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Approximate Analytical Solution of Fractional Lane-Emden Equation by Mittag-Leffler Function Method

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Abstract

The classical Lane-Emden differential equation, a nonlinear second-order differential equation, models the structure of an isothermal gas sphere in equilibrium under its own gravitation. In this paper, the Mittag-Leffler function expansion method is used to solve a class of fractional Lane-Emden differential equation. In the proposed differential equation, the polytropic term $f(y(x)) = y^m(x)$ (where $m = 0, 1, 2, \dots$ is the polytropic index; $0 < x \leq 1$) is replaced with a linear combination $f(y(x)) = a_0 + a_1 y(x) + a_2 y^2(x) + \cdots + a_m y^m(x) + \cdots + a_N y^N(x), 0 \leq m \leq N, N \in \mathbb{N}_0$. Explicit solutions of the fractional equation, when $f(y)$ are elementary functions are presented. In particular, we consider the special cases of the trigonometric, hyperbolic and exponential functions. Several examples are given to illustrate the method. Comparison of the Mittag-Leffler function method with other methods indicates that the method gives accurate and reliable approximate solutions of the fractional Lane-Emden differential equation.

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1. Introduction

Fractional calculus is a generalization and extension of classical calculus to non-integer orders. In recent times, fractional calculus has attracted the attention of researchers in several areas including mathematics, physics, biology, chemistry, engineering, economics and psychology ([1], [2], [3], [4], [5], [6], [7], [8], [9]). Several definitions of fractional calculus have been formulated by researchers. The most common and widely used definitions are the Caputo, Grünwald-Letnikov and Riemann-Liouville derivatives ([5], [8], [10]). The Grünwald-Letnikov derivative is mostly limited to numerical algorithms. The Riemann-Liouville fractional derivative has certain limitations and so becomes unsuitable in modeling some real-life phenomena since it requires the definition of fractional order initial conditions which have no physical meaningful explanation yet ([8]). The main advantage of the Caputo derivative is that it takes on the same form of initial conditions for the integer-order differential equations.

Recently, the Lane-Emden equation has been investigated by several researchers due to its significant applications in mathematical physics and astrophysics ([11]). The classical Lane-Emden equation, first introduced in 1870 by Lane and studied

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in 1907 by Emden is of the form:

$$y''(x) + \frac{2}{x}y'(x) + f(y(x)) = 0 \tag{1}$$

with initial conditions

$$y(0) = A, \quad y'(0) = B. \tag{2}$$

Equation (1) with various forms of $f(y(x))$ has been used to model several phenomena such as the theory of stellar structure, thermal explosions, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and thermionic currents ([12], [13]). Various methods have been presented by several authors to obtain the solution of the initial value problem (1) – (2). The Adomian decomposition method was employed in [11] to investigate the initial value problem (1) – (2). Solutions of the Lane-Emden equation have also been obtained via the series method (see e.g., [14], [15]). The series solutions obtained in [15] were compared with the results obtained using the homotopy perturbation method. A numerical algorithm was developed by Vanani and Aminataei [16] for solving the Lane-Emden equation. Ogunniran et al. in [17] investigated the linear stabilities of some explicit members of Runge-Kutta methods in integrating the Lane-Emden equation.

However, the standard Lane-Emden differential equation is the one given precisely by the initial value problem ([14])

$$\begin{aligned} &y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad m \geq 0; x > 0 \\ &y(0) = 1, \quad y'(0) = 0, \end{aligned} \tag{3}$$

where m is the polytropic index and $y = y(x)$ is the polytrope. Clearly, the equation (3) is linear when $m = 0, 1$ and as a result the analytical solutions of the corresponding equations are realisable in closed forms. By extension it is mentioned in [14] that a closed form solution is also possible for $m = 5$. For numerical solutions of second order ordinary differential equations, see [18] and [19].

As a result of the significant importance of fractional calculus in modeling real-life phenomena accurately, the fractional Lane-Emden equation has been formulated and studied by researchers in very recent times. By using the collocation method, a numerical solution of (3) was obtained in [20]. Some other researchers have also sought numerical solutions to the fractional Lane-Emden equation (see, e.g., [21], [22]). Approximate solutions based on orthonormal Bernoulli's polynomials method, Homotopy-Adomian decomposition method and the series expansion method were treated in [23], [24] and [25] respectively. Other treatise on the fractional Lane-Emden equation can be found in [26], [27], [28], [29] and [30]. In [28, Subsection 5.2.2], the authors considered the power series solutions of the fractional Lane-Emden equation with the polytropic term y^m with the fractional derivative described in the Caputo sense, while Malik and Mohammed [25] presented approximate solutions of fractional Lane-Emden equation using conformable Homotopy–Adomian decomposition method and conformable residual power series method.

Arafa et al. in [31] used the Mittag-Leffler function method to solve a simple fractional differential equation of the form

$$D^\alpha y = Ay^2, \quad 0 < \alpha \leq 1, \tag{4}$$

with the fractional derivative described in the Caputo sense. In this paper, the Mittag-Leffler function expansion method is employed to solve a class of fractional Lane-Emden initial value problem whose polytropic term $f(y(x)) = y^m(x)$ (where $m = 0, 1, 2, \ldots$ is the polytropic index; $0 < x \leq 1$) is replaced with a finite series $f(y(x)) = a_0 + a_1 y(x) + a_2 y^2(x) + \cdots + a_m y^m(x) + \cdots + a_N y^N(x), 0 \leq m \leq N, N \in \mathbb{N}_0$, with the fractional derivative described in the Caputo sense. The analytical solutions of the corresponding fractional equations, when $f(y)$ are elementary functions, are presented, from which several examples of fractional Lane-Emden equations are given. In particular, we consider the special cases where $f(y)$ are trigonometric, hyperbolic and exponential functions. Comparison of the Mittag-Leffler function method with other methods shows that the method is reliably capable of solving analytically, nonlinear fractional Lane-Emden differential equations, and by extension, one can apply the method to solve several nonlinear fractional Lane-Emden equations when the functions $f(y)$ are given by other special functions. The motivation for the forms of the nonlinear function $f(y)$ considered in this paper arises from the need to address those situations where the expansion coefficients $a_0, a_1, \ldots, a_{N-1}$ are not identically zero.

2. Caputo Fractional Derivative and Its Basic Properties

This section presents the definition and basic properties of Caputo derivative needed. For further discussions on fractional calculus and fractional differential equations, see, e.g., [5], [28], [32], [33], [34], [35], [36], [37], [38], [39], [40].

Definition 2.1. *For $\alpha > 0$, the Caputo fractional derivative of order α is defined as follows $(m \geq 1)$:*

$$\begin{aligned} {}^cD^\alpha f(x) =& \mathcal{J}^{m-\alpha} D^m f(x) \\ =& \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\xi)^{m-\alpha-1} f^{(m)}(\xi)\, d\xi, & m-1 < \alpha < m, \\ \frac{d^m}{dx^m} f(x) & \alpha = m, \end{cases} \end{aligned} \tag{5}$$

where

$$\mathcal{J}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi \tag{6}$$

is the Riemann-Liouville fractional integral of order α. Here $f^{(m)}(x) = \frac{d^m f(x)}{dx^m}$.

Remark 2.1. The Caputo derivative of the constant function C is zero, i.e ${}^cD^\alpha C = 0$.

Corollary 2.1. *For $m-1<\alpha<m$, we have*

$$
{}^cD^\alpha x^\beta = \begin{cases} 0, & \beta \in \{0,1,\cdots,m-1\}, \\ \frac{\Gamma(\nu+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta > m-1 \\ \text{non existing} & \text{otherwise}. \end{cases} \tag{7}
$$

The following properties of the Caputo derivative hold ([8], [10]).

Lemma 2.1. *Suppose $m-1<\alpha<m$; $n,m\in\mathbb{N}$; $\alpha\in\mathbb{R}$; $\lambda,\sigma\in\mathbb{C}$. Let $f(x)$ and $g(x)$ be such that ${}^cD^\alpha f(x)$ and ${}^cD^\alpha g(x)$ exist. We have that*

(*a*) $\lim_{\alpha\to n} {}^cD^\alpha f(x) = f^{(n)}(x)$

(*b*) $\lim_{\alpha\to n-1} {}^cD^\alpha f(x) = f^{(n-1)}(x) - f^{(n-1)}(0)$

(*c*) ${}^cD^\alpha(\lambda f(x)+\sigma g(x)) = \lambda\, {}^cD^\alpha f(x) + \sigma\, {}^cD^\alpha g(x)$

(*d*) ${}^cD^\alpha D^m f(x) = {}^cD^{\alpha+m} f(x) \neq D^m\, {}^cD^\alpha f(x)$.

3. Mittag-Leffler Function

In this section we examine the definition and properties of the Mittag-Leffler function as eigenfunctions of fractional derivatives ([1]). The Mittag-Leffler function plays an important role in the theory of fractional calculus. Just as the exponential function naturally comes out of the solutions of classical differential equations, the Mittag-Leffler function plays an analogous role in the solution of fractional differential equations. It is thus a generalization of the exponential function.

Definition 3.1. *The Mittag-Leffler function is defined by:*

$$
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0, \alpha\in\mathbb{R}, z\in\mathbb{C}. \tag{8}
$$

Definition 3.2. *The Mittag-Leffler function can also be represented in two arguments α and β as*

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta>0, \alpha,\beta\in\mathbb{R}, z\in\mathbb{C}; \tag{9}
$$

with $E_{\alpha,1} = E_\alpha$.

In particular, the significance of the Mittag-Leffler function is in the fact that it serves as the eigenfunction of the Caputo and Riemann-Liouville derivatives in fractional calculus ([1]). It is easy to see that ([8], [41])

$$
E_{1,1}(z) = E_1(z) = e^z, \;\; E_{1,2}(z) = \frac{e^z-1}{z}, \;\; E_{2,1}(z^2) = \cosh z, \;\; z\in\mathbb{C}.
$$

$$
E_{2,1}(-z^2) = \cos z, \;\; E_{2,2}(z^2) = \frac{\sinh z}{z}, \;\; E_{2,2}(-z^2) = \frac{\sin z}{z}, \;\; z\in\mathbb{C}.
$$

Corollary 3.1. *For $\alpha,\beta\in\mathbb{R}$, $z\in\mathbb{C}$ and $m\in\mathbb{N}$, the following recurrence relations hold for the Mittag-Leffler function* ([41]) :

(*a*) $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$

(*b*) $E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z)$

(*c*) $\left(\frac{d}{dz}\right)^m [z^{\beta-1}E_{\alpha,\beta}(z^\alpha)] = z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha)$.

4. Fractional Lane-Emden Equations and Their Approximate Solutions

In this section, we apply the Mittag-Leffler function discussed in Section 3 to obtain approximate solutions to the fractional Lane-Emden initial value problem

$$
\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\, {}^cD^\beta y(x) + f(y(x)) = 0 \\ &y(0) = A, \quad y'(0) = 0, \quad A\in[0,1], \end{aligned} \tag{10}
$$

with $0<x\le 1, 0<\beta\le 1, 1<\alpha\le 2, \omega\in\mathbb{R}$. Here ${}^cD^\mu$ denotes the Caputo fractional differential operator of order μ, $\mu>0$. The resulting solution is given as the Mittag-Leffler function. The function $f(y(x))$ will be chosen in such a way that it can be expanded in a power series, in particular, the Maclaurin series.

Towards this end, let the function $f(y(x))$ be given by the power series

$$
f(y(x)) = \sum_{m=0}^{\infty} a_m y^m(x), \tag{11}
$$

where a_m, $m = 0,1,2,3,\ldots$, are the expansion coefficients (constants). For computation reasons, one is interested in the approximation

$$
f_N(y(x)) = \sum_{m=0}^{N} a_m y^m(x), \quad a_N \neq 0. \tag{12}
$$

We proceed by giving the following result that will be needed in the sequel.

Proposition 4.1. *For $1<\alpha\le 2$, $0<x\le 1$, the function $f_N(y(x))$ admits the power series expansion*

$$
\mathsf{F}_N^\alpha(x) := f_N(y(x)) = \sum_{\ell=0}^{\infty} \mathsf{C}_{\ell,N}^\alpha x^{\alpha\ell}, \quad N\in\mathbb{N}_0, \tag{13}
$$

where the constants $\mathsf{C}_{\ell,N}^\alpha$ are given by

$$
\mathsf{C}_{\ell,N}^\alpha = \sum_{m=0}^{N} a_m \mathsf{B}_{\ell,m}^{\alpha,N}. \tag{14}
$$

Here the numbers $\mathsf{B}_{\ell,m}^{\alpha,N}$, $m = 3,4,5,\ldots$ are given by the finite series

$$
\mathsf{B}_{\ell,m}^{\alpha,N} = \sum_{p_{m-1}=0}^{\ell} \sum_{p_{m-2}=0}^{p_{m-1}} \cdots \sum_{p_1=0}^{p_2} \mathsf{A}_{\ell-p_{m-1},N}^\alpha \mathsf{A}_{p_{m-1}-p_{m-2},N}^\alpha \cdots \mathsf{A}_{p_1,N}^\alpha, \tag{15}
$$

with the Mittag-Leffler expansion coefficients given by

$$
\mathsf{A}_{\ell,N}^\alpha = \frac{b_N^\ell}{\Gamma(\alpha\ell+1)}, \quad b_N^\ell \text{ are constants.} \tag{16}
$$

In particular, the special cases $\mathsf{B}_{\ell,m}^{\alpha,N}$, $m = 0,1,2$, are given respectively by

$$
\mathsf{B}_{\ell,0}^{\alpha,N} = \begin{cases} 1, & \ell = 0, \\ 0, & \ell = 1,2,\cdots, \end{cases}
$$

$$
\mathsf{B}_{\ell,1}^{\alpha,N} = \mathsf{A}_{\ell,N}^\alpha, \quad \mathsf{B}_{\ell,2}^{\alpha,N} = \sum_{p_1=0}^{\ell} \frac{b_N^{p_1} b_N^{\ell-p_1}}{\Gamma(\alpha p_1+1)\Gamma(\alpha\ell-\alpha p_1+1)}. \tag{17}
$$

Proof. By definition, we use the Mittag-Leffler function

$$y(x) = E_\alpha(b_N x^\alpha) = \sum_{k=0}^{\infty} \frac{b_N^k x^{\alpha k}}{\Gamma(\alpha k+1)} \tag{18}$$

in (12) to see that

$$\begin{aligned} \mathsf{F}_N^\alpha(x) := f_N(y(x)) &= \sum_{m=0}^{N} a_m \left(\sum_{k=0}^{\infty} \frac{b_N^k x^{\alpha k}}{\Gamma(\alpha k+1)} \right)^m \\ &= \sum_{m=0}^{N} a_m \mathsf{S}_{m,N}^\alpha(x), \end{aligned} \tag{19}$$

where we have written ($m \geq 1$)

$$\mathsf{S}_{0,N}^\alpha = 1, \quad \mathsf{S}_{m,N}^\alpha(x) := y^m(x) = \left(\sum_{k=0}^{\infty} \frac{b_N^k x^{\alpha k}}{\Gamma(\alpha k+1)} \right)^m. \tag{20}$$

Let $\mathsf{A}_{k,N}^\alpha$ be given by

$$\mathsf{A}_{k,N}^\alpha = \frac{b_N^k}{\Gamma(\alpha k+1)}. \tag{21}$$

Then clearly,

$$\mathsf{S}_{1,N}^\alpha(x) = \sum_{\ell=0}^{\infty} \mathsf{A}_{\ell,N}^\alpha x^{\alpha\ell} = \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,1}^{\alpha,N} x^{\alpha\ell}. \tag{22}$$

Furthermore, the classical Cauchy product tells us that

$$\begin{aligned} \mathsf{S}_{2,N}^\alpha(x) = \left(\sum_{k=0}^{\infty} \mathsf{A}_{k,N}^\alpha x^{\alpha k} \right)^2 &= \sum_{m=0}^{\infty} \mathsf{A}_{m,N}^\alpha x^{\alpha m} \sum_{n=0}^{\infty} \mathsf{A}_{n,N}^\alpha x^{\alpha n} \\ &= \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,2}^{\alpha,N} x^{\alpha\ell}, \end{aligned} \tag{23}$$

where

$$\mathsf{B}_{\ell,2}^{\alpha,N} = \sum_{p_1=0}^{\ell} \mathsf{A}_{p_1,N}^\alpha \mathsf{A}_{\ell-p_1,N}^\alpha. \tag{24}$$

Also, in a similar way, we see that

$$\begin{aligned} \mathsf{S}_{3,N}^\alpha(x) &= \left(\sum_{k=0}^{\infty} \mathsf{A}_{k,N}^\alpha x^{\alpha k} \right)^3 \\ &= \left(\sum_{k=0}^{\infty} \mathsf{A}_{k,N}^\alpha x^{\alpha k} \right)^2 \sum_{n=0}^{\infty} \mathsf{A}_{n,N}^\alpha x^{\alpha n}. \end{aligned} \tag{25}$$

That is, one has

$$\begin{aligned} \mathsf{S}_{3,N}^\alpha(x) &= \mathsf{S}_{2,N}^\alpha(x) \mathsf{S}_{1,N}^\alpha(x) \\ &= \left(\sum_{m=0}^{\infty} \mathsf{B}_{m,2}^{\alpha,N} x^{\alpha m} \right) \left(\sum_{m=0}^{\infty} \mathsf{B}_{n,1}^{\alpha,N} x^{\alpha n} \right) = \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,3}^{\alpha,N} x^{\alpha\ell}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathsf{B}_{\ell,3}^{\alpha,N} &= \sum_{p_2=0}^{\ell} \mathsf{B}_{p_2,2}^{\alpha,N} \mathsf{B}_{\ell-p_2,1}^{\alpha,N} \\ &= \sum_{p_2=0}^{\ell} \sum_{p_1=0}^{p_2} \mathsf{A}_{p_1,N}^\alpha \mathsf{A}_{p_2-p_1,N}^\alpha \mathsf{A}_{\ell-p_2,N}^\alpha. \end{aligned} \tag{27}$$

Similarly, by taking more products, we see that

$$\begin{aligned} \mathsf{S}_{4,N}^\alpha(x) &= \left(\sum_{k=0}^{\infty} \mathsf{A}_{k,N}^\alpha x^{\alpha k} \right)^4 \\ &= \mathsf{S}_{3,N}^\alpha(x) \mathsf{S}_{1,N}^\alpha(x) = \left(\sum_{m=0}^{\infty} \mathsf{B}_{m,3}^{\alpha,N} x^{\alpha m} \right) \left(\sum_{n=0}^{\infty} \mathsf{B}_{n,1}^{\alpha,N} x^{\alpha n} \right) \\ &= \sum_{\ell=0}^{\infty} \sum_{p_3=0}^{\ell} \mathsf{B}_{p_3,3}^{\alpha,N} \mathsf{B}_{\ell-p_3,1}^{\alpha,N} x^{\alpha\ell} = \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,4}^{\alpha,N} x^{\alpha\ell}, \end{aligned} \tag{28}$$

where

$$\mathsf{B}_{\ell,4}^{\alpha,N} = \sum_{p_3=0}^{\ell} \sum_{p_2=0}^{p_3} \sum_{p_1=0}^{p_2} \mathsf{A}_{p_1,N}^\alpha \mathsf{A}_{p_2-p_1,N}^\alpha \mathsf{A}_{p_3-p_2,N}^\alpha \mathsf{A}_{\ell-p_3,N}^\alpha. \tag{29}$$

Continuing in this way, we obtain

$$\begin{aligned} \mathsf{S}_{m,N}^\alpha(x) &= \mathsf{S}_{m-1,N}^\alpha(x) \mathsf{S}_{1,N}^\alpha(x) \\ &= \left(\sum_{k=0}^{\infty} \mathsf{B}_{k,m-1}^{\alpha,N} x^{\alpha k} \right) \left(\sum_{n=0}^{\infty} \mathsf{B}_{n,1}^{\alpha,N} x^{\alpha n} \right) \\ &= \sum_{\ell=0}^{\infty} \sum_{p_{m-1}=0}^{\ell} \mathsf{B}_{p_{m-1},m-1}^{\alpha,N} \mathsf{B}_{\ell-p_{m-1},1}^{\alpha,N} x^{\alpha\ell} = \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,m}^{\alpha,N} x^{\alpha\ell}, \end{aligned} \tag{30}$$

where ($m = 3, 4, 5, \dots$)

$$\mathsf{B}_{\ell,m}^{\alpha,N} = \sum_{p_{m-1}=0}^{\ell} \sum_{p_{m-2}=0}^{p_{m-1}} \cdots \sum_{p_1=0}^{p_2} \mathsf{A}_{\ell-p_{m-1},N}^\alpha \mathsf{A}_{p_{m-1}-p_{m-2},N}^\alpha \cdots \mathsf{A}_{p_1,N}^\alpha. \tag{31}$$

Upon inserting (30) – (31) into (19), one gets

$$\mathsf{F}_N^\alpha(x) = \sum_{m=0}^{N} a_m \sum_{\ell=0}^{\infty} \mathsf{B}_{\ell,m}^{\alpha,N} x^{\alpha\ell} = \sum_{\ell=0}^{\infty} \mathsf{C}_{\ell,N}^\alpha x^{\alpha\ell}, \tag{32}$$

where

$$\mathsf{C}_{\ell,N}^\alpha = \sum_{m=0}^{N} a_m \mathsf{B}_{\ell,m}^{\alpha,N} \tag{33}$$

and we obtain the result as required. □

With the Proposition 4.1 in place we now present the main result of this paper.

Theorem 4.1. *For* $0 < x \leq 1$, $0 < \beta \leq 1$, $1 < \alpha \leq 2$, $\omega \in \mathbb{R}, N \in \mathbb{N}_0$*; the fractional initial value problem*

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}} {}^cD^\beta y(x) + \sum_{m=0}^{N} a_m y^m(x) = 0 \\ &y(0) = A, y'(0) = 0, \quad A \in [0,1], \end{aligned} \tag{34}$$

admits a series solution given by the Mittag-Leffler function

$$y(x)=\sum_{\ell=0}^{\infty}\frac{\mathscr{B}_N^{\ell}(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)}x^{\alpha\ell}, \tag{35}$$

where

$$\mathscr{B}_N^{0}(\alpha,\beta;\omega):=b_N^0=A, \tag{36}$$

$$\mathscr{B}_N^{\ell+1}(\alpha,\beta;\omega):=b_N^{\ell+1}=-\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,N}^{\alpha}}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)},\ \ell\geq 0. \tag{37}$$

The coefficients $\mathsf{C}_{\ell,N}^{\alpha}, \ell\geq 0$, *are as given in Proposition 4.1.*

Proof. The starting point is to substitute the Mittag-Leffler function (18) in the equation (34) to evaluate the Caputo fractional derivatives ${}^cD^{\alpha}y(x)$ and $x^{\beta-\alpha}\,{}^cD^{\beta}y(x)$. Towards this end, using Corollary 2.1, we have that

$$\begin{aligned}{}^cD^{\alpha}y(x)&=\sum_{k=0}^{\infty}\frac{b_N^k}{\Gamma(\alpha k+1)}\,{}^cD^{\alpha}x^{\alpha k}=\sum_{k=1}^{\infty}\frac{b_N^k}{\Gamma(\alpha k+1-\alpha)}x^{\alpha k-\alpha}\\ &=\sum_{k=0}^{\infty}\frac{b_N^{k+1}}{\Gamma(\alpha k+1)}x^{\alpha k}\end{aligned} \tag{38}$$

and similarly one sees that

$$\begin{aligned}x^{\beta-\alpha}\,{}^cD^{\beta}y(x)&=x^{\beta-\alpha}\sum_{k=0}^{\infty}\frac{b_N^k}{\Gamma(\alpha k+1)}\,{}^cD^{\beta}x^{\alpha k}\\ &=\sum_{k=0}^{\infty}\frac{b_N^{k+1}}{\Gamma(\alpha k+\alpha+1-\beta)}x^{\alpha k}.\end{aligned} \tag{39}$$

Substituting the series (38), (39) in (34) and then applying Proposition 4.1 gives

$$\sum_{\ell=0}^{\infty}\frac{b_N^{\ell+1}}{\Gamma(\alpha\ell+1)}x^{\alpha\ell}+\sum_{\ell=0}^{\infty}\frac{\omega b_N^{\ell+1}}{\Gamma(\alpha(\ell+1)-\beta+1)}x^{\alpha\ell}+\sum_{\ell=0}^{\infty}\mathsf{C}_{\ell,N}^{\alpha}x^{\alpha\ell}=0, \tag{40}$$

which implies that

$$\sum_{\ell=0}^{\infty}\left(\frac{b_N^{\ell+1}}{\Gamma(\alpha\ell+1)}+\frac{\omega b_N^{\ell+1}}{\Gamma(\alpha(\ell+1)-\beta+1)}+\mathsf{C}_{\ell,N}^{\alpha}\right)x^{\alpha\ell}=0. \tag{41}$$

Comparing the coefficients of $x^{\alpha\ell}, \ell=0,1,2,\cdots$ on both sides of (41) gives

$$b_N^{\ell+1}\left(\frac{1}{\Gamma(\alpha\ell+1)}+\frac{\omega}{\Gamma(\alpha(\ell+1)-\beta+1)}\right)=-\mathsf{C}_{\ell,N}^{\alpha}$$

and as a result,

$$b_N^{\ell+1}=-\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,N}^{\alpha}}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)}, \tag{42}$$

where the coefficients $\mathsf{C}_{\ell,N}^{\alpha}$ are as given in Proposition 4.1. □

Remark 4.1. For computation purposes, in addition to the first two coefficients $\mathsf{B}_{\ell,m}^{\alpha,N}, m=1,2$, given in Propostion 4.1, we present explicitly the coefficients $\mathsf{B}_{\ell,m}^{\alpha,N}, 3\leq m\leq 20$:

$$\begin{aligned}\mathsf{B}_{\ell,3}^{\alpha,N}&=\sum_{p_2=0}^{\ell}\sum_{p_1=0}^{p_2}\frac{b_N^{p_1}b_N^{p_2-p_1}b_N^{\ell-p_2}}{\Gamma(\alpha p_1+1)\Gamma(\alpha p_2-\alpha p_1+1)\Gamma(\alpha\ell-\alpha p_2+1)}\\ \mathsf{B}_{\ell,4}^{\alpha,N}&=\sum_{p_3=0}^{\ell}\sum_{p_2=0}^{p_3}\sum_{p_1=0}^{p_2}\frac{b_N^{p_1}b_N^{p_2-p_1}b_N^{p_3-p_2}b_N^{\ell-p_3}}{\Gamma(\alpha p_1+1)\Gamma(\alpha p_2-\alpha p_1+1)\Gamma(\alpha p_3-\alpha p_2+1)\Gamma(\alpha\ell-\alpha p_3+1)}\\ &\vdots\\ \mathsf{B}_{\ell,10}^{\alpha,N}&=\sum_{p_9=0}^{\ell}\sum_{p_8=0}^{p_9}\cdots\sum_{p_1=0}^{p_2}\frac{b_N^{\ell-p_9}b_N^{p_9-p_8}\cdots b_N^{p_1}}{\Gamma(\alpha\ell-\alpha p_9+1)\Gamma(\alpha p_9-\alpha p_8+1)\cdots\Gamma(\alpha p_1+1)}\\ &\vdots\\ \mathsf{B}_{\ell,20}^{\alpha,N}&=\sum_{p_{19}=0}^{\ell}\sum_{p_{18}=0}^{p_{19}}\cdots\sum_{p_1=0}^{p_2}\frac{b_N^{\ell-p_{19}}b_N^{p_{19}-p_{18}}\cdots b_N^{p_1}}{\Gamma(\alpha\ell-\alpha p_{19}+1)\Gamma(\alpha p_{19}-\alpha p_{18}+1)\cdots\Gamma(\alpha p_1+1)}.\end{aligned} \tag{43}$$

One sees that the fractional differential equation under consideration now takes the interesting form

$${}^cD^{\alpha}y(x)+\frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x)+a_0+a_1y(x)+a_2y^2(x)+\cdots+a_Ny^N(x)=0, \tag{44}$$

which clearly is a generalization of the fractional Lane-Emden equation

$${}^cD^{\alpha}y(x)+\frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x)+y^m(x)=0, \tag{45}$$

in the sense that the new function $f(y(x))$ is a linear combination of the classical polytropic term $y^m(x)$, $0\leq m\leq N$. Equation (45) is the one considered in [28, Subsection 5.2.2] using the power series method.

To make our calculations explicit and computationally interesting we consider those elementary functions whose expansion coefficients are explicitly known. Such elementary functions to be examined with their explicit associated expansion coefficients a_m, $m=0,1,2,3,\ldots$, are the trigonometric, hyperbolic and exponential functions. These functions are enumerated as follows.

(*a*) $f(y(x))=\sin y(x)$, $a_{2m}=0$, $a_{2m+1}=(-1)^m/(2m+1)!$;

(*b*) $f(y(x))=\cos y(x)$, $a_{2m+1}=0$, $a_{2m}=(-1)^m/(2m)!$;

(*c*) $f(y(x))=\sinh y(x)$, $a_{2m}=0$, $a_{2m+1}=1/(2m+1)!$;

(*d*) $f(y(x)) = \cosh y(x)$, $a_{2m+1} = 0$, $a_{2m} = 1/(2m)!$;

(*e*) $f(y(x)) = e^{\pm y(x)}$, $a_m = (\pm 1)^m/m!$.

For extensive discussions of models involving these functions as well as the physical interpretations of the solutions, see [12], [13], [42].

We proceed with the computation of the approximate solution of the problem (10) according to whether the functions $f(y)$ are trigonometric, hyperbolic or exponential functions.

4.1. Lane-Emden Equation Involving Trigonometric Functions

In this subsection, we treat the equation (10) as well as (34) for which the functions $f(y(x))$ are trigonometric functions, namely, the sine function and cosine function. The Maclaurin series representations for these trigonometric functions are employed.

4.1.1. The Special Case $f(y(x)) = \sin y(x)$

In this case the fractional initial value problem (10) reduces to a special one

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + \sin y(x) = 0,\\ &y(0) = 1, y'(0) = 0. \end{aligned} \tag{46}$$

In solving the initial value problem (46), we consider the series expansion of $\sin y(x)$, namely,

$$\sin y(x) = \sum_{m=0}^{N} a_{2m+1} y^{2m+1}(x), \quad a_{2m+1} = \frac{(-1)^m}{(2m+1)!}, \quad 0 \le m \le N, N \in \mathbb{N}_0. \tag{47}$$

In this case the problem (34) becomes

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + \sum_{m=0}^{N} (-1)^m \frac{y^{2m+1}(x)}{(2m+1)!} = 0\\ &y(0) = 1, y'(0) = 0. \end{aligned} \tag{48}$$

By Theorem 4.1, the analytical solution of the reduced fractional problem (48) is then given by the Mittag-Leffler function expansion

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_N^\ell(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)} x^{\alpha\ell}, \tag{49}$$

where the associated expansion coefficients are given explicitly by

$$\mathcal{B}_N^0(\alpha,\beta;\omega) := b_N^0 = 1, \tag{50}$$

$$\begin{aligned} \mathcal{B}_N^{\ell+1}(\alpha,\beta;\omega) &:= b_N^{\ell+1}\\ &= -\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,N}^\alpha}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)}, \quad \ell \ge 0. \end{aligned} \tag{51}$$

The coefficients $\mathsf{C}_{\ell,N}^\alpha, \ell \ge 0$, appearing in (51) take the formulation

$$\mathsf{C}_{\ell,N}^\alpha = \sum_{m=0}^{N} \frac{(-1)^m}{(2m+1)!} \mathsf{B}_{\ell,2m+1}^{\alpha,N}. \tag{52}$$

In order to see explicitly the values of the coefficients $\mathsf{C}_{\ell,N}^\alpha$, we consider the first values $N \in \mathbb{N}_0$ and this is presented as follows.

(a) $N = 0$. In this case, equation (48) reduces to the initial value problem

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + y(x) = 0,\\ &y(0) = 1, y'(0) = 0. \end{aligned} \tag{53}$$

It follows from Theorem 4.1 that the solution of the initial value problem (53) is given by the Mittag-Leffler expansion

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_0^\ell(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)} x^{\alpha\ell}, \tag{54}$$

where the expansion coefficients $\mathcal{B}_0^\ell$ admit the explicit formulation ($\ell \ge 0$)

$$\begin{aligned} &\mathcal{B}_0^0(\alpha,\beta;\omega) = 1,\\ &\mathcal{B}_0^{\ell+1}(\alpha,\beta;\omega) = -\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,0}^\alpha}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)}, \end{aligned} \tag{55}$$

with the numbers $\mathsf{C}_{\ell,0}^\alpha$, $\ell \ge 1$, given by

$$\mathsf{C}_{\ell,0}^\alpha = \mathsf{B}_{\ell,1}^{\alpha,0} = \mathsf{A}_{\ell,0}^\alpha = \frac{b_0^\ell}{\Gamma(\alpha\ell+1)}. \tag{56}$$

For further explicit calculations we proceed by assigning special values to the parameters α, β and ω and this is done in the following way as examples.

Example 4.1. *Setting $\alpha = 2, \beta = 1, \omega = 2$, we see in this case that equation (53) reduces to the initial value problem*

$$y'' + \frac{2}{x}y' + y = 0, \quad y(0) = 1, y'(0) = 0. \tag{57}$$

It is seen from the series (54) *that the solution of the initial value problem (57) takes the formulation*

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_0^\ell(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell}, \tag{58}$$

where the expansion coefficients $\mathcal{B}_0^\ell = \mathcal{B}_0^\ell(2,1;2)$ admit the explicit formulation ($\ell = 0, 1, 2, \dots$)

$$\begin{aligned} &\mathcal{B}_0^0(2,1;2) = 1,\\ &\mathcal{B}_0^{\ell+1}(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}_{\ell,0}^2}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \end{aligned} \tag{59}$$

and

$$\mathsf{C}^2_{\ell,0} = \mathsf{B}^{2,0}_{\ell,1} = \mathsf{A}^2_{\ell,0} = \frac{b^\ell_0}{\Gamma(2\ell+1)}, \qquad \ell = 0,1,2,\dots . \tag{60}$$

Clearly, the first Mittag-Leffler expansion coefficients $\mathcal{B}^\ell_0(2,1;2), \ell = 1,2,3,4$ yield the following values.

$$\begin{aligned}
\mathcal{B}^1_0(2,1;2) &= -\,\mathsf{C}^2_{0,0}\cdot\frac{1}{3} = -b^0_0\cdot\frac{1}{3} = -\frac{1}{3}\\
\mathcal{B}^2_0(2,1;2) &= -\,\mathsf{C}^2_{1,0}\cdot\frac{6}{5} = -\frac{\mathcal{B}^1_0}{2!}\cdot\frac{6}{5} = \frac{1}{6}\cdot\frac{6}{5} = \frac{1}{5}\\
\mathcal{B}^3_0(2,1;2) &= -\,\mathsf{C}^2_{2,0}\cdot\frac{5!}{7} = -\frac{\mathcal{B}^2_0}{4!}\cdot\frac{5!}{7} = -\frac{1}{5!}\cdot\frac{5!}{7} = -\frac{1}{7}\\
\mathcal{B}^4_0(2,1;2) &= -\,\mathsf{C}^2_{3,0}\cdot 560 = -\frac{\mathcal{B}^3_0}{6!}\cdot 560 = \frac{1}{7!}\cdot 560 = \frac{1}{9}.
\end{aligned} \tag{61}$$

Substituting these coefficients into (58) *gives the solution of the initial value problem* (57)*:*

$$y(x) = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \frac{1}{9!}x^8 + \cdots = \frac{\sin x}{x}. \tag{62}$$

Example 4.2. *Indeed if we set $\alpha = 3/2, \beta = 1/2, \omega = 2$, then one sees that equation* (53) *becomes the initial value problem*

$$\begin{aligned}
&{}^cD^{\frac{3}{2}}y(x) + \frac{2}{x}\,{}^cD^{\frac{1}{2}}y(x) + y(x) = 0,\\
&y(0) = 1, y'(0) = 0.
\end{aligned} \tag{63}$$

One clearly sees from the series (54) *that the solution of the initial value problem* (63) *gives*

$$y(x) = \sum_{\ell=0}^{\infty}\frac{\mathcal{B}^\ell_0\left(\frac{3}{2},\frac{1}{2};2\right)}{\Gamma\left(\frac{3\ell}{2}+1\right)}x^{\frac{3\ell}{2}}, \tag{64}$$

where the expansion coefficients $\mathcal{B}^\ell_0 = \mathcal{B}^\ell_0\left(\frac{3}{2},\frac{1}{2};2\right)$ admit the Gamma function representation

$$\begin{aligned}
&\mathcal{B}^0_0\left(\frac{3}{2},\frac{1}{2};2\right) = 1,\\
&\mathcal{B}^{\ell+1}_0\left(\frac{3}{2},\frac{1}{2};2\right) = -\frac{\Gamma(\frac{3}{2}\ell+1)\Gamma(\frac{3}{2}\ell+2)\mathsf{C}^{\frac{3}{2}}_{\ell,0}}{\Gamma(\frac{3}{2}\ell+2)+2\Gamma(\frac{3}{2}\ell+1)}, \ell \geq 0,
\end{aligned} \tag{65}$$

and

$$\mathsf{C}^{\frac{3}{2}}_{\ell,0} = \mathsf{B}^{\frac{3}{2},0}_{\ell,1} = \mathsf{B}^{\frac{3}{2},0}_{\ell,1} = \frac{b^\ell_0}{\Gamma\left(\frac{3}{2}\ell+1\right)}, \quad \ell = 0,1,2,\dots . \tag{66}$$

The first Mittag-Leffler expansion coefficients $\mathcal{B}^\ell_0\left(\frac{3}{2},\frac{1}{2};2\right), \ell = 1,2,3,4,5,6$, are computed as follows:

$$\begin{aligned}
\mathcal{B}^1_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{0,0}\cdot\frac{1}{3} = -b^0_0\cdot\frac{1}{3} = -\frac{1}{3}\\
\mathcal{B}^2_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{1,0}\cdot\frac{\Gamma(\frac{5}{2})\cdot\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2})+2\Gamma(\frac{5}{2})} = -\frac{\mathcal{B}^1_0}{\Gamma(\frac{5}{2})}\frac{5\sqrt{\pi}}{12} = \frac{5}{27}\\
\mathcal{B}^3_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{2,0}\cdot\frac{\Gamma(4)\cdot\Gamma(5)}{\Gamma(5)+2\Gamma(4)} = -\frac{\mathcal{B}^2_0}{\Gamma(4)}\cdot(4) = -\frac{10}{81}\\
\mathcal{B}^4_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{3,0}\cdot\frac{\Gamma(\frac{11}{2})\cdot\Gamma(\frac{13}{2})}{\Gamma(\frac{13}{2})+2\Gamma(\frac{11}{2})} = -\frac{\mathcal{B}^3_0}{\Gamma(\frac{11}{2})}\cdot\frac{693\sqrt{\pi}}{32}\\
&= \frac{22}{243}\\
\mathcal{B}^5_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{4,0}\cdot\frac{\Gamma(7)\cdot\Gamma(8)}{\Gamma(8)+2\Gamma(7)} = -\frac{\mathcal{B}^4_0}{\Gamma(7)}\cdot(560)\\
&= -\frac{154}{2187}\\
\mathcal{B}^6_0\left(\frac{3}{2},\frac{1}{2};2\right) &= -\,\mathsf{C}^{\frac{3}{2}}_{5,0}\cdot\frac{\Gamma\left(\frac{17}{2}\right)\cdot\Gamma(\frac{19}{2})}{\Gamma(\frac{19}{2})+2\Gamma(\frac{17}{2})}\\
&= -\frac{\mathcal{B}^5_0}{\Gamma\left(\frac{17}{2}\right)}\cdot\frac{1640925\sqrt{\pi}}{256} = \frac{374}{6561}.
\end{aligned} \tag{67}$$

Upon substituting these coefficients into (64) *yields the solution*

$$\begin{aligned}
y(x) = 1 &- \frac{4x^{\frac{3}{2}}}{9\sqrt{\pi}} + \frac{5x^3}{162} - \frac{64x^9/2}{15309\sqrt{\pi}} + \frac{11x^6}{87480}\\
&- \frac{512x^{15/2}}{57572775\sqrt{\pi}} + \frac{187x^9}{1190427840} + \cdots .
\end{aligned} \tag{68}$$

(b) $N = 1$. Indeed it is understood here that equation (48) reduces to the initial value problem

$$\begin{aligned}
&{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + y(x) - \frac{y^3(x)}{3!} = 0,\\
&y(0) = 1, y'(0) = 0.
\end{aligned} \tag{69}$$

It is also seen here that the solution of the fractional problem (69) gives

$$y(x) = \sum_{\ell=0}^{\infty}\frac{\mathcal{B}^\ell_1(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)}x^{\alpha\ell}, \tag{70}$$

with the expansion coefficients $\mathcal{B}^\ell_1 = \mathcal{B}^\ell_1(\alpha,\beta;\omega)$ ($\ell \geq 0$) given by

$$\begin{aligned}
&\mathcal{B}^0_1(\alpha,\beta;\omega) = 1,\\
&\mathcal{B}^{\ell+1}_1(\alpha,\beta;\omega) = -\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}^\alpha_{\ell,1}}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)}.
\end{aligned} \tag{71}$$

Here the coefficients $\mathsf{C}^\alpha_{\ell,1}$, $\ell \geq 0$, on the right of (71) are given by

$$\mathsf{C}^\alpha_{\ell,1} = \mathsf{B}^{\alpha,1}_{\ell,1} - \frac{1}{3!}\mathsf{B}^{\alpha,1}_{\ell,3}, \tag{72}$$

where the numbers $\mathsf{B}^{\alpha,N}_{\ell,m}, m = 1,3$, are as illustrated in (16) and (27) respectively.

Example 4.3. *In this case, another special Lane-Emden equation can be obtained by setting $\alpha = 2, \beta = 1, \omega = 2$; and as a result, equation (69) reduces to*

$$y'' + \frac{2}{x}y' + y - \frac{y^3}{3!} = 0, \quad y(0) = 1, y'(0) = 0, \tag{73}$$

whose solution admits the series representation

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}^{\ell}_1(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell}. \tag{74}$$

From equations (71) and (72), we respectively have

$$\mathcal{B}^0_1(2,1;2) = 1,$$
$$\mathcal{B}^{\ell+1}_1(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,1}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell \geq 0; \tag{75}$$

and

$$\mathsf{C}^2_{\ell,1} = \frac{b^{\ell}_1}{\Gamma(2\ell+1)} - \frac{1}{6}\left(\sum_{p_2=0}^{\ell}\sum_{p_1=0}^{p_2} \frac{b^{p_1}_1 b^{p_2-p_1}_1 b^{\ell-p_2}_1}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2\ell-2p_2+1)}\right). \tag{76}$$

It is straightforward to see that the first Mittag-Leffler expansion coefficients $\mathcal{B}^{\ell}_1(2,1;2)$, $\ell = 1,2,3,4,5,6$, are calculated as follows.

$$\begin{aligned}
\mathcal{B}^1_1(2,1;2) &= -\mathsf{C}^2_{0,1}\cdot\frac{1}{3} = -\frac{5b^0_1}{6}\cdot\frac{1}{3} = -\frac{5}{6}\cdot\frac{1}{3} = -\frac{5}{18}\\
\mathcal{B}^2_1(2,1;2) &= -\mathsf{C}^2_{1,1}\cdot\frac{6}{5} = -\frac{\mathcal{B}^1_1}{3}\cdot\frac{6}{5} = \frac{5}{54}\cdot\frac{6}{5} = \frac{1}{9}\\
\mathcal{B}^3_1(2,1;2) &= -\mathsf{C}^2_{2,1}\cdot\frac{5!}{7} = -\left(\frac{\mathcal{B}^2_1}{36} - \frac{\mathcal{B}^1_1\mathcal{B}^1_1}{4}\right)\cdot\frac{120}{7}\\
&= \frac{1}{7776}\cdot\frac{120}{7} = \frac{5}{2268}\\
\mathcal{B}^4_1(2,1;2) &= -\mathsf{C}^2_{3,1}\cdot 560 = -\left(\frac{\mathcal{B}^3_1}{1080} - \frac{\mathcal{B}^1_1\mathcal{B}^2_1}{144}\right)\cdot 560\\
&= -\frac{53}{244944}\cdot 560 = -\frac{265}{2187}\\
\mathcal{B}^5_1(2,1;2) &= -\mathsf{C}^2_{4,1}\cdot\frac{362880}{11}\\
&= -\left(\frac{\mathcal{B}^4_1}{60480} - \frac{\mathcal{B}^1_1\mathcal{B}^3_1}{4320} - \frac{\mathcal{B}^2_1\mathcal{B}^2_1}{3456}\right)\cdot\frac{362880}{11} = \frac{2875}{16038}\\
\mathcal{B}^6_1(2,1;2) &= -\mathsf{C}^2_{5,1}\cdot\frac{39916800}{13}\\
&= -\left(\frac{\mathcal{B}^5_1}{5443200} - \frac{\mathcal{B}^1_1\mathcal{B}^4_1}{241920} - \frac{\mathcal{B}^2_1\mathcal{B}^3_1}{51840}\right)\cdot\frac{39916800}{13} = \frac{29050}{85293}.
\end{aligned} \tag{77}$$

Upon inserting the first coefficients (77) in (74) gives the solution

$$\begin{aligned}
y(x) =& 1 - \frac{5}{18}\cdot\frac{1}{2!}x^2 + \frac{1}{9}\cdot\frac{1}{4!}x^4 + \frac{5}{2268}\cdot\frac{1}{6!}x^6\\
&- \frac{265}{2187}\cdot\frac{1}{8!}x^8 + \frac{2875}{16038}\cdot\frac{1}{10!}x^{10} + \frac{29050}{85293}\cdot\frac{1}{12!}x^{12}\cdots\\
=& 1 - 0.138889x^2 + 0.00347222x^4 + 0.0000030619x^6\\
&- 0.0000030052x^8 + 0.0000000494x^{10}\\
&+ 0.0000000007x^{12} + \cdots
\end{aligned} \tag{78}$$

(c) $N = 2$. Another approximate solution of the problem(46) can be obtained in this case in view of the reduced problem (48):

$$\begin{aligned}
&{}^cD^{\alpha}y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x) + y(x) - \frac{y^3(x)}{3!} + \frac{y^5(x)}{5!} = 0,\\
&y(0) = 1, y'(0) = 0.
\end{aligned} \tag{79}$$

Example 4.4. *Similarly, in this case of $N = 2$, we specialise to the values $\alpha = 2, \beta = 1, \omega = 2$; to see that equation (79) becomes*

$$y'' + \frac{2}{x}y' + y - \frac{y^3}{3!} + \frac{y^5}{5!} = 0, \quad y(0) = 1, y'(0) = 0. \tag{80}$$

Clearly, the solution of the problem (80) takes the series formula

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}^{\ell}_2(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell}, \tag{81}$$

with the coefficients described as follows:

$$\begin{aligned}
&\mathcal{B}^0_2(2,1;2) = 1,\\
&\mathcal{B}^{\ell+1}_2(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,2}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)},\\
&\mathsf{C}^2_{\ell,2} = a_1\mathsf{B}^{2,2}_{\ell,1} + a_3\mathsf{B}^{2,2}_{\ell,3} + a_5\mathsf{B}^{2,2}_{\ell,5} = \mathsf{B}^{2,2}_{\ell,1} - \frac{1}{3!}\mathsf{B}^{2,2}_{\ell,3} + \frac{1}{5!}\mathsf{B}^{2,2}_{\ell,5}.
\end{aligned} \tag{82}$$

Here, one clearly understands that

$$\begin{aligned}
\mathsf{B}^{2,2}_{\ell,1} &= \frac{b^{\ell}_2}{\Gamma(2\ell+1)},\\
\mathsf{B}^{2,2}_{\ell,3} &= \sum_{p_2=0}^{\ell}\sum_{p_1=0}^{p_2} \frac{b^{p_1}_2 b^{p_2-p_1}_2 b^{\ell-p_2}_2}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2\ell-2p_2+1)}\\
\mathsf{B}^{2,2}_{\ell,5} &= \sum_{p_4=0}^{\ell}\sum_{p_3=0}^{p_4}\sum_{p_2=0}^{p_3}\sum_{p_1=0}^{p_2}\\
&\frac{b^{p_1}_2 b^{p_2-p_1}_2 b^{p_3-p_2}_2 b^{p_4-p_3}_2 b^{\ell-p_4}_2 b^{\ell}_2[\Gamma(2\ell-2p_4+1)]^{-1}}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2p_3-2p_2+1)\Gamma(2p_4-2p_3+1)}.
\end{aligned} \tag{83}$$

As a result, the first Mittag-Leffler expansion coefficients $\mathcal{B}_3^\ell(2,1;2)$, $\ell=1,2,3,4$, are computed as follows.

$$\begin{aligned}
\mathcal{B}_2^1(2,1;2) &= -\,\mathsf{C}_{0,2}^2\cdot\frac{1}{3} = -\frac{101}{120}\cdot\frac{1}{3} = -\frac{101}{360}\\
\mathcal{B}_2^2(2,1;2) &= -\,\mathsf{C}_{1,2}^2\cdot\frac{6}{5} = -\frac{41\mathcal{B}_2^1}{120}\cdot\frac{6}{5} = \frac{4141}{36000}\\
\mathcal{B}_2^3(2,1;2) &= -\,\mathsf{C}_{2,2}^2\cdot\frac{5!}{7} = -\left(\frac{41\mathcal{B}_2^2}{1440}-\frac{19\mathcal{B}_2^1\mathcal{B}_2^1}{480}\right)\cdot\frac{5!}{7}\\
&=\frac{49591}{18144000}\\
\mathcal{B}_2^4(2,1;2) &= -\,\mathsf{C}_{3,2}^2\cdot(560)\\
&= -\left(\frac{41\mathcal{B}_2^3}{43200}-\frac{19\mathcal{B}_2^1\mathcal{B}_2^2}{2880}\right)\cdot 560 = -\frac{16891139}{139968000}\\
\mathcal{B}_2^5(2,1;2) &= -\,\mathsf{C}_{4,2}^2\cdot\frac{362880}{11}\\
&= -\left(\frac{41\mathcal{B}_2^4}{2419200}-\frac{19\mathcal{B}_2^1\mathcal{B}_2^3}{86400}-\frac{19\mathcal{B}_2^2\mathcal{B}_2^2}{69120}\right)\cdot\frac{362880}{11}\\
&=\frac{93349921751}{513216000000}.
\end{aligned}\tag{84}$$

Upon substituting these coefficients into the series formula (81) *gives the solution*

$$\begin{aligned}
y(x) =&1-\frac{101}{360}\cdot\frac{1}{2!}x^2+\frac{4141}{36000}\cdot\frac{1}{4!}x^4+\frac{49591}{18144000}\cdot\frac{1}{6!}x^6\\
&-\frac{16891139}{139968000}\cdot\frac{1}{8!}x^8+\frac{93349921751}{513216000000}\cdot\frac{1}{10!}x^{10}+\cdots\\
=&1-0.14027778x^2+0.00479282x^4\\
&+0.000003796x^6+0.000002993x^8\\
&+0.0000000501x^{10}+\cdots
\end{aligned}\tag{85}$$

4.1.2. The Special Case $f(y(x))=\cos y(x)$

It is seen here that the fractional Lane-Emden equation under consideration is

$$\begin{aligned}
&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x)+\cos y(x)=0,\\
&y(0)=1,y'(0)=0.
\end{aligned}\tag{86}$$

Finding the approximate solution of the initial value problem (86) amounts to solving the problem

$$\begin{aligned}
&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x)+\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}y^{2m}(x)=0,\\
&y(0)=1,y'(0)=0,
\end{aligned}\tag{87}$$

where we have used the truncated power series

$$\cos y(x)=\sum_{m=0}^{N}a_{2m}y^{2m}(x)=\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}y^{2m}(x),\ \ N\in\mathbb{N}_0.\tag{88}$$

Theorem 4.1 tells us that the analytical solution of the initial value problem (87) admits the series solution

$$y(x)=\sum_{\ell=0}^{\infty}\frac{\mathcal{B}_N^\ell(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)}x^{\alpha\ell},\tag{89}$$

where we have

$$\mathcal{B}_N^0(\alpha,\beta;\omega):=b_N^0=1,\tag{90}$$

$$\begin{aligned}
\mathcal{B}_N^{\ell+1}&(\alpha,\beta;\omega):=b_N^{\ell+1}\\
&=-\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,N}^\alpha}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)},\quad \ell\geq 0.
\end{aligned}\tag{91}$$

with

$$\mathsf{C}_{\ell,N}^\alpha=\sum_{m=0}^{N}\frac{(-1)^m}{(2m)!}\mathsf{B}_{\ell,2m}^{\alpha,N}.\tag{92}$$

We now proceed to the explicit computation of the first expansion coefficients $\mathcal{B}_N^\ell(\alpha,\beta;\omega)$. This is carried out by first considering the first values of $N\in\mathbb{N}_0$.

(a) $N=0$. In this case, equation (86) reduces to the initial value problem

$$\begin{aligned}
&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x)+1=0,\\
&y(0)=1,y'(0)=0.
\end{aligned}\tag{93}$$

One sees from (89) that the solution of the initial value problem (93) is given by

$$y(x)=\sum_{\ell=0}^{\infty}\frac{\mathcal{B}_0^\ell(\alpha,\beta;\omega)}{\Gamma(\alpha\ell+1)}x^{\alpha\ell},\tag{94}$$

where ($\ell\geq 0$)

$$\begin{aligned}
&\mathcal{B}_0^0(\alpha,\beta;\omega)=1,\\
&\mathcal{B}_0^{\ell+1}(\alpha,\beta;\omega)=-\frac{\Gamma(\alpha\ell+1)\Gamma(\alpha(\ell+1)-\beta+1)\mathsf{C}_{\ell,0}^\alpha}{\Gamma(\alpha(\ell+1)-\beta+1)+\omega\Gamma(\alpha\ell+1)},
\end{aligned}\tag{95}$$

with the numbers $\mathsf{C}_{\ell,0}^\alpha$, $\ell\geq 0$, given by

$$\mathsf{C}_{\ell,0}^\alpha=a_0\mathsf{B}_{\ell,0}^{\alpha,0}=\mathsf{B}_{\ell,0}^{\alpha,0}=\delta_{\ell 0}.\tag{96}$$

Example 4.5. *With the special values $\alpha=2,\beta=1$ and $\omega=2$, we obtain a special case of the standard Lane-Emden equation*

$$y''+\frac{2}{x}y'+1=0,\quad y(0)=1,y'(0)=0,\tag{97}$$

whose solution is of the form

$$y(x)=\sum_{\ell=0}^{\infty}\frac{\mathcal{B}_0^\ell(2,1;2)}{\Gamma(2\ell+1)}x^{2\ell}.\tag{98}$$

Here we have

$$\begin{aligned}
&\mathcal{B}_0^0(2,1;2)=1,\\
&\mathcal{B}_0^{\ell+1}(2,1;2)=-\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}_{\ell,0}^2}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)},\ \ \ell\geq 0,
\end{aligned}\tag{99}$$

with $\mathsf{C}^2_{\ell,0} = \delta_{\ell 0}$*. Interestingly one has the formula*

$$\mathcal{B}^1_0(2,1;2) = \begin{cases} -\frac{1}{3}, & \ell = 0, \\ 0, & \ell = 1,2,\cdots, \end{cases} \tag{100}$$

and as a result we obtain the closed form solution

$$y(x) = 1 - \frac{1}{6}x^2. \tag{101}$$

(b) $N = 1$. In this case, equation (87) reduces to the initial value problem

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + 1 - \frac{y^2}{2!} = 0, \\ &y(0) = 1, y'(0) = 0. \end{aligned} \tag{102}$$

Example 4.6. *Considering the the special values* $\alpha = 2, \beta = 1$ *and* $\omega = 2$*; equation* (102) *becomes the initial value problem*

$$y'' + \frac{2}{x}y' + 1 - \frac{y^2}{2!} = 0, \quad y(0) = 1, y'(0) = 0, \tag{103}$$

with the solution

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}^\ell_1(2,1;2)}{\Gamma(2\ell+1)}x^{2\ell}. \tag{104}$$

It follows that

$$\begin{aligned} &\mathcal{B}^0_1(2,1;2) = 1, \\ &\mathcal{B}^{\ell+1}_1(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,1}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell \geq 0 \end{aligned} \tag{105}$$

with

$$\begin{aligned} \mathsf{C}^2_{\ell,1} &= a_0\mathsf{B}^{2,1}_{\ell,0} + a_2\mathsf{B}^{2,1}_{\ell,2} \\ &= \delta_{\ell 0} - \frac{1}{2}\sum_{p_1=0}^{\ell} \frac{b_1^{p_1} b_1^{\ell-p_1}}{\Gamma(2p_1+1)\Gamma(2\ell-2p_1+1)}. \end{aligned}$$

Consequently, the first Mittag-Leffler expansion coefficients $\mathcal{B}^\ell_1(2,1;2), \ell = 1,2,3,4,5,$ *are now computed as follows.*

$$\begin{aligned} \mathcal{B}^1_1(2,1;2) &= -\mathsf{C}^2_{0,1}\cdot\frac{1}{3} = -\frac{\mathcal{B}^0_1}{2}\cdot\frac{1}{3} = -\frac{1}{6} \\ \mathcal{B}^2_1(2,1;2) &= -\mathsf{C}^2_{1,1}\cdot\frac{6}{5} = \frac{\mathcal{B}^1_1}{2}\cdot\frac{6}{5} = -\frac{1}{10} \\ \mathcal{B}^3_1(2,1;2) &= -\mathsf{C}^2_{2,1}\cdot\frac{5!}{7} = \left(\frac{\mathcal{B}^2_1}{24} + \frac{\mathcal{B}^1_1\mathcal{B}^1_1}{8}\right)\cdot\frac{5!}{7} = -\frac{1}{84} \\ \mathcal{B}^4_1(2,1;2) &= -\mathsf{C}^2_{3,1}\cdot 560 = \left(\frac{\mathcal{B}^3_1}{720} + \frac{\mathcal{B}^1_1\mathcal{B}^2_1}{48}\right)\cdot 560 = \frac{5}{27} \\ \mathcal{B}^5_1(2,1;2) &= -\mathsf{C}^2_{4,1}\cdot\frac{362880}{11} \\ &= \left(\frac{\mathcal{B}^4_1}{40320} + \frac{\mathcal{B}^1_1\mathcal{B}^3_1}{1440} + \frac{\mathcal{B}^2_1\mathcal{B}^2_1}{1152}\right)\cdot\frac{362880}{11} = \frac{29}{60}. \end{aligned} \tag{106}$$

Upon using the coefficients (106)*, one obtains the solution*

$$\begin{aligned} y(x) =& 1 - \frac{1}{6}\cdot\frac{1}{2!}x^2 - \frac{1}{10}\cdot\frac{1}{4!}x^4 - \frac{1}{84}\cdot\frac{1}{6!}x^6 + \frac{5}{27}\cdot\frac{1}{8!}x^8 \\ &+ \frac{29}{60}\cdot\frac{1}{10!}x^{10} + \cdots \\ =& 1 - 0.0833333x^2 - 0.00416667x^4 - 0.00001653x^6 \\ &+ 0.00000459x^8 + 0.00000013x^{10} + \cdots \end{aligned} \tag{107}$$

(c) $N = 2$. Here, equation (87) becomes the initial value problem

$$\begin{aligned} &{}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + 1 - \frac{y^2}{2!} + \frac{y^4}{4!} = 0, \\ &y(0) = 1, y'(0) = 0. \end{aligned} \tag{108}$$

Example 4.7. *Consider the problem*

$$y'' + \frac{2}{x}y' + 1 - \frac{y^2}{2!} + \frac{y^4}{4!} = 0, \quad y(0) = 1, y'(0) = 0. \tag{109}$$

The solution of (109) *takes the form*

$$y(x) = \sum_{\ell=0}^{\infty} \frac{\mathcal{B}^\ell_2(2,1;2)}{\Gamma(2\ell+1)}x^{2\ell}, \tag{110}$$

where the coefficients $\mathcal{B}^\ell_2(2,1;2)$ *are given by*

$$\begin{aligned} &\mathcal{B}^0_2(2,1;2) = 1, \\ &\mathcal{B}^{\ell+1}_2(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,2}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, (\ell \geq 0) \\ &\mathsf{C}^2_{\ell,2} = a_0\mathsf{B}^{2,2}_{\ell,0} + a_2\mathsf{B}^{2,2}_{\ell,2} + a_4\mathsf{B}^{2,2}_{\ell,4} = \mathsf{B}^{2,2}_{\ell,0} - \frac{1}{2!}\mathsf{B}^{2,2}_{\ell,2} + \frac{1}{4!}\mathsf{B}^{2,2}_{\ell,4}, \end{aligned} \tag{111}$$

with

$$\begin{aligned} &\mathsf{B}^{2,2}_{\ell,0} = \delta_{\ell 0},\ \mathsf{B}^{2,2}_{\ell,2} = \sum_{p_1=0}^{\ell} \frac{b_2^{p_1} b_2^{\ell-p_1}}{\Gamma(2p_1+1)\Gamma(2\ell-2p_1+1)} \\ &\mathsf{B}^{2,2}_{\ell,4} = \sum_{p_3=0}^{\ell}\sum_{p_2=0}^{p_3}\sum_{p_1=0}^{p_2} \\ &\frac{b_2^{p_1} b_2^{p_2-p_1} b_2^{p_3-p_2} b_2^{\ell-p_3}}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2p_3-2p_2+1)\Gamma(2\ell-2p_3+1)}. \end{aligned} \tag{112}$$

The first Mittag-Leffler expansion coefficients $\mathcal{B}^{\ell+1}_2(2,1;2), \ell = 0,1,2,\ldots$ *are computed as follows.*

$$\begin{aligned} \mathcal{B}^1_2(2,1;2) &= -\mathsf{C}^2_{0,2}\cdot\frac{1}{3} = -\left(1 - \frac{\mathcal{B}^0_2}{2} + \frac{\mathcal{B}^0_2}{24}\right)\cdot\frac{1}{3} = -\frac{13}{72} \\ \mathcal{B}^2_2(2,1;2) &= -\mathsf{C}^2_{1,2}\cdot\frac{6}{5} = -\left(-\frac{\mathcal{B}^1_2}{2} + \frac{\mathcal{B}^1_2}{24}\right)\cdot\frac{6}{5} = -\frac{143}{1440} \end{aligned}$$

$$\begin{aligned}\mathcal{B}_2^3(2,1;2) &= -\mathsf{C}_{2,2}^2\cdot\frac{5!}{7} = \left(\frac{11\mathcal{B}_2^2}{288}+\frac{11\mathcal{B}_2^1\mathcal{B}_2^1}{96}\right)\cdot\frac{120}{7}\\ &= -\frac{143}{145152}\\ \mathcal{B}_2^4(2,1;2) &= -\mathsf{C}_{3,2}^2\cdot 560 = \left(\frac{11\mathcal{B}_2^3}{8640}+\frac{11\mathcal{B}_2^1\mathcal{B}_2^2}{576}\right)\cdot 560\\ &= \frac{26741}{139968}\\ \mathcal{B}_2^5(2,1;2) &= -\mathsf{C}_{4,2}^2\cdot\frac{362880}{11}\\ &= \left(\frac{11\mathcal{B}_2^4}{483840}+\frac{11\mathcal{B}_2^1\mathcal{B}_2^3}{17280}+\frac{11\mathcal{B}_2^2\mathcal{B}_2^2}{1152}\right)\cdot\frac{362880}{11}\\ &= \frac{6059911}{14929920}.\end{aligned} \tag{113}$$

Upon substituting these coefficients in (110) *yields the series solution*

$$\begin{aligned}y(x) =& 1-\frac{13}{72}\cdot\frac{1}{2!}x^2-\frac{143}{1440}\cdot\frac{1}{4!}x^4-\frac{143}{145152}\cdot\frac{1}{6!}x^6\\ &+\frac{26741}{139968}\cdot\frac{1}{8!}x^8+\frac{6059911}{14929920}\cdot\frac{1}{10!}x^{10}+\cdots\\ =& 1-0.09027778x^2-0.00413773x^4-0.00000137x^6\\ &+0.000004738x^8+0.000000112x^{10}+\cdots.\end{aligned} \tag{114}$$

4.2. Lane-Emden Equation Involving Hyperbolic Functions

This subsection discusses the approximate solution of the initial value problems (10) and (34) for which the functions $f(y(x))$ are the hyperbolic sine function and the hyperbolic cosine function. We also make use of the Maclaurin series representations for these hyperbolic functions.

4.2.1. The Special Case $f(y(x)) = \sinh y(x)$

We consider the fractional Lane-Emden initial value problem

$$\begin{aligned}&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}{}^cD^\beta y(x)+\sinh y(x)=0,\\ &y(0)=1, y'(0)=0.\end{aligned} \tag{115}$$

In solving the initial value problem (115), we are interested in the series expansion of $\sinh y(x)$, namely ($0\le m\le N, N\in\mathbb{N}_0$),

$$\sinh y(x)=\sum_{m=0}^{N}a_{2m+1}y^{2m+1}(x), \qquad a_{2m+1}=\frac{1}{(2m+1)!}, \tag{116}$$

so that finding the approximate solution of the problem (115) amounts to solving the corresponding problem

$$\begin{aligned}&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}{}^cD^\beta y(x)+\sum_{m=0}^{N}\frac{1}{(2m+1)!}y^{2m+1}(x)=0,\\ &y(0)=1, y'(0)=0.\end{aligned} \tag{117}$$

We proceed to obtaining the solutions of the initial value problem (117) for the first values of $N\in\mathbb{N}$.

(a) $N = 0$. Clearly, in this case $a_1 = 1$ and as a result the resulting initial value problem coincides with the problem in (53).

(b) $N = 1$. Here, we have from (117) the problem

$$\begin{aligned}&{}^cD^\alpha y(x)+\frac{\omega}{x^{\alpha-\beta}}{}^cD^\beta y(x)+y(x)+\frac{y^3(x)}{3!}=0,\\ &y(0)=1, y'(0)=0.\end{aligned} \tag{118}$$

Example 4.8. *Consider the initial value problem*

$$y''+\frac{2}{x}y'+y+\frac{y^3}{3!}=0, \quad y(0)=1, y'(0)=0 \tag{119}$$

Clearly the expansion coefficients appearing in the series solution of (119) *are given by*

$$\begin{aligned}&\mathcal{B}_1^0(2,1;2)=1,\\ &\mathcal{B}_1^{\ell+1}(2,1;2)=-\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}_{\ell,1}^2}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell\ge 0.\end{aligned} \tag{120}$$

with

$$\begin{aligned}\mathsf{C}_{\ell,1}^2 &= a_1\mathsf{B}_{\ell,1}^{2,1}+a_3\mathsf{B}_{\ell,3}^{2,1}\\ &= \frac{b_1^\ell}{\Gamma(2\ell+1)}\\ &+\frac{1}{6}\left(\sum_{p_2=0}^{\ell}\sum_{p_1=0}^{p_2}\frac{b_1^{p_1}b_1^{p_2-p_1}b_1^{\ell-p_2}}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2\ell-2p_2+1)}\right).\end{aligned} \tag{121}$$

The first Mittag-Leffler expansion coefficients $\mathcal{B}_1^\ell(2,1;2), \ell = 0,1,2,3,4,5$, are presented as follows.

$$\begin{aligned}\mathcal{B}_1^1(2,1;2) &= -\mathsf{C}_{0,1}^2\cdot\frac{1}{3} = -\frac{7\mathcal{B}_1^0}{6}\cdot\frac{1}{3} = -\frac{7}{18}\\ \mathcal{B}_1^2(2,1;2) &= -\mathsf{C}_{1,1}^2\cdot\frac{6}{5} = -\frac{2\mathcal{B}_1^1}{3}\cdot\frac{6}{5} = \frac{14}{45}\\ \mathcal{B}_1^3(2,1;2) &= -\mathsf{C}_{2,1}^2\cdot\frac{5!}{7} = -\left(\frac{\mathcal{B}_1^2}{18}-\frac{\mathcal{B}_1^1\mathcal{B}_1^1}{24}\right)\cdot\frac{120}{7} = -\frac{131}{324}\\ \mathcal{B}_1^4(2,1;2) &= -\mathsf{C}_{3,1}^2\cdot 560 = -\left(\frac{\mathcal{B}_1^3}{540}+\frac{\mathcal{B}_1^1\mathcal{B}_1^2}{144}\right)\cdot 560 = \frac{1946}{2187}\\ \mathcal{B}_1^5(2,1;2) &= -\mathsf{C}_{4,1}^2\cdot\frac{362880}{11}\\ &= -\left(\frac{\mathcal{B}_1^4}{30240}+\frac{\mathcal{B}_1^1\mathcal{B}_1^3}{4320}+\frac{\mathcal{B}_1^2\mathcal{B}_1^2}{3456}\right)\cdot\frac{362880}{11} = -\frac{248213}{80190}\end{aligned} \tag{122}$$

Upon substituting the first coefficients (122) *we obtain the* *solution*

$$\begin{aligned}y(x) =& \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_1^{\ell}(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell}\\ =&1 - \frac{7}{18}\cdot\frac{1}{2!}x^2 + \frac{14}{45}\cdot\frac{1}{4!}x^4 - \frac{131}{324}\cdot\frac{1}{6!}x^6 + \frac{1946}{2187}\cdot\frac{1}{8!}x^8\\ &- \frac{248213}{80190}\cdot\frac{1}{10!}x^{10} + \cdots\\ =&1 - 0.19444444x^2 + 0.01296296x^4 - 0.00056156x^6\\ &+ 0.000022069x^8 - 0.000000853x^{10} + \cdots . \quad (123)\end{aligned}$$

(c) $N = 2$. Indeed it is easy to understand here that from (117) one obtains the initial value problem

$$\begin{aligned}&{}^cD^{\alpha}y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x) + y(x) + \frac{y^3(x)}{3!} + \frac{y^5(x)}{5!} = 0,\\ &y(0) = 1, y'(0) = 0.\end{aligned} \tag{124}$$

Example 4.9. *Given the problem*

$$y'' + \frac{2}{x}y' + y + \frac{y^3}{3!} + \frac{y^5}{5!} = 0, \quad y(0) = 1, y'(0) = 0. \tag{125}$$

Clearly, the expansion coefficients of the series solution of (125) *are*

$$\begin{aligned}&\mathcal{B}_2^0(2,1;2) = 1,\\ &\mathcal{B}_2^{\ell+1}(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}_{\ell,2}^2}{\Gamma(2\ell+2) + 2\Gamma(2\ell+1)},\\ &\mathsf{C}_{\ell,2}^2 = a_1\mathsf{B}_{\ell,1}^{2,2} + a_3\mathsf{B}_{\ell,3}^{2,2} + a_5\mathsf{B}_{\ell,5}^{2,2} = \mathsf{B}_{\ell,1}^{2,2} + \frac{1}{3!}\mathsf{B}_{\ell,3}^{2,2} + \frac{1}{5!}\mathsf{B}_{\ell,5}^{2,2},\end{aligned} \tag{126}$$

where

$$\begin{aligned}&\mathsf{B}_{\ell,1}^{2,2} = \frac{b_2^{\ell}}{\Gamma(2\ell+1)},\\ &\mathsf{B}_{\ell,3}^{2,2} = \sum_{p_2=0}^{\ell}\sum_{p_1=0}^{p_2} \frac{b_2^{p_1} b_2^{p_2-p_1} b_2^{\ell-p_2}}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2\ell-2p_2+1)}\\ &\mathsf{B}_{\ell,5}^{2,2} = \sum_{p_4=0}^{\ell}\sum_{p_3=0}^{p_4}\sum_{p_2=0}^{p_3}\sum_{p_1=0}^{p_2}\\ &\frac{b_2^{p_1} b_2^{p_2-p_1} b_2^{p_3-p_2} b_2^{p_4-p_3} b_2^{\ell-p_4}[\Gamma(2\ell-2p_4+1)]^{-1}}{\Gamma(2p_1+1)\Gamma(2p_2-2p_1+1)\Gamma(2p_3-2p_2+1)\Gamma(2p_4-2p_3+1)}.\end{aligned} \tag{127}$$

The first expansion coefficients $\mathcal{B}_1^{\ell}(2,1;2), \ell = 1,2,3,4,5$ *are computed as follows.*

$$\begin{aligned}&\mathcal{B}_2^1(2,1;2) = -\mathsf{C}_{0,2}^2\cdot\frac{1}{3} = -\frac{47b_2^0}{40}\cdot\frac{1}{3} = -\frac{47}{120}\\ &\mathcal{B}_2^2(2,1;2) = -\mathsf{C}_{1,2}^2\cdot\frac{6}{5} = -\frac{27\mathcal{B}_2^1}{3}\cdot\frac{6}{5} = \frac{1269}{4000}\\ &\mathcal{B}_2^3(2,1;2) = -\mathsf{C}_{2,2}^2\cdot\frac{5!}{7} = -\left(\frac{9\mathcal{B}_2^2}{160} - \frac{7\mathcal{B}_2^1\mathcal{B}_2^1}{160}\right)\cdot\frac{120}{7}\\ &= -\frac{282893}{672000}\\ &\mathcal{B}_2^4(2,1;2) = -\mathsf{C}_{3,2}^2\cdot 560 = -\left(\frac{3\mathcal{B}_2^3}{1600} + \frac{7\mathcal{B}_2^1\mathcal{B}_2^2}{960}\right)\cdot 560\\ &= \frac{4747}{5000}\\ &\mathcal{B}_2^5(2,1;2) = -\mathsf{C}_{4,2}^2\cdot\frac{362880}{11}\\ &= -\left(\frac{3\mathcal{B}_2^4}{89600} + \frac{7\mathcal{B}_2^1\mathcal{B}_2^3}{28800} + \frac{7\mathcal{B}_2^2\mathcal{B}_2^2}{23040}\right)\cdot\frac{362880}{11} = -\frac{2379140611}{704000000}\end{aligned} \tag{128}$$

Upon inserting the first coefficients into the series solution one obtains

$$\begin{aligned}y(x) =& \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_2^{\ell}(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell}\\ =& 1 - \frac{47}{120}\cdot\frac{1}{2!}x^2 + \frac{1269}{4000}\cdot\frac{1}{4!}x^4 - \frac{282893}{672000}\cdot\frac{1}{6!}x^6 + \frac{4747}{5000}\cdot\frac{1}{8!}x^8\\ &- \frac{2379140611}{704000000}\cdot\frac{1}{10!}x^{10} + \cdots\\ =& 1 - 0.19583333x^2 + 0.01321875x^4 - 0.00058468x^6\\ &+ 0.000023547x^8 - 0.000000931x^{10} + \cdots . \quad (129)\end{aligned}$$

4.2.2. *The Special Case* $f(y(x)) = \cosh y(x)$

We consider the fractional Lane-Emden equation

$$\begin{aligned}&{}^cD^{\alpha}y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x) + \cosh y(x) = 0,\\ &y(0) = 1, y'(0) = 0.\end{aligned} \tag{130}$$

In order to solve the initial value problem (130), we consider the series expansion of $\cosh y(x)$:

$$\cosh y(x) = \sum_{m=0}^{N} a_{2m}y^{2m}(x) = \sum_{m=0}^{N} \frac{y^{2m}(x)}{(2m)!}, 0 \le m \le N, N \in \mathbb{N}_0, \tag{131}$$

and as a result we have the associated problem

$$\begin{aligned}&{}^cD^{\alpha}y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^{\beta}y(x) + \sum_{m=0}^{N} \frac{y^{2m}(x)}{(2m)!} = 0,\\ &y(0) = 1, y'(0) = 0.\end{aligned} \tag{132}$$

For $N = 0$, we observe that the resulting initial value problem coincides with the problem in (93).

Example 4.10. *Given the initial value problem*

$$y'' + \frac{2}{x}y' + 1 + \frac{y^2}{2!} = 0, \quad y(0) = 1, y'(0) = 0. \tag{133}$$

It follows immediately that the solution to the problem (133) *is given by*

$$\begin{aligned} y(x) =& 1 - \frac{1}{2}\cdot\frac{1}{2!}x^2 + \frac{3}{10}\cdot\frac{1}{4!}x^4 - \frac{3}{4}\cdot\frac{1}{6!}x^6 + \frac{7}{3}\cdot\frac{1}{8!}x^8 \\ & - \frac{2877}{220}\cdot\frac{1}{10!}x^{10} + \cdots \\ =& 1 - 0.25000000x^2 + 0.01250000x^4 - 0.00104167x^6 \\ & + 0.00005787x^8 - 0.00000360x^{10} + \cdots \end{aligned} \tag{134}$$

Example 4.11. *Consider the problem*

$$y'' + \frac{2}{x}y' + 1 + \frac{y^2}{2!} + \frac{y^4}{4!} = 0, \quad y(0) = 1, y'(0) = 0. \tag{135}$$

One easily sees that the solution to the problem (135) *gives*

$$\begin{aligned} y(x) =& 1 - \frac{37}{72}\cdot\frac{1}{2!}x^2 + \frac{481}{1440}\cdot\frac{1}{4!}x^4 - \frac{126503}{145152}\cdot\frac{1}{6!}x^6 \\ & + \frac{406445}{139968}\cdot\frac{1}{8!}x^8 - \frac{256054097}{14929920}\cdot\frac{1}{10!}x^{10}\cdots \\ =& 1 - 0.25694444x^2 + 0.01391782x^4 - 0.00121045x^6 \\ & + 0.00007202x^8 - 0.00000473x^{10} + \cdots. \end{aligned} \tag{136}$$

4.3. The Case of Exponential Functions $f(y) = \exp(\pm y)$

In this subsection, we compute the approximate solution of the fractional Lane-Emden problem (10) as well as the initial problem (34) for which the functions $f(y(x))$ are exponential functions. The Maclaurin series expansion for these exponential functions are applied.

4.3.1. Fractional Isothermal Gas Sphere Equation

The fractional isothermal gas sphere equation is given by

$$\begin{aligned} & {}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + e^{y(x)} = 0, \\ & y(0) = 0, y'(0) = 0. \end{aligned} \tag{137}$$

Also in this case, we look for the approximate solution of the problem (137) by solving the complementary equation

$$\begin{aligned} & {}^cD^\alpha y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^cD^\beta y(x) + \sum_{m=0}^{N}\frac{y^m(x)}{m!} = 0, \\ & y(0) = 0, y'(0) = 0. \end{aligned} \tag{138}$$

The problem

$$y'' + \frac{2}{x}y' + e^y = 0, \quad y(0) = 0, y'(0) = 0 \tag{139}$$

models the isothermal gas spheres where the temperature remains constant and the index m is infinite ([42]).

Example 4.12. *Given the problem*

$$y'' + \frac{2}{x}y' + 1 = 0, \quad y(0) = 0, y'(0) = 0. \tag{140}$$

It is straightforward to see by the initial conditions in (140) *that the expansion coefficients appearing in the series solution of* (140) *are given by*

$$\begin{aligned} & \mathscr{B}_0^0(2,1;2) = 0, \\ & \mathscr{B}_0^{\ell+1}(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,0}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell \geq 0, \end{aligned} \tag{141}$$

with $\mathsf{C}^2_{\ell,0} = \delta_{\ell 0}$. Thus one has the Mittag-Leffler expansion coefficient

$$\mathscr{B}_0^1(2,1;2) = \begin{cases} -\frac{1}{3}, & \ell = 0, \\ 0, & \ell = 1,2,\cdots. \end{cases} \tag{142}$$

Consequently we obtain the closed form solution

$$y(x) = \sum_{\ell=0}^{\infty}\frac{\mathscr{B}_0^\ell(2,1;2)}{\Gamma(2\ell+1)}x^{2\ell} = -\frac{1}{6}x^2. \tag{143}$$

Example 4.13. $N = 1$*: It is seen here that*

$$y'' + \frac{2}{x}y' + 1 + y = 0, \quad y(0) = 0, y'(0) = 0. \tag{144}$$

The coefficients of the series solution of (144) *has the formulation*

$$\begin{aligned} & \mathscr{B}_1^0(2,1;2) = 0, \\ & \mathscr{B}_1^{\ell+1}(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}^2_{\ell,1}}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell \geq 0 \end{aligned} \tag{145}$$

with

$$\mathsf{C}^2_{\ell,1} = \delta_{\ell 0} + \frac{b_1^\ell}{\Gamma(2\ell+1)}. \tag{146}$$

Further simplification gives the following first expansion coefficients $\mathscr{B}_1^\ell(2,1;2), \ell = 1,2,3,4,5.$

$$\begin{aligned} & \mathscr{B}_1^1(2,1;2) = -\mathsf{C}^2_{0,1}\cdot\frac{1}{3} = -(1+\mathscr{B}_1^0)\cdot\frac{1}{3} = -\frac{1}{3} \\ & \mathscr{B}_1^2(2,1;2) = -\mathsf{C}^2_{1,1}\cdot\frac{6}{5} = -\frac{\mathscr{B}_1^1}{2}\cdot\frac{6}{5} = \frac{1}{5} \\ & \mathscr{B}_1^3(2,1;2) = -\mathsf{C}^2_{2,1}\cdot\frac{5!}{7} = -\frac{\mathscr{B}_1^2}{24}\cdot\frac{120}{7} = -\frac{1}{7} \\ & \mathscr{B}_1^4(2,1;2) = -\mathsf{C}^2_{3,1}\cdot 560 = -\frac{\mathscr{B}_1^3}{720}\cdot 560 = \frac{1}{9} \\ & \mathscr{B}_1^5(2,1;2) = -\mathsf{C}^2_{4,1}\cdot\frac{362880}{11} = -\frac{\mathscr{B}_1^4}{40320}\cdot\frac{362880}{11} = -\frac{1}{11} \end{aligned} \tag{147}$$

Upon substituting we obtain the solution

$$\begin{aligned} y(x) &= \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_1^{\ell}(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell} \\ &= -\frac{1}{3}\cdot\frac{1}{2!}x^2 + \frac{1}{5}\cdot\frac{1}{4!}x^4 - \frac{1}{7}\cdot\frac{1}{6!}x^6 \\ &\quad + \frac{1}{9}\cdot\frac{1}{8!}x^8 - \frac{1}{11}\cdot\frac{1}{10!}x^{10} + \cdots \\ &= -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \frac{1}{9!}x^8 - \frac{1}{11!}x^{10} + \cdots \end{aligned} \tag{148}$$

Example 4.14. $N = 2$: *Consider the initial value problem*

$$y'' + \frac{2}{x}y' + 1 + y + \frac{y^2}{2!} = 0, \quad y(0) = 0, y'(0) = 0. \tag{149}$$

The series solution of (149) *has the following expansion coefficients.*

$$\begin{aligned} &\mathcal{B}_2^0(2,1;2) = 0, \\ &\mathcal{B}_2^{\ell+1}(2,1;2) = -\frac{\Gamma(2\ell+1)\Gamma(2\ell+2)\mathsf{C}_{\ell,2}^2}{\Gamma(2\ell+2)+2\Gamma(2\ell+1)}, \quad \ell \geq 0, \end{aligned} \tag{150}$$

where

$$\begin{aligned} \mathsf{C}_{\ell,2}^2 = \delta_{\ell 0} &+ \frac{b_2^{\ell}}{\Gamma(2\ell+1)} \\ &+ \frac{1}{2}\left(\sum_{p_1=0}^{\ell} \frac{b_2^{p_1} b_2^{\ell-p_1}}{\Gamma(2p_1+1)\Gamma(2\ell-2p_1+1)}\right). \end{aligned} \tag{151}$$

Simplifying further gives the following first expansion coefficients $\mathcal{B}_2^{\ell}(2,1;2)$, $\ell = 0,1,2,\ldots$.

$$\begin{aligned} &\mathcal{B}_2^1(2,1;2) = -\mathsf{C}_{0,2}^2\cdot\frac{1}{3} = -1\cdot\frac{1}{3} = -\frac{1}{3} \\ &\mathcal{B}_2^2(2,1;2) = -\mathsf{C}_{1,2}^2\cdot\frac{6}{5} = -\frac{\mathcal{B}_2^1}{2}\cdot\frac{6}{5} = \frac{1}{5} \\ &\mathcal{B}_2^3(2,1;2) = -\mathsf{C}_{2,2}^2\cdot\frac{5!}{7} = -\left(\frac{\mathcal{B}_2^2}{24} + \frac{\mathcal{B}_2^1\mathcal{B}_2^1}{8}\right)\cdot\frac{120}{7} = -\frac{8}{21} \\ &\mathcal{B}_2^4(2,1;2) = -\mathsf{C}_{3,2}^2\cdot 560 = -\left(\frac{\mathcal{B}_2^3}{720} + \frac{\mathcal{B}_2^1\mathcal{B}_2^2}{48}\right)\cdot 560 = \frac{29}{27} \\ &\mathcal{B}_2^5(2,1;2) = -\mathsf{C}_{4,2}^2\cdot\frac{362880}{11} \\ &= -\left(\frac{\mathcal{B}_2^4}{40320} + \frac{\mathcal{B}_2^1\mathcal{B}_2^3}{1440} + \frac{\mathcal{B}_2^2\mathcal{B}_2^2}{576}\right)\cdot\frac{362880}{11} = -\frac{40320}{8173}. \end{aligned} \tag{152}$$

As a result we obtain the solution

$$\begin{aligned} y(x) &= \sum_{\ell=0}^{\infty} \frac{\mathcal{B}_2^{\ell}(2,1;2)}{\Gamma(2\ell+1)} x^{2\ell} \\ &= -\frac{1}{3}\cdot\frac{1}{2!}x^2 + \frac{1}{5}\cdot\frac{1}{4!}x^4 - \frac{8}{21}\cdot\frac{1}{6!}x^6 + \frac{29}{27}\cdot\frac{1}{8!}x^8 - \frac{40320}{8173}\cdot\frac{1}{10!}x^{10} + \cdots \\ &= -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{8}{3\cdot 7!}x^6 + \frac{29}{3\cdot 9!}x^8 - \frac{40320}{743\cdot 11!}x^{10} + \cdots \end{aligned} \tag{153}$$

4.3.2. *Richardson's Model of Thermionic Current*

Here, we consider the initial value problem

$$\begin{aligned} &{}^{c}D^{\alpha}y(x) + \frac{\omega}{x^{\alpha-\beta}}\,{}^{c}D^{\beta}y(x) + e^{-y(x)} = 0, \\ &y(0) = 0, y'(0) = 0. \end{aligned} \tag{154}$$

The problem (Richardson's model of thermionic current)

$$y'' + \frac{2}{x}y' + e^{-y} = 0, \quad y(0) = 0, y'(0) = 0 \tag{155}$$

models the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium ([13], [42]).

Clearly, for $N = 0$ with $\alpha = 2, \beta = 1$ and $\omega = 2$, (154) coincides with (140).

Example 4.15. *Consider the problem*

$$y'' + \frac{2}{x}y' + 1 - y = 0, \quad y(0) = 0, y'(0) = 0. \tag{156}$$

As in the preceding calculations we easily see that the initial value problem (156) *has the solution*

$$\begin{aligned} y(x) &= -\frac{1}{3}\cdot\frac{1}{2!}x^2 - \frac{1}{5}\cdot\frac{1}{4!}x^4 - \frac{1}{7}\cdot\frac{1}{6!}x^6 - \frac{1}{9}\cdot\frac{1}{8!}x^8 - \frac{1}{11}\cdot\frac{1}{10!}x^{10} + \cdots \\ &= -\frac{1}{3!}x^2 - \frac{1}{5!}x^4 - \frac{1}{7!}x^6 - \frac{1}{9!}x^8 - \frac{1}{11!}x^{10} + \cdots. \end{aligned} \tag{157}$$

Example 4.16. $N = 2$: *Given the initial value problem*

$$y'' + \frac{2}{x}y' + 1 - y + \frac{y^2}{2!} = 0, \quad y(0) = 0, y'(0) = 0. \tag{158}$$

It is also easy to show that the solution to (158) *yields*

$$\begin{aligned} y(x) &= -\frac{1}{3}\cdot\frac{1}{2!}x^2 - \frac{1}{5}\cdot\frac{1}{4!}x^4 - \frac{8}{21}\cdot\frac{1}{6!}x^6 - \frac{29}{27}\cdot\frac{1}{8!}x^8 \\ &\quad - \frac{40320}{8173}\cdot\frac{1}{10!}x^{10} + \cdots \\ &= -\frac{1}{3!}x^2 - \frac{1}{5!}x^4 - \frac{8}{3\cdot 7!}x^6 - \frac{29}{3\cdot 9!}x^8 - \frac{40320}{743\cdot 11!}x^{10} + \cdots. \end{aligned} \tag{159}$$

5. Concluding Remarks

In this paper, we have used the Mittag-Leffler function expansion method to find approximate solutions of a class of fractional Lane-Emden equation whose nonlinear forms of $f(y)$ are expressible as $f(y(x)) = a_0 + a_1 y(x) + a_2 y^2(x) + \cdots + a_m y^m(x) + \cdots + a_N y^N(x), 0 \leq m \leq N, N \in \mathbb{N}_0$; the values of the expansion coefficients a_m, $0 \leq m \leq N$, were explicitly provided. We considered the cases for which the functions $f(y(x))$ are trigonometric, hyperbolic and exponential functions. In all these cases, the Lane-Emden equations for the special values $N = 0; \alpha = 2, \beta = 1, \omega = 2$, coincide with the classical and standard ones ((57), (97), (140)) and consequently the corresponding solutions are given in closed forms ((62), (101), (143)).

In the case where the functions $f(y)$ are exponential functions, our results can be compared with those of [11, equations

(59) and (70)] (see also [25, equation (25)]). The other nonlinear forms of $f(y)$ are the trigonometric and hyperbolic functions. Our approximate solutions in the case of trigonometric functions are comparable with the solutions [11, equations (81) and (83)]; and in the case of hyperbolic functions, see [11, equations (89) and (91)] (see also [22]).

The method employed in this paper can be applied to other similar cases in applied sciences where the models are given as strongly nonlinear ordinary differential equations. By extension, the method, can therefore, be accurately and reliably used to compute approximate solutions of nonlinear fractional differential equations of Lane-Emden type where the nonlinear forms of $f(y)$ involve several other special functions.

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References

[1] R. Garrappa, E. Kaslik & M. Popolizio, "Evaluation of fractional integrals and derivatives of elementary functions: overview and tutorial", Mathematics **7**, (2019) 407.

[2] G. Jumarie, "A Fokker-Planck equation of fractional order with respect to time", Journal of Math. Physics **33** (1992) 3536.

[3] G. Jumarie, "Fractional Fokker-Planck equation, solutions and applications", Physical Review, **63** (2001) 1.

[4] G. Jumarie, "Schrödinger equation for quantum-fractal space-time of order n via the complex-valued fractional Brownian motion", Intern. Y. of Modern Physics A **16** (2001) 5061.

[5] A. A. Kilbas, *Theory and applications of fractional differential equations*, Elsevier, (2006).

[6] F. Mainardi & R. Gorenflo, "Time-fractional derivatives in relaxation processes: A tutorial survey", Fractional Calculus and Applied Analysis **10** (2007) 269.

[7] M. D. Ortigueira, *Fractional calculus for scientists and engineers* Springer (2011).

[8] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999).

[9] M. Žecová & J. Terpák, "Heat conduction modeling by using fractional-order derivatives", Applied Mathematics and Computation **257** (2015) 365.

[10] K. Diethelm, *The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type*, Springer-Verlag Berlin Heidelberg (2010).

[11] A. M. Wazwaz, "A new algorithm for solving differential equations of Lane-Emden type", Appl. Math. Comput. **118** (2001) 287.

[12] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, (1967).

[13] O. U. Richardson, *The Emission of Electricity from Hot Bodies*, Longman, Green and Co., London, New York, (1921).

[14] C. Mohan & A. R. Al-Bayaty, "Power series solutions of the Lane-Emden equation", Astrophysics and Space Science **73** (1980) 227.

[15] J. I. Ramos, "Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method", Chaos, Solitons & Fractals **38** (2008) 400.

[16] S. K. Vanani & A. Aminataei, "On the numerical solutions of differential equations of Lane-Emden type", Computers and Mathematics with Applications **59** (2010) 2815.

[17] M. O. Ogunniran, O. A. Tayo, Y. Haruna & A. F. Adebisi, "Linear stability analysis of Runge-Kutta methods for singular Lane-Emden equations", Journal of the Nigerian Society of Physical Sciences **2** (2020) 134.

[18] E. O. Adeyefa & O. S. Esan, "Exponentially fitted Chebyshev based algorithm as second order initial value solver", Journal of the Nigerian Society of Physical Sciences **2** (2020) 51.

[19] S. E. Fadugba, S.N. Ogunyebi & B.O. Falodun, "An examination of a second order numerical method for solving initial value problems", Journal of the Nigerian Society of Physical Sciences **2** (2020) 120.

[20] M. S. Mechee & N. Senu, "Numerical study of fractional differential equations of Lane-Emden type by method of collocation", Applied Mathematics **3** (2012) 851.

[21] A. Akgül, M. Inc, E. Karatas & D. Baleanu, "Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique", Advances in Difference Equations **2015** (2015) 220.

[22] A. Saadatmandi, A. Ghasemi-Nasrabady & A. Eftekhari, "Numerical study of singular fractional Lane-Emden type equations arising in astrophysics", J. Astrophys. Astr. (2019) **40** 27.

[23] P. K. Sahu & B. Mallick, "Approximate solution of fractional order Lane–Emden type differential equation by orthonormal Bernoulli's polynomials", Int. J. Appl. Comput. Math (2019) **5** 89.

[24] M. I. Nouh & E. A.-B. Abdel-Salam, "Approximate Solution to the Fractional Lane–Emden Type Equations", Iran J. Sci. Tech. Trans. Sci. **42**, (2018) 2199.

[25] A. M. Malik & O.H. Mohammed, "Two efficient methods for solving fractional Lane–Emden equations with conformable fractional derivative", Journal of the Egyptian Mathematical Society (2020) 28.

[26] B. Căruntu, C. Bota, M. Lăpădat & M. S. Paşca, "Polynomial least squares method for fractional Lane–Emden equations", Symmetry **11** (2019) 479.

[27] J. Davila, L. Dupaigne & J. Wei, "On the fractional Lane-Emden equation", Trans. Am. Math. Soc. **369** (2017) 6087.

[28] C. Milici, G. Drăgănescu & J.T. Machado, "Introduction to fractional differential equations", Nonlinear Systems and Complexity, **25** (2019).

[29] U. Saeed, "Haar Adomian method for the solution of fractional nonlinear Lane-Emden type equations arising in astrophysics", Taiwanese Journal of Mathematics **21** (2017) 1175.

[30] O. A. Uwaheren, A. F. Adebisi & O. A. Taiwo, "Perturbed collocation method for solving singular multi-order fractional differential equations of Lane-Emden type", Journal of the Nigerian Society of Physical Sciences **2** (2020) 141.

[31] A. A. M. Arafa & S. Z. Rida, A. A. Mohammadein, H. M. Ali, "Solving nonlinear fractional differential equation by generalized Mittag-Leffler function method", Communications in Theoretical Physics **59** (2013) 661.

[32] A. Atangana & A. Secer, "A note on fractional order derivatives and table of fractional derivatives of some special functions", Abstract and Applied Analysis **2013** (2013) 1.

[33] M. Caputo & M. Fabrizio, "A new definition of fractional derivative without singular kernel", Progr. Fract. Differ. Appl. **1** (2015) 73.

[34] S. Das, "Functional Fractional Calculus", Springer (2011).

[35] M. Davison & C. Essex, "Fractional differential equations and

initial value problems", The Mathematical Scientist **23** (1998) 108.

[36] S. E. Fadugba, "Solution of fractional order equations in the domain of the Mellin transform", Journal of the Nigerian Society of Physical Sciences **1** (2019) 138.

[37] R. Herrmann, Fractional calculus, an introduction for Phycists, World Scientific, (2011).

[38] J. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results", Computers and Mathematics with Applications **51** (2006) 1367.

[39] J. Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions", Applied Mathematics Letters **22** (2009) 378.

[40] E. C. de Oliveira & J. A. T. Machado, "A review of definitions for fractional derivatives and integral", Mathematical Problems in Engineering **2014** (2014) 1.

[41] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity*, Imperial College Press (2010).

[42] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, (1962).