RJKL Estimator

The Bias of an estimator $\widetilde{\beta}$ is expressed in equation (20), its Mean Squared Error Matrix (MSEM) in equation (21) and its Scalar Mean Squared Error (MSE) in equation (22).

$$Bias\left(\widetilde{\beta}\right) = E\left(\widetilde{\beta}\right) - \beta \quad (20)$$

$$MSEM\left(\widetilde{\beta}\right) = E\left(\widetilde{\beta}-\beta\right)\left(\widetilde{\beta}-\beta\right)' = D\left(\widetilde{\beta}\right) + Bias\left(\widetilde{\beta}\right) Bias\left(\widetilde{\beta}\right)' \quad (21)$$

$$MSE\left(\widetilde{\beta}\right) = E\left(\widetilde{\beta}-\beta\right)\left(\widetilde{\beta}-\beta\right)' = tr\left(D\left(\widetilde{\beta}\right) + Bias\left(\widetilde{\beta}\right) Bias\left(\widetilde{\beta}\right)'\right) \quad (22)$$

where $D\left(\widetilde{\beta}\right)$ is the variance matrix of $\widetilde{\beta}$, $E\left(\widetilde{\beta}\right)$ is the expectation of $\widetilde{\beta}$ and $tr(A)$ is the trace of a matrix, A. Also, for $\widetilde{\alpha} = T\widetilde{\beta}$, $MSEM(\widetilde{\alpha}) = T'MSEM\left(\widetilde{\beta}\right)T$ and $MSE(\widetilde{\alpha}) = MSE\left(\widetilde{\beta}\right)$. Additionally, for two estimators $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$, $\widetilde{\beta}_1$ is said to be superior to $\widetilde{\beta}_2$ with respect to the MSEM criterion, if and only if, $MSEM\left(\widetilde{\beta}_1\right) - MSEM\left(\widetilde{\beta}_2\right) \geq 0$. If $MSEM\left(\widetilde{\beta}_1\right) - MSEM\left(\widetilde{\beta}_2\right) \geq 0$ then $MSE\left(\widetilde{\beta}_1\right) - MSE\left(\widetilde{\beta}_2\right) \geq 0$ for $j = 1, 2$. The converse is not true. We also made use of the following lemmas for the theoretical comparison:

Lemma 2.1 [19] For some vector, α and a positive definite matrix, A (that is, A > 0); $A - \alpha\alpha' \geq 0$ if and only if $\alpha A^{-1}\alpha' \leq 1$.

Lemma 2.2 [20] Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be two competing estimators of β. Suppose that the difference between the covariance of the two estimators, $D = Cov\left(\widetilde{\beta}_1\right) - Cov\left(\widetilde{\beta}_2\right) > 0$, then $\Delta\left(\widetilde{\beta}_1, \widehat{\beta}_2\right) = MSEM\left(\widetilde{\beta}_1\right) - MSEM\left(\widetilde{\beta}_2\right) \geq 0$ if and only if $d_2'(D + d_1 d_1')^{-1} d_1 \leq 1$. $MSEM\left(\widetilde{\beta}_j\right)$ and d_j denote the mean squared error matrix and bias vector of $\widetilde{\beta}_j$ respectively for $j = 1, 2$.

We used the MSE to prove the superiority of the RJKL. The Mean Squared Error of the OLS estimator and the M-Estimator is expressed in equation (23) and (24), respectively.

$$MSE(\widehat{\alpha}) = \sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i} \quad (23)$$

$$MSE(\widehat{\alpha}_M) = \sum_{i=1}^{p} \Omega_{ii} \quad (24)$$

where $\Omega_{ii} = Cov(\widehat{\alpha}_M)$.

The mean squared error matrix and the scalar mean squared error of the JKL is defined as follows:

$$MSEM(\widehat{\alpha}_{JKL}) = \widehat{\sigma}^2\left(I - (2k(\Lambda + kI)^{-1})^2\right)^2 \Lambda^{-1}\left(I - 2k(\Lambda + kI)^{-1}\right)^2 + Bias(\widehat{\alpha}_{JKL})' Bias(\widehat{\alpha}_{JKL}) \quad (25)$$

$$MSE(\widehat{\alpha}_{JKL}) = \widehat{\sigma}^2 \sum_{i=1}^{p} \frac{\left((\lambda_i + k)^2 - 4k^2\right)^2 (\lambda_i - k)^2}{\lambda_i(\lambda_i + k)^6} + \sum_{i=1}^{p} \frac{\left((\lambda_i - k)^2 (\lambda_i + 3k) - (\lambda_i + k)^3\right)^2 \alpha_i^2}{(\lambda_i + k)^6} \quad (26)$$

where

$Bias(\widehat{\alpha}_{JKL}) = \left[\left(I - 2k(\Lambda + kI)^{-1}\right)^2 \left(I + 2k(\Lambda + kI)^{-1}\right) - I\right]\alpha.$

The corresponding mean square error matrix and the scalar mean square error for the RJKL estimator is:

$$MSEM(\widehat{\alpha}_{RJKL}) = \left(I - (2k(\Lambda + kI)^{-1})^2\right)^2 \Omega\left(I + 2k(\Lambda + kI)^{-1}\right)^2 + Bias(\widehat{\alpha}_{JKL})' Bias(\widehat{\alpha}_{JKL}) \quad (27)$$

$$MSE(\widehat{\alpha}_{RJKL}) = \sum_{i=1}^{p} \frac{\Omega_{ii}\left((\lambda_i + k)^2 - 4k^2\right)^2 (\lambda_i - k)^2}{(\lambda_i + k)^6} + \sum_{i=1}^{p} \frac{\left((\lambda_i - k)^2 (\lambda_i + 3k) - (\lambda_i + k)^3\right)^2 \alpha_i^2}{(\lambda_i + k)^6} \quad (28)$$

We only consider the JKL and the robust estimators in the theoretical comparison and we impose the following conditions to present the main theorems:

1. φ is skew-symmetric and non-decreasing;
2. The errors are symmetric;
3. O is finite.

2.2.1. Superiority of the RJKL estimator over the JKL estimator

We state a theorem for RJKL to be superior to the JKL estimator:

Theorem 2.1: The proposed estimator, $\widehat{\alpha}_{RJKL}$ is superior to the jackknife Kibria Lukman estimator, $\widehat{\alpha}_{\text{JKL}}$ if and only if then $O < \sigma^2\Lambda^{-1}$ for every k > 0.

Proof: The difference between the MSE of the RJKL and the JKL estimator from (28) and (26) is

$$\Delta_1 = MSE(\widehat{\alpha}_{RJKL}) - MSE(\widehat{\alpha}_{\text{JKL}}) = \sum_{i=1}^{p} \frac{\Omega_{ii}\left((\lambda_i + k)^2 - 4k^2\right)^2 (\lambda_i - k)^2}{(\lambda_i + k)^6} - \frac{\widehat{\sigma}^2\left((\lambda_i + k)^2 - 4k^2\right)^2 (\lambda_i - k)^2}{\lambda_i(\lambda_i + k)^6}$$

$$= \sum_{i=1}^{p} \left(\Omega_{ii} - \frac{\sigma^2}{\lambda_i}\right)\left(\frac{\left((\lambda_i + k)^2 - 4k^2\right)^2 (\lambda_i - k)^2}{(\lambda_i + k)^6}\right) \quad (29)$$

For $\Delta_1 < 0$ will imply that $\sum_{i=1}^{p} \Omega_{ii} < \sum_{i=1}^{p} \sigma^2/\lambda_i, k > 0$.

2.2.2. Superiority of the RJKL estimator over the robust KL estimator

The scalar mean square error of the KL estimator is

$$MSE(\widehat{\alpha}_{\text{M\_KL}}) = \sum_{i=1}^{p} \frac{(\lambda_i - k)^2}{(\lambda_i + k)^2}\Omega_{ii} + \sum_{i=1}^{p} \frac{4\alpha_i^2 k^2}{(\lambda_i + k)^2} \tag{30}$$

Theorem 2.2: The proposed estimator, $\widehat{\alpha}_{RJKL}$ is superior to the robust Kibria Lukman estimator, $\widehat{\alpha}_{\text{M\_KL}}$ if and only if

$$\alpha\Omega^{-1}\left[2\left(I - 2k(\Lambda + kI)^{-1}\right)^{-1}\left(\left(I - 2k(\Lambda + kI)^{-1}\right)\left(I + 2k(\Lambda + kI)^{-1}\right) + 1\right)^{-1} - I\right]\alpha' < 1$$

for $k > 0$.

Proof: The difference between the MSE of the RJKL and the KL estimator from (28) and (30) is

$$\begin{aligned}\Delta_2 &= MSE(\widehat{\alpha}_{\text{RJKL}}) - MSE(\widehat{\alpha}_{M_{KL}}) \\ &= \sum_{i=1}^{p} \frac{\Omega_{ii}(\lambda_i - k)^2}{(\lambda_i + k)^2}\left(\frac{\left((\lambda_i + k)^2 - 4k^2\right)^2}{(\lambda_i + k)^4} - 1\right) \\ &\quad + \sum_{i=1}^{p} \frac{\alpha_i^2}{(\lambda_i + k)^2}\left(\frac{\left((\lambda_i - k)^2(\lambda_i + 3k) - (\lambda_i + k)^3\right)^2}{(\lambda_i + k)^4} - 4k^2\right)\end{aligned} \tag{31}$$

Consequently,

$$\Omega\left[2\left(I - 2k(\Lambda + kI)^{-1}\right)^{-1}\left(\left(I - 2k(\Lambda + kI)^{-1}\right)\left(I + 2k(\Lambda + kI)^{-1}\right) + 1\right)^{-1} - I\right]^{-1}$$

is positive definite, provided that $\left((\lambda_i + k)^2 - 4k^2\right)^2 > (\lambda_i + k)^4$ and $\left((\lambda_i - k)^2(\lambda_i + 3k) - (\lambda_i + k)^3\right)^2 > 4k^2(\lambda_i + k)^4$.

2.2.3. Superiority of the RJKL estimator over the Ridge M-estimator

The scalar mean square error of the Ridge M-estimator is

$$MSE(\widehat{\alpha}_M(k)) = \sum_{i=1}^{p} \frac{\lambda_i^2}{(\lambda_i + k)^2}\Omega_{ii} + \sum_{i=1}^{p} \frac{k^2\alpha_i^2}{(\lambda_i + k)^2} \tag{32}$$

Theorem 2.3: The proposed estimator, $\widehat{\alpha}_{RJKL}$ is superior to the Ridge M-estimator, $\widehat{\alpha}_{\text{M\_K}}$ if and only if

$$\alpha\Omega^{-1}\left[\left(\left(I - 2k(\Lambda + kI)^{-1}\right)^2\left(I + 2k(\Lambda + kI)^{-1}\right) - I\right)^2 - \left(\left(I - k(\Lambda + kI)^{-1}\right) - I\right)^2\right] \left[\left(I - k(\Lambda + kI)^{-1}\right)^2 - \left(I - (2k(\Lambda + kI)^{-1})^2\right)^2\left(I + 2k(\Lambda + kI)^{-1}\right)^2\right]^{-1}\alpha' < 1$$

for $k > 0$.

Proof: The difference between the MSE of the RJKL and the Ridge M-estimator from (28) and (32) is

$$\begin{aligned}\Delta_3 &= MSE(\widehat{\alpha}_{\text{RJKL}}) - MSE(\widehat{\alpha}_{M\_K}) \\ &= \sum_{i=1}^{p} \frac{\Omega_{ii}}{(\lambda_i + k)^2}\left(\frac{\left((\lambda_i + k)^2 - 4k^2\right)^2(\lambda_i - k)^2}{(\lambda_i + k)^4} - \lambda_i^2\right) \\ &\quad + \sum_{i=1}^{p} \frac{\alpha_i^2}{(\lambda_i + k)^2}\left(\frac{\left((\lambda_i - k)^2(\lambda_i + 3k) - (\lambda_i + k)^3\right)^2}{(\lambda_i + k)^4} - k^2\right)\end{aligned} \tag{33}$$

Consequently,

$$\Omega\left[\left(\left(I - 2k(\Lambda + kI)^{-1}\right)^2\left(I + 2k(\Lambda + kI)^{-1}\right) - I\right)^2 - \left(\left(I - k(\Lambda + kI)^{-1}\right) - I\right)^2\right]^{-1} \left[\left(I - k(\Lambda + kI)^{-1}\right)^2 - \left(I - (2k(\Lambda + kI)^{-1})^2\right)^2\left(I + 2k(\Lambda + kI)^{-1}\right)^2\right]$$

is positive definite, provided that $\left((\lambda_i + k)^2 - 4k^2\right)^2(\lambda_i - k)^2 > \lambda_i^2(\lambda_i + k)^4$ and $\left((\lambda_i - k)^2(\lambda_i + 3k) - (\lambda_i + k)^3\right)^2 > k^2(\lambda_i + k)^4$.

2.2.4. Superiority of the RJKL estimator over the M-estimator

We state a theorem for RJKL to be superior to the M-estimator:

Theorem 2.4: The proposed estimator, $\widehat{\alpha}_{RJKL}$ is superior to the M-estimator, $\widehat{\alpha}_{\mathrm{M}}$ if and only if

$$\alpha\Omega^{-1}\left(I-\left(I-\left(2k(\Lambda+kI)^{-1}\right)^{2}\right)^{2}\left(I+2k(\Lambda+kI)^{-1}\right)^{2}\right)^{-1}$$
$$\left(\left(I-2k(\Lambda+kI)^{-1}\right)^{2}\left(I+2k(\Lambda+kI)^{-1}\right)-I\right)^{2}\alpha' < 1$$

for k > 0.

Proof: The difference between the MSE of the RJKL and the M estimator from (28) and (24) is

$$\begin{aligned}\Delta_4 &= MSE\left(\widehat{\alpha}_{RJKL}\right)-MSE\left(\widehat{\alpha}_{\mathrm{M}}\right)\\ &= \sum_{i=1}^{p}\Omega_{ii}\left(\frac{\left((\lambda_i+k)^2-4k^2\right)^2(\lambda_i-k)^2}{(\lambda_i+k)^6}-1\right)\\ &\quad+\sum_{i=1}^{p}\frac{\left((\lambda_i-k)^2(\lambda_i+3k)-(\lambda_i+k)^3\right)^2\alpha_i^2}{(\lambda_i+k)^6} \qquad (34)\end{aligned}$$

Consequently,

$$\Omega\left(I-\left(I-\left(2k(\Lambda+kI)^{-1}\right)^{2}\right)^{2}\left(I+2k(\Lambda+kI)^{-1}\right)^{2}\right)$$
$$\left(\left(I-2k(\Lambda+kI)^{-1}\right)^{2}\left(I+2k(\Lambda+kI)^{-1}\right)-I\right)^{2-1}$$

is positive definite, provided that $\left((\lambda_i+k)^2-4k^2\right)^2(\lambda_i-k)^2 > (\lambda_i+k)^6$. Notice that the right hand expression after the addition sign is positive, this follows from bias squared.

3. Robust choice of the biasing parameter

It is customary to use the optimization procedure to obtain the biasing parameter of an estimator [3]. This is done by minimizing equation (28), which is rewritten in (35), with respect to k.

$$\begin{aligned}f(k) = &\sum_{i=1}^{p}\frac{\Omega_{ii}\left((\lambda_i+k)^2-4k^2\right)^2(\lambda_i-k)^2}{(\lambda_i+k)^6}\\ &+\sum_{i=1}^{p}\frac{\left((\lambda_i-k)^2(\lambda_i+3k)-(\lambda_i+k)^3\right)^2\alpha_i^2}{(\lambda_i+k)^6} \qquad (35)\end{aligned}$$

This can be obtained by setting

$$\frac{\partial f(k)}{\partial k} = 0 \qquad (36)$$

Proceeding with (36) will yield a rather complex estimation for k. Thus, we propose to use the robust version of the biasing parameter used for the jackknife KL estimator [15]. The biasing parameter used for the jackknife KL estimator is presented in (37).

$$\hat{k} = \sqrt{\frac{p\widehat{\sigma}^2}{\sum_{i=1}^{p}\widehat{\alpha}_i^2}} \qquad (37)$$

This parameter (37) is the squared root of harmonic mean of the biasing parameter used for ridge estimator. The robust equivalence of the parameter, $\hat{k}$ is

$$\hat{k}_M = \sqrt{\frac{p\hat{A}^2}{\sum_{i=1}^{p}\widehat{\alpha}_{Mi}^2}} \qquad (38)$$

where $\hat{A}^2$ is given by Huber [8] as

$$\hat{A}^2 = \frac{s^2(n-p)^{-1}\sum_{i=1}^{p}\left(\varphi(e_i/s)\right)^2}{\left(\sum_{i=1}^{p}(1/n)\varphi'(e_i/s)\right)^2} \qquad (39)$$

With the assumption that $\widehat{\alpha}_M \sim N(\alpha, A^2\Lambda^{-1})$. And, this holds since $n^{\frac{1}{2}}\left(\widehat{\alpha}_M^2-\alpha\right)\Lambda N\left(0, A^2\Lambda^{-1}\right)$, where

$$A^2 = \frac{s_o^2 E(\varphi^2(\varepsilon/s_o))}{\left(E(\varphi'(\varepsilon/s_o))\right)^2}, \qquad (40)$$

with the scale estimate s_o.

4. Monte Carlo Simulation Study

We adopt the Monte Carlo simulation design [21, 22] to observe the superiority of the Robust Jackknife KL estimator. The design was also recently adopted in related studies [23–28]. R programming language was used for the simulation.

The following equation is used to generate the predictors:

$$x_{ij} = (1-\rho^2)^{1/2}z_{ij}+\rho z_{i,p+1}, \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,p \qquad (41)$$

where ρ^2 denotes the correlation between independent variables and z_{ij} are pseudo-random numbers from the standard normal distribution. The coefficients $\beta_1, \beta_2, \ldots, \beta_p$ are selected as the normalized Eigenvectors corresponding to the largest eigenvalue of $X'X$ such that $\beta'\beta = 1$. This is a common restriction in simulation studies of this kind [3, 25–26, 29–30].

The dependent variable determined by

$$y_i = \beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}+\cdots+\beta_p x_{ip}+\varepsilon_i, \quad i = 1,2,\ldots,n \qquad (42)$$

where the $\varepsilon_i' s$ are independently generated from $N(0, \sigma^2)$. We consider number of independent variables, p = 3 and p = 7. We introduced 10%, 20% and 30% outlier into each sample size considered in the simulation study [31]. The other specifications considered in the simulation design is as follows:

1. $\rho = 0.70,\ 0.80,\ 0.90,\ 0.99$
2. $\sigma = 1,\ 5,\ 10$

Table 1: Estimated MSEs of the estimators when there are no outliers and p=3

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	0.0906	0.0945	0.0817	0.0856	0.0743	0.0781	0.0640	0.0675
		0.8	0.1201	0.1250	0.1033	0.1082	0.0886	0.0935	0.0663	0.0708
		0.9	0.2145	0.2232	0.1633	0.1720	0.1201	0.1286	0.0618	0.0689
		0.99	1.9546	2.0313	0.5968	0.6575	0.0671	0.0920	0.0373	0.0400
	5	0.7	2.2644	2.3616	1.7243	1.8253	1.2825	1.3828	0.7235	0.8078
		0.8	3.0013	3.1259	2.1305	2.2606	1.4436	1.5713	0.6654	0.7631
		0.9	5.3623	5.5804	3.2613	3.4870	1.7660	1.9747	0.5260	0.6413
		0.99	48.8650	50.7823	13.0782	14.5443	2.0620	2.4665	5.4359	5.1779
	10	0.7	9.0574	9.4463	6.6716	7.0819	4.7377	5.1484	2.3424	2.6843
		0.8	12.0052	12.5036	8.2686	8.7944	5.3753	5.8912	2.2291	2.6120
		0.9	21.4491	22.3218	12.7356	13.6432	6.6773	7.5085	1.8716	2.3134
		0.99	195.4598	203.1294	52.0364	57.8922	8.4338	10.0135	23.8768	22.6897
100	1	0.7	0.0470	0.0492	0.0443	0.0464	0.0418	0.0438	0.0376	0.0395
		0.8	0.0637	0.0666	0.0583	0.0611	0.0533	0.0560	0.0445	0.0470
		0.9	0.1159	0.1213	0.0985	0.1034	0.0826	0.0871	0.0562	0.0600
		0.99	1.0714	1.1209	0.4493	0.4816	0.1042	0.1212	0.0088	0.0096
	5	0.7	1.1756	1.2303	0.9732	1.0242	0.7955	0.8426	0.5246	0.5635
		0.8	1.5919	1.6658	1.2475	1.3150	0.9527	1.0137	0.5344	0.5815
		0.9	2.8982	3.0318	2.0106	2.1281	1.3059	1.4063	0.5010	0.5642
		0.99	26.7861	28.0236	9.3727	10.1409	1.7215	2.0322	0.7853	0.7860
	10	0.7	4.7023	4.9212	3.7496	3.9546	2.9304	3.1197	1.7457	1.8980
		0.8	6.3674	6.6630	4.8106	5.0824	3.5146	3.7596	1.7988	1.9809
		0.9	11.5929	12.1274	7.7879	8.2604	4.8593	5.2603	1.7529	1.9918
		0.99	107.1445	112.0945	37.1706	40.2385	6.9026	8.1198	4.0429	3.9826
200	1	0.7	0.0229	0.0240	0.0222	0.0233	0.0216	0.0226	0.0205	0.0215
		0.8	0.0310	0.0325	0.0297	0.0311	0.0284	0.0298	0.0260	0.0273
		0.9	0.0566	0.0593	0.0520	0.0545	0.0475	0.0499	0.0394	0.0415
		0.99	0.5249	0.5498	0.2979	0.3151	0.1365	0.1474	0.0149	0.0181
	5	0.7	0.5723	0.6000	0.5105	0.5363	0.4536	0.4775	0.3558	0.3763
		0.8	0.7761	0.8133	0.6660	0.6999	0.5659	0.5968	0.3999	0.4251
		0.9	1.4158	1.4833	1.1108	1.1697	0.8468	0.8978	0.4615	0.4986
		0.99	13.1226	13.7447	5.9327	6.3356	1.8403	2.0648	0.1855	0.2230
	10	0.7	2.2894	2.4000	1.9666	2.0687	1.6748	1.7687	1.1970	1.2755
		0.8	3.1043	3.2531	2.5609	2.6958	2.0798	2.2015	1.3328	1.4300
		0.9	5.6634	5.9333	4.2718	4.5066	3.1064	3.3082	1.5428	1.6854
		0.99	52.4905	54.9787	23.3878	24.9955	7.1815	8.0669	0.9502	1.0706
250	1	0.7	0.0185	0.0196	0.0180	0.0191	0.0176	0.0186	0.0168	0.0178
		0.8	0.0250	0.0265	0.0241	0.0256	0.0232	0.0246	0.0216	0.0229
		0.9	0.0456	0.0484	0.0424	0.0450	0.0393	0.0418	0.0337	0.0358
		0.99	0.4227	0.4494	0.2553	0.2738	0.1308	0.1426	0.0200	0.0236
	5	0.7	0.4614	0.4892	0.4180	0.4438	0.3775	0.4014	0.3062	0.3266
		0.8	0.6248	0.6630	0.5462	0.5810	0.4738	0.5055	0.3503	0.3760
		0.9	1.1391	1.2097	0.9160	0.9777	0.7197	0.7730	0.4200	0.4584
		0.99	10.5672	11.2344	5.0044	5.4380	1.6925	1.9306	0.1565	0.1963
	10	0.7	1.8457	1.9567	1.6107	1.7131	1.3959	1.4901	1.0355	1.1138
		0.8	2.4993	2.6521	2.0988	2.2374	1.7399	1.8648	1.1655	1.2645
		0.9	4.5565	4.8390	3.5136	3.7599	2.6261	2.8375	1.3832	1.5302
		0.99	42.2689	44.9375	19.6719	21.4040	6.5464	7.4852	0.7302	0.8829

3. $n = 50,\ 100,\ 200,\ 250$

The biasing parameter, $\hat{k}$ was used for the ridge, KL and Jackknife KL estimator while the robust version of the biasing parameter, $\hat{k}_M$ is used for the robust version of the estimators, including the robust jackknife KL. The simulation was done 2000 times by generating new pseudo-random numbers and the esti-

Table 2: Estimated MSEs of the estimators when there is 10% outlier and p=3

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	13.6963	0.1479	10.0387	0.1313	7.0596	0.1169	3.3265	0.0949
		0.8	19.9972	0.1966	13.7140	0.1671	8.8318	0.1409	3.4809	0.1002
		0.9	39.1477	0.3526	23.1621	0.2666	12.0224	0.1943	3.1734	0.0968
		0.99	384.1922	3.2231	101.8056	1.0736	15.5236	0.1687	43.3645	0.0674
	5	0.7	137.1126	3.6961	99.5904	2.8692	69.2274	2.1761	31.7675	1.2357
		0.8	184.4161	4.9145	125.4914	3.5931	80.0792	2.5253	31.1814	1.2177
		0.9	334.6314	8.8142	196.9320	5.6335	101.7029	3.2920	27.0002	1.0943
		0.99	3091.902	80.5657	819.9766	24.6852	126.6444	4.3096	359.7422	6.3585
	10	0.7	515.965	14.7841	374.3237	11.2446	259.8384	8.3034	119.0229	4.3970
		0.8	689.075	19.6578	468.3979	14.1016	298.5428	9.6738	116.2113	4.4187
		0.9	1239.973	35.2564	729.1771	22.1921	376.3083	12.7275	100.0509	4.1134
		0.99	11372.96	322.2612	3017.1655	98.4042	465.9051	17.3591	1325.6418	27.3046
100	1	0.7	7.6998	0.0710	6.0621	0.0665	4.6620	0.0622	2.6673	0.0549
		0.8	11.3464	0.0960	8.4537	0.0872	6.0668	0.0790	2.9758	0.0645
		0.9	22.3993	0.1742	14.8158	0.1462	9.0488	0.1209	3.1156	0.0796
		0.99	221.3106	1.6064	75.0130	0.6830	13.3166	0.1701	9.6412	0.0139
	5	0.7	79.8287	1.7757	62.0647	1.4746	46.9970	1.2094	25.9263	0.8017
		0.8	109.6894	2.3985	80.9090	1.8927	57.3702	1.4589	27.4833	0.8379
		0.9	202.3623	4.3550	133.0403	3.0711	80.7291	2.0451	27.6453	0.8410
		0.99	1888.093	40.1531	641.9309	14.9983	116.0079	3.2139	84.8115	1.0515
	10	0.7	300.1889	7.1028	233.0442	5.7389	176.1599	4.5550	96.8446	2.8014
		0.8	409.2441	9.5939	301.4447	7.3756	213.4082	5.5098	102.0062	2.9647
		0.9	749.2885	17.4197	492.0093	12.0162	298.1691	7.7904	102.0787	3.0693
		0.99	6946.875	160.6115	2359.3943	59.6581	427.1866	12.8245	315.4987	5.0330
200	1	0.7	4.4751	0.0355	3.7892	0.0344	3.1754	0.0333	2.1944	0.0314
		0.8	6.5295	0.0480	5.3021	0.0457	4.2303	0.0434	2.6193	0.0394
		0.9	12.9185	0.0872	9.5866	0.0793	6.8417	0.0719	3.2967	0.0584
		0.99	129.3138	0.8062	56.8508	0.4555	17.1766	0.2086	2.6167	0.0263
	5	0.7	46.0261	0.8880	38.4685	0.7913	31.7480	0.7021	21.1678	0.5494
		0.8	62.9938	1.1988	50.6022	1.0295	39.8739	0.8757	24.0539	0.6218
		0.9	116.2244	2.1794	85.6010	1.7212	60.5908	1.3247	28.8401	0.7453
		0.99	1090.0300	20.1500	479.2827	9.6267	145.7462	3.4039	23.5782	0.4220
	10	0.7	175.6885	3.5519	146.6835	3.0737	120.9005	2.6397	80.3387	1.9222
		0.8	238.7105	4.7953	191.5810	3.9980	150.8015	3.2894	90.7374	2.1769
		0.9	436.7494	8.7175	321.4645	6.6959	227.3712	4.9901	108.0523	2.6457
		0.99	4062.8734	80.5996	1786.2549	38.1570	543.0836	13.4103	87.9686	1.8838
250	1	0.7	4.0422	0.0296	3.4584	0.0288	2.9318	0.0280	2.0741	0.0266
		0.8	5.8499	0.0403	4.8094	0.0387	3.8919	0.0371	2.4794	0.0341
		0.9	11.4636	0.0738	8.6490	0.0682	6.2987	0.0627	3.1569	0.0527
		0.99	114.3099	0.6868	52.1028	0.4135	16.7411	0.2115	2.3191	0.0344
	5	0.7	38.3098	0.7405	32.3421	0.6688	26.9940	0.6018	18.4195	0.4841
		0.8	52.5104	1.0077	42.6718	0.8802	34.0728	0.7629	21.1009	0.5630
		0.9	96.9599	1.8450	72.5000	1.4935	52.2625	1.1835	25.7424	0.7081
		0.99	910.4138	17.1656	413.6027	8.6770	132.6110	3.3776	19.6279	0.4017
	10	0.7	145.3148	2.9619	122.5263	2.5978	102.1118	2.2638	69.4115	1.6977
		0.8	197.7114	4.0306	160.4884	3.4168	127.9780	2.8640	79.0100	1.9680
		0.9	362.2294	7.3800	270.6038	5.8033	194.8606	4.4479	95.7777	2.4934
		0.99	3377.0204	68.6613	1533.5258	34.3510	491.4155	13.2504	72.9195	1.7006

mated MSE calculated as:

$$mse\left(\widehat{\alpha}\right) = \frac{1}{2000}\sum_{j=1}^{2000}\left(\widehat{\alpha}_{ij}-\alpha_i\right)'\left(\widehat{\alpha}_{ij}-\alpha_i\right) \quad (43)$$

As the standard deviation, σ and the degree of multicollinearity, ρ increases, the mean square error, MSEs of the estimators, $\widehat{\alpha}$, $\widehat{\alpha}_M$, $\widehat{\alpha}_K$, $\widehat{\alpha}_{M\_K}$, $\widehat{\alpha}_{KL}$, $\widehat{\alpha}_{M\_KL}$, $\widehat{\alpha}_{JKL}$ and $\widehat{\alpha}_{RJKL}$ also increases. The mean square error, MSEs of the estimators,

Table 3: Estimated MSEs of the estimators when there are 20% outliers and p=3

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	27.4909	0.2521	20.2574	0.2197	14.3415	0.1908	6.8243	0.1449
		0.8	40.3804	0.3364	27.9507	0.2809	18.2475	0.2316	7.4111	0.1545
		0.9	79.5740	0.6035	47.8373	0.4495	25.5539	0.3206	7.1857	0.1496
		0.99	790.303	5.5425	218.992	1.9246	35.8040	0.3346	86.3257	0.1074
	5	0.7	272.653	6.3015	199.545	4.9528	140.154	3.8028	65.9791	2.1712
		0.8	368.888	8.4073	253.726	6.2640	164.505	4.4998	66.6474	2.2272
		0.9	672.473	15.0834	402.222	9.9382	213.812	6.0549	60.7675	2.1345
		0.99	6245.01	138.519	1727.07	45.6565	287.645	8.3804	699.754	7.5586
	10	0.7	1025.56	25.2056	749.801	19.5587	526.109	14.7806	247.809	8.1339
		0.8	1377.33	33.6286	946.375	24.7504	613.041	17.5204	248.908	8.4380
		0.9	2490.26	60.3325	1488.33	39.3541	790.911	23.7011	226.165	8.2671
		0.99	22949.5	554.071	6349.96	182.212	1065.63	33.6397	2586.37	32.0533
100	1	0.7	15.1429	0.1200	11.8102	0.1110	8.9723	0.1026	4.9686	0.0879
		0.8	22.3024	0.1624	16.4695	0.1456	11.6798	0.1300	5.5478	0.1027
		0.9	43.9747	0.2953	28.8485	0.2439	17.4039	0.1979	5.7482	0.1243
		0.99	433.363	2.7235	145.132	1.1828	24.3159	0.3201	20.1608	0.0265
	5	0.7	156.829	2.9992	121.501	2.5008	91.5606	2.0585	49.7936	1.3663
		0.8	215.022	4.0580	157.956	3.2314	111.340	2.5171	52.3434	1.4738
		0.9	396.164	7.3797	259.142	5.3047	155.911	3.6269	51.5900	1.5822
		0.99	3695.54	68.0707	1242.22	27.1764	212.261	6.7520	172.944	1.4096
	10	0.7	593.815	11.9965	459.802	9.8262	346.290	7.9204	188.139	5.0142
		0.8	808.576	16.2319	593.809	12.7085	418.459	9.7046	196.747	5.4576
		0.9	1479.48	29.5183	967.986	20.9228	582.716	14.0706	193.291	5.9866
		0.99	13717.4	272.281	4619.94	108.333	794.266	26.9217	641.662	6.2819
200	1	0.7	8.5590	0.0546	7.1615	0.0526	5.9191	0.0507	3.9641	0.0472
		0.8	12.4987	0.0740	10.0274	0.0700	7.8863	0.0661	4.7286	0.0589
		0.9	24.7452	0.1350	18.1275	0.1214	12.7229	0.1086	5.8933	0.0856
		0.99	247.230	1.2522	106.201	0.6986	30.1372	0.3139	4.8016	0.0399
	5	0.7	88.0770	1.3649	73.1962	1.2110	59.9960	1.0693	39.3377	0.8269
		0.8	120.611	1.8505	96.2486	1.5846	75.2182	1.3437	44.4431	0.9475
		0.9	222.578	3.3753	162.506	2.6714	113.635	2.0624	52.3102	1.1721
		0.99	2086.82	31.2969	897.195	15.4657	255.893	5.8296	41.8475	0.7606
	10	0.7	336.691	5.4597	279.756	4.7377	229.235	4.0805	150.121	2.9871
		0.8	457.457	7.4019	365.064	6.2033	285.290	5.1345	168.490	3.4419
		0.9	836.317	13.5011	610.777	10.4839	427.256	7.9212	196.828	4.3292
		0.99	7773.24	125.185	3346.15	61.5152	957.602	23.0985	155.941	3.2643
250	1	0.7	7.6189	0.0448	6.4429	0.0434	5.3895	0.0421	3.7021	0.0396
		0.8	11.067	0.0609	8.9921	0.0581	7.1774	0.0554	4.4397	0.0502
		0.9	21.7733	0.1115	16.2260	0.1019	11.6384	0.0927	5.6469	0.0760
		0.99	218.570	1.0377	97.8022	0.6138	30.2036	0.3058	4.1657	0.0496
	5	0.7	72.4863	1.1186	60.8732	1.0056	50.4973	0.9004	33.9836	0.7165
		0.8	99.6846	1.5225	80.5435	1.3247	63.8719	1.1432	38.9391	0.8365
		0.9	184.767	2.7871	137.222	2.2539	98.0540	1.7852	47.2803	1.0714
		0.99	1741.86	25.9362	780.620	13.4438	242.517	5.5131	34.7890	0.7911
	10	0.7	274.650	4.4745	230.508	3.9307	191.072	3.4313	128.323	2.5835
		0.8	374.584	6.0901	302.465	5.1794	239.672	4.3583	145.838	3.0239
		0.9	687.970	11.1482	510.550	8.8300	364.476	6.8313	175.387	3.9233
		0.99	6427.93	103.743	2876.86	53.4205	891.498	21.7854	128.508	3.2871

$\widehat{\alpha}$, $\widehat{\alpha}_M$, $\widehat{\alpha}_K$, $\widehat{\alpha}_{M\_K}$, $\widehat{\alpha}_{KL}$, $\widehat{\alpha}_{M\_KL}$, $\widehat{\alpha}_{JKL}$ and $\widehat{\alpha}_{RJKL}$ decrease as the sample size increases. As expected, the MSEs increases as the outliers increases. This is what births the purpose of our study, to propose a better estimator which handles outliers jointly with multicollinearity by yielding a minimum MSE. As we know it, the performance of the OLS is poor in the presence of outliers and multicollinearity. This can be observed in Table (1-8) and when there is no outlier (Table 1 & 5), the jackknife

Table 4: Estimated MSEs of the estimators when there are 30% outliers and p=3

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	42.8128	1.1147	31.6134	0.9605	22.4115	0.8201	10.6056	0.5859
		0.8	63.1076	1.5114	43.7884	1.2524	28.6259	1.0210	11.5133	0.6538
		0.9	124.554	2.8334	75.1313	2.1200	40.1568	1.5214	10.9966	0.7209
		0.99	1235.20	26.6215	344.127	11.0545	51.1102	2.7550	114.977	0.3498
	5	0.7	425.609	22.2058	313.429	18.2857	221.616	14.8223	104.952	9.4545
		0.8	577.241	30.1395	399.724	23.7643	260.936	18.2928	105.598	10.4202
		0.9	1054.87	53.3573	635.995	37.9961	340.290	25.6981	94.3308	11.0380
		0.99	9841.98	506.709	2750.33	201.981	405.678	48.7122	923.202	12.2529
	10	0.7	1603.19	88.0314	1180.05	72.1510	834.108	58.1816	395.727	36.7551
		0.8	2157.41	115.778	1493.33	90.7417	974.720	69.3911	395.734	39.1248
		0.9	3908.96	210.583	2356.57	149.327	1261.65	100.545	351.986	43.0182
		0.99	36170.2	1987.74	10123.2	789.441	1503.91	189.681	3415.795	50.4719
100	1	0.7	21.9449	0.2312	16.9722	0.2108	12.7585	0.1917	6.8909	0.1579
		0.8	32.4753	0.3138	23.7871	0.2769	16.6932	0.2427	7.7505	0.1834
		0.9	64.4363	0.5716	41.9240	0.4641	25.0097	0.3685	8.1158	0.2194
		0.99	638.769	5.2860	211.730	2.3982	35.7890	0.7310	30.2949	0.0588
	5	0.7	229.929	5.7788	176.924	4.8626	132.203	4.0421	70.5340	2.7288
		0.8	315.839	7.8429	230.364	6.3371	160.890	5.0219	74.1414	3.0440
		0.9	582.709	14.2855	378.129	10.5470	224.968	7.4702	72.9364	3.5126
		0.99	5437.17	132.0971	1804.28	57.3248	306.655	16.8488	262.016	2.5864
	10	0.7	870.874	23.1149	669.761	19.2563	500.176	15.8246	266.649	10.4209
		0.8	1187.82	31.3710	865.995	25.1117	604.560	19.6900	278.519	11.6933
		0.9	2175.94	57.1419	1411.80	41.8709	839.947	29.4090	272.570	13.6617
		0.99	20172.5	528.373	6699.34	228.894	1142.26	67.2673	975.172	11.0599
200	1	0.7	12.3718	0.1028	10.2729	0.0983	8.4133	0.0940	5.5140	0.0860
		0.8	18.2202	0.1403	14.5162	0.1312	11.3207	0.1225	6.6581	0.1065
		0.9	36.2226	0.2572	26.3754	0.2276	18.3733	0.2001	8.3854	0.1516
		0.99	362.358	2.3898	154.744	1.3440	43.7583	0.6179	7.0546	0.0898
	5	0.7	127.504	2.5702	105.398	2.2789	85.8416	2.0101	55.4493	1.5485
		0.8	175.938	3.5056	139.749	3.0109	108.592	2.5619	63.3166	1.8203
		0.9	326.571	6.4269	237.550	5.1467	165.332	4.0331	75.3718	2.3770
		0.99	3073.00	59.7253	1318.54	31.2958	377.572	13.1203	61.8303	1.9138
	10	0.7	485.060	10.2807	400.873	8.9939	326.389	7.8142	210.598	5.8199
		0.8	664.064	14.0223	527.384	11.8961	409.708	9.9837	238.676	6.8910
		0.9	1221.52	25.7073	888.342	20.3805	618.079	15.7932	281.502	9.1218
		0.99	11394.4	238.904	4885.92	124.855	1397.64	52.2367	229.319	7.8085
250	1	0.7	10.0629	0.0740	8.4145	0.0713	6.9507	0.0687	4.6546	0.0639
		0.8	14.6414	0.1007	11.7454	0.0952	9.2400	0.0899	5.5572	0.0800
		0.9	28.8334	0.1844	21.1552	0.1661	14.8922	0.1487	6.9958	0.1176
		0.99	288.820	1.7173	125.332	1.0015	37.0075	0.4891	6.3407	0.0793
	5	0.7	95.2388	1.8496	79.1862	1.6547	64.9614	1.4739	42.7579	1.1599
		0.8	130.944	2.5165	104.553	2.1822	81.7941	1.8767	48.5731	1.3643
		0.9	242.623	4.6092	177.371	3.7318	124.329	2.9620	57.9075	1.7933
		0.99	2288.42	42.9196	991.234	22.8923	291.752	9.9008	51.2254	1.5034
	10	0.7	361.592	7.3981	300.594	6.5134	246.530	5.6998	162.094	4.3143
		0.8	493.526	10.0661	394.058	8.5973	308.269	7.2706	182.987	5.1045
		0.9	907.090	18.4365	663.258	14.7336	465.033	11.5268	216.678	6.8007
		0.99	8483.85	171.678	3677.92	91.2278	1084.86	39.3404	190.115	6.1180

KL estimator performs better than other non-robust estimator as argued by Ugwuowo et al. [15]. The proposed estimator in this study, $\widehat{\alpha}_{\text{RJKL}}$ performs much better than the existing estimators considered in this study. That is, the Robust Jackknife Kibria Lukman estimator has the smallest mean square error.

Table 5: Estimated MSEs of the estimators when there are no outliers and p=7

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	0.2697	0.2861	0.2208	0.2391	0.1776	0.1971	0.1106	0.1302
		0.8	0.3781	0.4014	0.2898	0.3163	0.2144	0.2426	0.1084	0.1346
		0.9	0.7106	0.7545	0.4639	0.5153	0.2738	0.3261	0.0779	0.1134
		0.99	6.7379	7.1589	1.7016	2.0728	0.2011	0.2786	0.4034	0.2812
	5	0.7	6.7424	7.1523	4.6574	5.1544	3.0057	3.5329	1.1284	1.5434
		0.8	9.4532	10.0345	6.0780	6.7871	3.5309	4.2619	1.0342	1.5256
		0.9	17.7638	18.8637	9.7972	11.1263	4.4276	5.6791	0.7544	1.3060
		0.99	168.448	178.973	40.1434	49.2488	5.8421	7.1385	16.9292	11.8110
	10	0.7	26.9695	28.6090	18.3182	20.3240	11.5532	13.6742	4.1293	5.7498
		0.8	37.8129	40.1378	23.9652	26.8178	13.6418	16.5678	3.8478	5.7570
		0.9	71.0552	75.4550	38.7965	44.1234	17.2632	22.2467	2.8872	5.0310
		0.99	673.791	715.892	160.257	196.655	23.5174	28.6142	68.9781	48.1329
100	1	0.7	0.1398	0.1480	0.1250	0.1330	0.1112	0.1190	0.0868	0.0942
		0.8	0.1966	0.2081	0.1687	0.1799	0.1431	0.1540	0.1000	0.1099
		0.9	0.3704	0.3920	0.2855	0.3062	0.2123	0.2318	0.1072	0.1228
		0.99	3.5231	3.7282	1.2434	1.3929	0.1913	0.2582	0.0273	0.0273
	5	0.7	3.4947	3.6994	2.7012	2.9052	2.0214	2.2188	1.0551	1.2194
		0.8	4.9144	5.2019	3.5908	3.8762	2.4962	2.7675	1.0860	1.2920
		0.9	9.2599	9.8007	5.9888	6.5146	3.4971	3.9695	0.9865	1.2604
		0.99	88.0776	93.2049	28.4307	32.0951	3.8710	5.2275	1.7745	1.5159
	10	0.7	13.9790	14.7976	10.5672	11.3906	7.6909	8.4885	3.7718	4.4183
		0.8	19.6575	20.8078	14.0815	15.2312	9.5407	10.6323	3.9301	4.7328
		0.9	37.0396	39.2029	23.6031	25.7142	13.4985	15.3861	3.6637	4.7223
		0.99	352.311	372.820	113.347	127.995	15.4177	20.7938	7.4482	6.3335
200	1	0.7	0.0647	0.0683	0.0614	0.0649	0.0582	0.0616	0.0522	0.0554
		0.8	0.0909	0.0959	0.0844	0.0892	0.0782	0.0829	0.0667	0.0710
		0.9	0.1710	0.1805	0.1498	0.1587	0.1300	0.1383	0.0955	0.1027
		0.99	1.6255	1.7153	0.8128	0.8796	0.2906	0.3334	0.0165	0.0237
	5	0.7	1.6178	1.7071	1.3832	1.4686	1.1688	1.2499	0.8092	0.8805
		0.8	2.2718	2.3972	1.8662	1.9850	1.5036	1.6150	0.9280	1.0217
		0.9	4.2758	4.5118	3.2115	3.4303	2.3108	2.5089	1.0747	1.2205
		0.99	40.6370	42.8824	17.8112	19.4537	4.8583	5.8012	0.2790	0.3684
	10	0.7	6.4711	6.8284	5.3885	5.7326	4.4155	4.7435	2.8485	3.1339
		0.8	9.0871	9.5887	7.2798	7.7584	5.6921	6.1419	3.2789	3.6510
		0.9	17.1032	18.0474	12.5815	13.4614	8.8149	9.6115	3.8625	4.4339
		0.99	162.548	171.530	70.8341	77.4027	19.1209	22.8692	1.1435	1.4841
250	1	0.7	0.0507	0.0535	0.0486	0.0513	0.0466	0.0492	0.0427	0.0452
		0.8	0.0712	0.0752	0.0671	0.0710	0.0632	0.0668	0.0557	0.0590
		0.9	0.1341	0.1417	0.1205	0.1275	0.1076	0.1141	0.0843	0.0898
		0.99	1.2751	1.3466	0.6942	0.7454	0.2948	0.3277	0.0261	0.0336
	5	0.7	1.2667	1.3375	1.1070	1.1734	0.9588	1.0211	0.7022	0.7559
		0.8	1.7806	1.8802	1.5009	1.5932	1.2466	1.3317	0.8261	0.8965
		0.9	3.3536	3.5415	2.6044	2.7738	1.9560	2.1063	1.0100	1.1204
		0.99	31.8780	33.6647	14.9915	16.2498	4.7865	5.5181	0.2975	0.3886
	10	0.7	5.0670	5.3499	4.3067	4.5737	3.6140	3.8643	2.4612	2.6752
		0.8	7.1226	7.5208	5.8442	6.2154	4.7030	5.0456	2.8969	3.1764
		0.9	13.4145	14.1658	10.1797	10.8600	7.4268	8.0308	3.5881	4.0222
		0.99	127.512	134.659	59.5520	64.5848	18.7782	21.6873	1.1794	1.5301

5. Real-Life Application

We adopted the Hussein data for the data analysis

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i, \qquad i = 1, 2, \ldots, 31 \quad (44)$$

Hussein and Zari [32] were the first to originally adopt the Hussein data and was recently used to test the performance of a proposed two parameter estimator in the presence of outlier [3]. The data contains 31 observations and three independent vari-

Table 6: Estimated MSEs of the estimators when there is 10% outlier and p=7

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	60.8029	0.4791	40.8878	0.3938	25.4231	0.3180	8.8012	0.1997
		0.8	99.2107	0.6718	62.2013	0.5205	34.8383	0.3907	9.5212	0.2056
		0.9	216.044	1.2625	116.569	0.8524	50.8361	0.5308	8.2968	0.1780
		0.99	2333.39	11.9803	545.305	3.6804	84.8065	0.5415	260.208	0.3678
	5	0.7	422.505	11.9761	283.381	8.7943	175.556	6.1779	60.2840	2.8467
		0.8	611.046	16.7925	382.392	11.6306	213.564	7.5504	57.9704	2.9104
		0.9	1187.80	31.5545	640.163	19.2652	278.541	10.3922	45.1488	2.6870
		0.99	11631.7	299.420	2716.67	89.2637	420.106	13.4739	1286.53	13.2986
	10	0.7	1510.69	47.9040	1013.53	34.8725	628.169	24.2323	215.967	10.9305
		0.8	2136.76	67.1685	1337.73	46.1804	747.614	29.6979	203.205	11.2630
		0.9	4054.03	126.217	2185.96	76.6677	951.854	41.0706	154.244	10.5271
		0.99	38783.8	1197.65	9062.67	356.691	1393.78	53.9159	4259.15	53.9040
100	1	0.7	33.3801	0.2180	24.8173	0.1937	17.6889	0.1711	8.2997	0.1315
		0.8	54.6386	0.3065	38.4207	0.2612	25.3986	0.2199	9.9254	0.1510
		0.9	119.500	0.5773	74.5251	0.4430	41.2638	0.3280	10.4418	0.1645
		0.99	1282.08	5.4894	397.063	2.0845	50.4080	0.4107	34.2417	0.0428
	5	0.7	239.390	5.4501	177.582	4.2958	126.232	3.2974	58.9316	1.8394
		0.8	344.928	7.6607	242.290	5.7469	159.995	4.1420	62.5281	1.9869
		0.9	668.299	14.4296	416.904	9.7203	231.095	6.0453	58.9708	2.0327
		0.99	6539.09	137.201	2032.17	49.2646	262.700	8.9174	174.470	1.9351
	10	0.7	873.665	21.8004	647.966	16.9514	460.494	12.7991	214.904	6.8875
		0.8	1236.35	30.6427	868.354	22.7161	573.360	16.1320	224.127	7.5045
		0.9	2347.62	57.7179	1464.59	38.5455	812.026	23.6971	207.553	7.8010
		0.99	22517.5	548.797	7002.33	196.679	908.997	35.5307	602.934	7.9639
200	1	0.7	26.5675	0.1008	21.7406	0.0952	17.4507	0.0898	10.7326	0.0796
		0.8	43.6640	0.1415	34.3492	0.1307	26.2661	0.1203	14.3672	0.1011
		0.9	94.7433	0.2663	68.3657	0.2314	46.6883	0.1991	19.2553	0.1433
		0.99	988.387	2.5302	417.755	1.3002	104.913	0.4966	6.8410	0.0385
	5	0.7	138.754	2.5188	113.288	2.1673	90.6932	1.8448	55.4578	1.2999
		0.8	200.219	3.5364	157.283	2.9357	120.070	2.3959	65.4601	1.5274
		0.9	387.860	6.6543	279.733	5.1010	190.928	3.7714	78.7039	1.8875
		0.99	3780.44	63.2284	1600.02	29.7487	403.397	9.6212	26.2144	0.6796
	10	0.7	499.716	10.0752	407.821	8.5153	326.312	7.1013	199.297	4.7765
		0.8	706.782	14.1453	555.039	11.5504	423.557	9.2456	230.727	5.6423
		0.9	1341.19	26.6170	967.189	20.1369	660.061	14.6469	272.047	7.0741
		0.99	12846.0	252.909	5439.26	118.575	1372.75	38.1181	89.1150	2.7126
250	1	0.7	23.7443	0.0798	19.9966	0.0761	16.6024	0.0726	11.0348	0.0658
		0.8	39.3865	0.1121	32.0454	0.1051	25.5322	0.0983	15.3790	0.0855
		0.9	86.3019	0.2113	65.0884	0.1882	47.1256	0.1665	22.3962	0.1279
		0.99	926.147	2.0085	432.778	1.1122	136.614	0.4907	8.4626	0.0533
	5	0.7	125.902	1.9932	105.646	1.7495	87.3520	1.5231	57.5402	1.1285
		0.8	183.632	2.8026	149.001	2.3814	118.342	1.9971	70.8069	1.3553
		0.9	359.783	5.2793	270.848	4.1691	195.660	3.1997	92.5738	1.7509
		0.99	3548.57	50.1888	1656.48	25.1650	522.139	9.2833	32.6286	0.7798
	10	0.7	445.692	7.9727	373.720	6.8642	308.752	5.8468	203.024	4.1248
		0.8	633.414	11.2102	513.634	9.3527	407.647	7.6792	243.524	4.9686
		0.9	1206.99	21.1167	908.186	16.4252	655.669	12.3754	309.832	6.5003
		0.99	11598.9	200.749	5412.89	100.221	1705.35	36.7024	106.558	3.0723

ables. Details description can be found in [32,3].

The data contains multicollinearity with its variance inflation factor, VIF>10 and about 19.4% outliers in the y-direction at observations 12, 14, 15, 16, 30 and 31. Hence, it is sufficient to use the data in this study. The output of the analysis is presented in Table 9.

The RJKL estimator has the smallest mean square value. Thus, the RJKL estimator performed better. The percentage MSE

Table 7: Estimated MSEs of the estimators when there are 20% outliers and p=7

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	119.914	1.0249	80.5354	0.8315	50.0589	0.6608	17.4943	0.3986
		0.8	196.356	1.4329	123.079	1.1009	69.0594	0.8176	19.2267	0.4192
		0.9	428.611	2.9971	231.566	2.0723	101.677	1.3390	17.2449	0.5015
		0.99	4636.04	29.4453	1092.23	10.7202	177.533	2.2999	545.556	0.5486
	5	0.7	818.077	25.1608	547.263	19.0514	338.257	13.9238	116.863	7.0046
		0.8	1184.59	35.2298	739.444	25.3346	412.225	17.3117	113.255	7.4746
		0.9	2304.40	66.2106	1238.94	42.5563	539.034	24.8324	90.0158	7.5425
		0.99	22578.0	629.137	5270.26	209.884	866.359	36.1990	2712.14	16.1937
	10	0.7	2921.54	100.502	1955.11	75.7647	1209.05	55.0819	418.169	27.4365
		0.8	4139.97	140.781	2585.75	100.868	1442.75	68.6166	397.063	29.4010
		0.9	7866.74	264.728	4232.82	169.731	1843.98	98.7288	308.658	29.8633
		0.99	75384.1	2517.78	17619.9	839.856	2884.57	145.069	8964.29	65.3513
100	1	0.7	64.3033	0.3636	47.2958	0.3187	33.2345	0.2772	15.0659	0.2057
		0.8	105.357	0.5111	73.2423	0.4290	47.6494	0.3549	17.9011	0.2338
		0.9	230.299	0.9633	141.697	0.7288	76.7823	0.5297	18.3979	0.2537
		0.99	2467.38	9.1679	743.891	3.6115	89.5646	0.7829	73.7920	0.0692
	5	0.7	451.313	9.0874	331.732	7.2409	232.916	5.6325	105.360	3.2388
		0.8	652.082	12.7723	453.225	9.7249	294.798	7.1444	110.705	3.5839
		0.9	1265.65	24.0712	778.849	16.5974	422.200	10.6691	101.254	3.8743
		0.99	12401.6	229.079	3743.17	87.2753	451.618	17.9354	373.088	2.3922
	10	0.7	1645.11	36.3492	1209.82	28.7364	850.020	22.1430	385.168	12.4717
		0.8	2330.35	51.0881	1620.83	38.6357	1055.34	28.1486	397.402	13.8775
		0.9	4431.27	96.2822	2729.89	66.0659	1482.56	42.2002	357.451	15.1433
		0.99	42575.8	916.292	12885.5	348.721	1564.79	71.5612	1266.77	9.7050
200	1	0.7	51.8951	0.1592	42.1616	0.1490	33.5487	0.1391	20.2075	0.1207
		0.8	85.2974	0.2235	66.6005	0.2040	50.4516	0.1855	26.9671	0.1516
		0.9	185.026	0.4206	132.354	0.3597	89.3118	0.3039	35.7030	0.2093
		0.99	1931.06	3.9945	801.767	2.0537	192.314	0.7878	13.5335	0.0631
	5	0.7	265.637	3.9787	215.353	3.4238	170.926	2.9148	102.374	2.0546
		0.8	384.132	5.5850	299.383	4.6468	226.286	3.8029	120.372	2.4415
		0.9	745.649	10.5089	532.507	8.1126	358.563	6.0534	142.701	3.1030
		0.99	7282.62	99.7940	3017.36	48.3717	720.906	16.6721	51.9463	1.2305
	10	0.7	956.170	15.9145	774.997	13.5292	614.952	11.3592	368.095	7.7612
		0.8	1354.06	22.3397	1055.12	18.3830	797.330	14.8522	423.951	9.2666
		0.9	2571.73	42.0342	1836.38	32.1730	1236.35	23.7566	491.943	11.9066
		0.99	24647.7	399.164	10211.6	193.054	2439.54	66.2922	174.788	4.9018
250	1	0.7	49.7937	0.1376	42.0037	0.1305	34.9404	0.1235	23.3224	0.1103
		0.8	82.8689	0.1937	67.6386	0.1800	54.0985	0.1669	32.8819	0.1424
		0.9	181.992	0.3651	138.031	0.3218	100.670	0.2813	48.7098	0.2099
		0.99	1954.84	3.4733	927.931	1.9613	303.128	0.8999	18.7292	0.1124
	5	0.7	266.658	3.4385	224.513	3.0327	186.357	2.6544	123.821	1.9913
		0.8	389.032	4.8396	317.098	4.1467	253.220	3.5115	153.400	2.4393
		0.9	761.738	9.1229	577.253	7.3240	420.579	5.7381	203.068	3.3072
		0.99	7506.45	86.7781	3562.79	46.1198	1163.49	19.0945	71.3877	2.0149
	10	0.7	948.340	13.7537	798.193	11.9834	662.298	10.3470	439.725	7.5317
		0.8	1349.17	19.3579	1099.43	16.4008	877.710	13.7119	531.431	9.2588
		0.9	2572.53	36.4902	1949.31	29.0336	1420.11	22.5058	685.605	12.6739
		0.99	24732.5	347.098	11741.3	184.017	3836.20	75.8862	234.766	7.9755

change of OLS estimator to the proposed is about 73%. The jackknife KL estimator performs better than the ridge and the KL estimator[15]. However, when there is both multicollinearity and outlier, the proposed RJKL study dominates other estimators. The estimator with the closest performance is the Robust Kibria-Lukman estimator with about 8% difference. The application result agrees with the theoretical proofs and the simulation. All the estimators produced the same regression sign.

Table 8: Estimated MSEs of the estimators when there are 30% outliers and p=7

n	$\widehat{\sigma}$	ρ	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
50	1	0.7	176.029	25.4476	117.630	20.5623	72.4841	16.2500	24.5439	9.6180
		0.8	289.091	40.7835	180.230	31.6595	100.078	23.8026	26.7521	12.4852
		0.9	632.508	87.9825	339.479	62.5223	146.664	41.7969	23.2801	16.3585
		0.99	6842.66	927.927	1587.39	379.980	250.968	88.9143	789.588	5.9982
	5	0.7	1198.50	249.099	799.409	198.493	491.401	154.303	165.777	88.1495
		0.8	1738.52	356.258	1082.10	272.434	599.513	201.112	159.759	101.484
		0.9	3389.34	676.860	1816.30	472.164	782.782	308.093	124.102	114.791
		0.99	33298.5	6511.84	7712.37	2575.63	1228.32	565.161	3889.30	51.3169
	10	0.7	4302.62	939.749	2870.54	747.659	1765.13	580.130	596.058	330.089
		0.8	6101.25	1325.89	3799.03	1012.04	2106.00	745.378	562.091	374.207
		0.9	11603.0	2464.02	6221.37	1714.74	2683.96	1115.34	426.552	412.784
		0.99	111319.1	23366.9	25810.1	9187.00	4102.53	1989.79	12934.1	190.624
100	1	0.7	87.9633	0.7618	63.8612	0.6566	44.1094	0.5603	19.2165	0.3977
		0.8	144.012	1.0742	98.6666	0.8876	62.8889	0.7205	22.5087	0.4541
		0.9	314.994	2.0255	190.263	1.5195	100.081	1.0925	22.3053	0.5107
		0.99	3387.24	19.2659	983.693	8.0896	111.755	2.0612	121.875	0.1558
	5	0.7	628.585	19.0303	456.143	15.4409	314.869	12.2794	136.945	7.4362
		0.8	907.147	26.8310	621.375	20.9103	395.976	15.8243	141.763	8.5224
		0.9	1759.23	50.5780	1062.86	36.0667	559.477	24.2995	125.289	9.8913
		0.99	17235.4	481.069	5012.56	198.963	576.233	49.5735	624.349	4.4673
	10	0.7	2290.08	76.1193	1661.75	61.5544	1146.97	48.7586	498.577	29.2766
		0.8	3245.87	107.318	2223.17	83.3977	1416.52	62.8993	506.765	33.6416
		0.9	6171.17	202.300	3727.83	143.967	1961.68	96.7536	438.654	39.2026
		0.99	59262.7	1924.14	17227.7	795.448	1975.93	198.076	2146.44	17.9635
200	1	0.7	73.1493	0.2996	58.8538	0.2771	46.2797	0.2555	27.1012	0.2157
		0.8	120.479	0.4212	93.0287	0.3793	69.4843	0.3398	35.8860	0.2685
		0.9	261.752	0.7935	184.570	0.6684	122.085	0.5546	46.3870	0.3661
		0.99	2734.52	7.5420	1099.80	3.9736	242.901	1.6104	18.6298	0.1589
	5	0.7	375.503	7.4839	301.842	6.4787	237.110	5.5537	138.592	3.9787
		0.8	543.927	10.5212	419.764	8.8351	313.348	7.3109	161.745	4.8223
		0.9	1057.87	19.8201	745.870	15.5525	493.402	11.8510	187.766	6.4108
		0.99	10343.8	188.360	4164.78	96.1097	924.537	36.8939	71.6864	3.4590
	10	0.7	1349.72	29.9343	1084.69	25.7389	851.829	21.8968	497.604	15.4259
		0.8	1912.79	42.0830	1475.82	35.1269	1101.40	28.8672	568.271	18.7578
		0.9	3636.66	79.2775	2563.59	61.9265	1695.46	46.9327	645.146	25.1003
		0.99	34893.8	753.406	14047.6	383.982	3119.88	147.128	242.676	13.7813
250	1	0.7	77.3227	0.2738	65.2093	0.2573	54.2173	0.2414	36.1045	0.2114
		0.8	128.843	0.3857	105.175	0.3550	84.1090	0.3256	51.0087	0.2711
		0.9	283.208	0.7277	214.894	0.6342	156.734	0.5474	75.5029	0.3965
		0.99	3040.22	6.9228	1442.93	4.0329	467.134	1.9628	26.1510	0.2998
	5	0.7	408.258	6.8396	343.645	6.0789	285.092	5.3668	188.916	4.1066
		0.8	597.276	9.6350	486.802	8.3489	388.581	7.1625	234.658	5.1307
		0.9	1172.97	18.1740	888.968	14.8722	647.371	11.9286	310.650	7.2851
		0.99	11587.6	172.879	5493.24	97.5796	1773.18	45.1740	98.5925	6.3387
	10	0.7	1439.39	27.3570	1210.94	24.1521	1004.01	21.1663	664.426	15.9385
		0.8	2051.01	38.5383	1670.75	33.1929	1332.80	28.2843	803.719	19.9655
		0.9	3915.69	72.6924	2965.88	59.2099	2158.22	47.2351	1033.85	28.5155
		0.99	37674.9	691.468	17842.1	389.808	5746.51	180.118	318.295	25.2059

Though, the intercept value of OLS estimator is the highest. We noticed a sharp reduction in the intercept for other estimators, especially the jackknife KL and the proposed.

6. Conclusion

We introduced a new robust estimator for the linear regression model in this study and named it the Robust Jackknife Kibria

Table 9: Regression coefficients and MSEs of estimator adopting the Hussein data.

Coefficients	$\widehat{\alpha}$	$\widehat{\alpha}_M$	$\widehat{\alpha}_K$	$\widehat{\alpha}_{M\_K}$	$\widehat{\alpha}_{KL}$	$\widehat{\alpha}_{M\_KL}$	$\widehat{\alpha}_{JKL}$	$\widehat{\alpha}_{RJKL}$
β_0	208.8853	173.3445	178.6308	155.4823	148.3763	137.6202	92.9456	103.4135
β_1	0.6130	0.9976	0.8627	1.1450	1.1124	1.2924	1.5698	1.5747
β_2	1.2563	1.1153	1.1574	1.0569	1.0584	0.9985	0.8772	0.8867
β_3	-1.2213	-1.1159	-1.2635	-1.1408	-1.3058	-1.1658	-1.3832	-1.2135
MSEs	1850.4816	864.4366	1353.9507	695.6977	935.0263	545.3211	785.1599	500.0968

Lukman (RJKL) Estimator. The RJKL estimator was proposed to handle outlier and multicollinearity together. The estimator was formed by grafting the M-estimator into the Jackknife Kibria Lukman (JKL) estimator. We presented theorems that state the necessary conditions for the new estimator to perform better than the JKL estimator and other existing robust estimators discussed. We observed a good performance in the simulation study of the new estimator and the real-life data analysis. Both results supports the efficiency of the new estimator.

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