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A Higher-order Block Method for Numerical Approximation of Third-order Boundary Value Problems in ODEs

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Abstract

In recent times, numerical approximation of 3rd-order boundary value problems (BVPs) has attracted great attention due to its wide applications in solving problems arising from sciences and engineering. Hence, A higher-order block method is constructed for the direct solution of 3rd-order linear and non-linear BVPs. The approach of interpolation and collocation is adopted in the derivation. Power series approximate solution is interpolated at the points required to suitably handle both linear and non-linear third-order BVPs while the collocation was done at all the multi-derivative points. The three sets of discrete schemes together with their first, and second derivatives formed the required higher-order block method (HBM) which is applied to standard third-order BVPs. The HBM is self-starting since it doesn't need any separate predictor or starting values. The investigation of the convergence analysis of the HBM is completely examined and discussed. The improving tactics are fully considered and discussed which resulted in better performance of the HBM. Three numerical examples were presented to show the performance and the strength of the HBM over other numerical methods. The comparison of the HBM errors and other existing work in the literature was also shown in curves.

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1. Introduction

Numerous common happenings in connection with physical sciences, and engineering are modeled in form of linear and nonlinear BVPs. Although, some modeled problems do not have theoretical solution or closed form. Consequently, numerical method is employed to solve such class of modeled problems. In this article, the numerical solution of third-order BVPs of the type

$$y'''(x) = v(x, y, y', y''), \tag{1}$$

with the initial conditions, boundary conditions or any other form as follow,

$$y(a) = \alpha_a, \quad y'(a) = \beta_a, \quad y'(x_b) = v_b, \tag{2}$$

$$y(x_a) = \alpha_0, \quad y'(a) = \beta_a, \quad y(b) = v_b, \tag{3}$$

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$$y(x_a) = \alpha_a, \quad y'(b) = \beta_b, \quad y(b) = v_b, \tag{4}$$

The constants parameters in equations (2) - (4) are taken to be continuous functions and v fulfils the condition for the existence and uniqueness of the problem. It follows that (1) is a third-order ordinary differential equations with initial, and boundary conditions (3) - (4). Modeled equation (1) is of great importance to scientists and engineers due to its numerous usage in sciences and engineering. Scholars have developed numerous techniques for solving (1). The conventional methods of solving (1) could be; by reducing it to the system of first-order ordinary differential equations and a suitable numerical methods for first-order ODEs would be applied to solve the system of equations. The shooting method, and finite different method. The limitations of these methods have been discussed by numerous scholars [1–4]. For instance, The reduction approach have been reported by various literatures to have alot of limitations such computational burden, lots of human efforts, requires lots of time for the computation, and complexity in the computer computation which affects the accuracy and efficiency of the method in terms of error and time of execution. On the other hand, the shooting method suffers inaccuracy and instability while the finite different method is very demanding and does not give a satisfactory results. Other numerical methods for solving ordinary differential equations also exist in literature [5–15].

In other to improve the limitations and the weakness associated with the conventional methods, we present a higher-order block method capable of solving (1) directly, accurately and reduces computational time. The new method namely HBM is expected to improve the accuracies of the existing methods in the literature.

2. Methodology

In this section, a new method capable of handing (1) is constructed. We considered an approximate equation as the power series of the form,

$$y(x) = \sum_{z=0}^{3k+1} m_z x^z \tag{5}$$

where the step-number ($k = 3$) is taken into consideration, 3k+1 is equivalent to $r + 2s - 1$, where r is the interpolation points and s is the collocation points. The third and fourth derivative of (5) yields

$$\begin{cases} y'''(x) = \displaystyle\sum_{z=3}^{3k+1} z(z-1)(z-2)m_z x^{z-3} \\ y^{(iv)}(x) = \displaystyle\sum_{z=4}^{3k+1} z(z-1)(z-2)(z-3)m_z x^{z-4} \end{cases} \tag{6}$$

Now, interpolating (5) at $x_{n+r}, r = 0(1)k-1$ and collocating both the third and fourth derivatives in (6) at $x_{n+s}, s = 0(1)k$ to generate the system of ten by ten equations written in matrix form as

$$X_1 A_1 = B_1 \tag{7}$$

$$X_1 = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & \cdots & x_n^9 & x_n^{10} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & \cdots & 9x_n^8 & 10x_n^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & \cdots & 72x_n^7 & 90x_n^8 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & \cdots & 504x_n^6 & 720x_n^7 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & \cdots & 504x_{n+1}^6 & 720x_{n+1}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & \cdots & 504x_{n+2}^6 & 720x_{n+2}^7 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & \cdots & 504x_{n+3}^6 & 720x_{n+3}^7 \\ 0 & 0 & 0 & 0 & 24 & 120x_n & \cdots & 3024x_n^5 & 5040x_n^6 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & \cdots & 3024x_{n+1}^5 & 5040x_{n+1}^6 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+2} & \cdots & 3024x_{n+2}^5 & 5040x_{n+2}^6 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+3} & \cdots & 3024x_{n+3}^5 & 5040x_{n+3}^6 \end{pmatrix}$$

$B_1 = (y_n, y_{n+1}, y_{n+2}, v_n, v_{n+1}, v_{n+2}, v_{n+3}, w_n, w_{n+1}, w_{n+2}, w_{n+3})^T$
$A_1 = (m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10})^T$

Solving the system of equations above, we obtained the values of the coefficients $m_n, n = 0, ..., 10$ as follows.

After obtaining the values of these coefficients and changing the variable, $x = xn + lh$ using the appropriate transformation, the polynomial in (5) may be written as,

$$\begin{aligned} y(l) &= a_0 y_n + a_1 y_{n+1} + a_2 y_{n+2} + h^3 (b_0 v_n + b_1 v_{n+1} + b_2 v_{n+2} + b_3 v_{n+3}) \\ &\quad + h^4 (c_0 w_n + c_1 w_{n+1} + c_2 w_{n+2} + c_3 w_{n+3}) \end{aligned} \tag{8}$$

where

$$\begin{cases} a_0 = \dfrac{de}{2}, a_1 = -el, a_2 = \dfrac{dl}{2}, d = (l-1), e = (l-2) \\ b_0 = \dfrac{h^2}{544320}\, de(77\,l^7 - 1059\,l^6 + 5699\,l^5 - 14625\,l^4 + 15749\,l^3 + 3165\,l^2 - 22003\,l + 18381) \\ b_1 = \dfrac{lh^3}{20160}\, de(7\,l^7 - 79\,l^6 + 304\,l^5 - 370\,l^4 - 206\,l^3 + 122\,l^2 + 778\,l + 2090) \\ b_2 = -\dfrac{lh^3}{20160}\, de(7\,l^7 - 89\,l^6 + 409\,l^5 - 755\,l^4 + 319\,l^3 + 199\,l^2 - 41\,l - 521) \\ b_3 = -\dfrac{lh^3}{544320}\, de(77\,l^7 - 789\,l^6 + 2864\,l^5 - 4230\,l^4 + 1574\,l^3 + 1086\,l^2 + 110\,l - 1842) \\ c_0 = \dfrac{lh^4}{181440}\, de(7\,l^7 - 99\,l^6 + 559\,l^5 - 1581\,l^4 + 2245\,l^3 - 1191\,l^2 - 503\,l + 873) \\ c_1 = \dfrac{lh^4}{20160}\, de(7\,l^7 - 89\,l^6 + 424\,l^5 - 878\,l^4 + 550\,l^3 + 382\,l^2 + 46\,l - 626) \\ c_2 = \dfrac{lh^4}{20160}\, de(7\,l^7 - 79\,l^6 + 319\,l^5 - 517\,l^4 + 205\,l^3 + 137\,l^2 + l - 271) \\ c_3 = \dfrac{lh^4}{181440}\, de(7\,l^7 - 69\,l^6 + 244\,l^5 - 354\,l^4 + 130\,l^3 + 90\,l^2 + 10\,l - 150) \end{cases} \tag{9}$$

Remark 1. *The variable coefficients functions in* (8) *are continuous and differentiable within the interval of solution of* $[a, b]$ *with a step size given by the function* $h = \frac{b-a}{N}$. *It follows that N is the number of sub-interval of the solution. The continuous function* (8) *and its first* $y'(x)$ *and second derivatives* $y''(x)$ *were used to generate the main and auxiliary methods which produces a sum of nine equations jointed together to supply the entire approximations on the interval for the direct solution of third-order BVPs of the type* (1). *Furthermore, by evaluating* (8)*, its first and second derivatives at* $x_{n+l}, l = 0(1)k$*, The following nine equations or otherwise called HBM were acquired.*

$$\begin{aligned} y_{n+1} &= y_n + y_n' h + \frac{1}{2} y_n'' h^2 + \frac{62387}{544320} h^3 v_n + \frac{89}{3360} h^3 v_{n+1} + \frac{439}{20160} h^3 v_{n+2} + \\ &\quad \frac{1031}{272160} h^3 v_{n+3} + \frac{1879}{181440} h^4 w_n - \frac{359}{10080} h^4 w_{n+1} - \frac{13}{960} h^4 w_{n+2} - \frac{17}{18144} h^4 w_{n+3} \\ y_{n+2} &= y_n + 2 y_n' h + 2 y_n'' h^2 + \frac{5048}{8505} h^3 v_n + \frac{164}{315} h^3 v_{n+1} + \frac{4}{21} h^3 v_{n+2} + \frac{244}{8505} h^3 v_{n+3} \\ &\quad + \frac{172}{2835} h^4 w_n - \frac{4}{15} h^4 w_{n+1} - \frac{34}{315} h^4 w_{n+2} - \frac{4}{567} h^4 w_{n+3} \\ y_{n+3} &= y_n + 3 y_n' h + \frac{9}{2} y_n'' h^2 + \frac{657}{448} h^3 v_n + \frac{2187}{1120} h^3 v_{n+1} + \frac{2187}{2240} h^3 v_{n+2} + \frac{117}{1120} h^3 v_{n+3} \\ &\quad + \frac{351}{2240} h^4 w_n - \frac{729}{1120} h^4 w_{n+1} - \frac{729}{2240} h^4 w_{n+2} - \frac{27}{1120} h^4 w_{n+3} \end{aligned} \tag{10}$$

$$\begin{aligned} y_{n+1}' &= y_n' + h y_n'' + \frac{19519}{68040} h^2 v_n + \frac{1301}{10080} h^2 v_{n+1} + \frac{181}{2520} h^2 v_{n+2} + \\ &\quad \frac{3329}{272160} h^2 v_{n+3} + \frac{371}{12960} h^3 w_n - \frac{313}{2520} h^3 w_{n+1} - \frac{89}{2016} h^3 w_{n+2} - \frac{137}{45360} h^3 w_{n+3} \\ y_{n+2}' &= y_n' + 2h y_n'' + \frac{5731}{8505} h^2 v_n + \frac{296}{315} h^2 v_{n+1} + \frac{109}{315} h^2 v_{n+2} + \frac{344}{8505} h^2 v_{n+3} \\ &\quad + \frac{206}{2835} h^3 w_n - \frac{20}{63} h^3 w_{n+1} - \frac{52}{315} h^3 w_{n+2} - \frac{4}{405} h^3 w_{n+3} \end{aligned} \tag{11}$$

$$
\begin{aligned}
y'_{n+3} &= y'_n + 3h\,y''_n + \frac{603}{560}h^2 v_n + \frac{2187}{1120}h^2 v_{n+1} + \frac{729}{560}h^2 v_{n+2} + \frac{27}{160}h^2 v_{n+3} \\
&\quad + \frac{27}{224}h^3 w_n - \frac{243}{560}h^3 w_{n+1} - \frac{243}{1120}h^3 w_{n+2} - \frac{9}{280}h^3 w_{n+3} \\
y''_{n+1} &= y''_n + \frac{6893}{18144}hv_n + \frac{313}{672}hv_{n+1} + \frac{89}{672}hv_{n+2} + \frac{397}{18144}hv_{n+3} \\
&\quad + \frac{1283}{30240}h^2 w_n - \frac{851}{3360}h^2 w_{n+1} - \frac{269}{3360}h^2 w_{n+2} - \frac{163}{30240}h^2 w_{n+3} \\
y''_{n+2} &= y''_n + \frac{223}{567}hv_n + \frac{20}{21}hv_{n+1} + \frac{13}{21}hv_{n+2} + \frac{20}{567}hv_{n+3} \\
&\quad + \frac{43}{945}h^2 w_n - \frac{16}{105}h^2 w_{n+1} - \frac{19}{105}h^2 w_{n+2} - \frac{8}{945}h^2 w_{n+3} \qquad (12) \\
y''_{n+3} &= y''_n + \frac{93}{224}hv_n + \frac{243}{224}hv_{n+1} + \frac{243}{224}hv_{n+2} + \frac{93}{224}hv_{n+3} \\
&\quad \frac{57}{1120}h^2 w_n - \frac{81}{1120}h^2 w_{n+1} + \frac{81}{1120}h^2 w_{n+2} - \frac{57}{1120}h^2 w_{n+3}
\end{aligned}
$$

3. The Properties of the HBM

In this section, It is important to examine and discuss the convergence analysis of the HBM such as the order & the error constants, convergence, zero stability, and convergence.

3.1. Order & Error Constant of the HBM

According to [16–18], the linear difference operator L in respect to equation (10) is defined by

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left\{a_j y(x_n + jh) - h^3 v_j y'''(x_n + jh) - h^4 w_j y''''(x_n + jh)\right\} \tag{13}$$

$y(x)$ is assumed to be continuously differentiable function. Therefore function (13) can be maximize in taylor series about x to obtain

$$L(y(x);h) = D_0 y(x) + D_1 h y'(x) + D_2 h^2 y''(x) + ... + D_q h^q y^{(q)}(x) \tag{14}$$

where $D_q, q = 1, 2, ...$ are constants in such that,

$$D_0 = D_1 = ... = D_q = 0, D_{p+3} \neq 0 \tag{15}$$

The method (10) is of uniform order 11 with the following error constants
$D_{q+3} = (-5.143239 \times 10^{-10}, -1.312531 \times 10^{-09}, -2.394622 \times 10^{-09})^T$.

3.2. Consistency

As stated by [18–20], a LMM of the form (10) is said to be consistent if it has order greater than or equals to one. The HBM satisfies the condition for consistency since its order, is 11 which is greater than one.

3.3. Zero-stability of the HBM

Taking into consideration method (10) could be written in matrix difference form as ,

$$
\begin{aligned}
A^{(0)}Y_m &= A^{(1)}Y_{m-1} + h^3\left[C^{(0)}G_m + C^{(3)}G_{m-3}\right] + \\
&\quad h^4\left[D^{(0)}H_m + D^{(4)}H_{m-3}\right] \qquad (16)
\end{aligned}
$$

The matrix parameter $A^{(0)}, A^{(1)}, C^{(0)}, C^{(3)}, D^{(0)}, D^{(4)}, H^{(0)}, H^{(1)}$ are the square matrices whose arrays are the coefficients (10) and are defined as below.

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C^{(0)} = \begin{pmatrix} \frac{89}{3360} & \frac{439}{20160} & \frac{1031}{272160} \\ \frac{164}{315} & \frac{4}{21} & \frac{244}{8505} \\ \frac{2187}{1120} & \frac{2187}{2240} & \frac{117}{1120} \end{pmatrix}$$

$$C^{(3)} = \begin{pmatrix} 0 & 0 & \frac{62387}{544320} \\ 0 & 0 & \frac{5048}{8505} \\ 0 & 0 & \frac{657}{448} \end{pmatrix},$$

$$D^{(0)} = \begin{pmatrix} -\frac{359}{10080} & -\frac{13}{960} & -\frac{17}{18144} \\ -\frac{4}{15} & -\frac{34}{315} & -\frac{4}{567} \\ -\frac{729}{1120} & -\frac{729}{2240} & -\frac{27}{1120} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 0 & \frac{1879}{181440} \\ 0 & 0 & \frac{172}{2835} \\ 0 & 0 & \frac{351}{2240} \end{pmatrix},$$

The limit of (16) is taken as $h \to 0$, to obtain the difference system

$$A^{(0)} Y_m - A^{(1)} Y_{m-1} = 0 \tag{17}$$

The first characteristics of (17) is given by

$$\rho(F) = \det\left(F A^{(0)} - A^{(1)}\right) = F^2 (F-1) = 0 \tag{18}$$

Hence, $F = 0, 0, 1$

The block method of the form (10) is said to be zero stable if as $\rho(F) = 0$, then $\left|F_j\right| \leq 1, \; j = 0, 1, ...$ for those roots with $\left|F_j\right| = 1$, the multiplicity does not exceed 1 [21–26]. Also the block method (10) is consistent since p>1. Since (10) is consistent and zero stable, it is also convergence [27].

3.4. Convergence

In respect to the claim of Lambert [18] which also corroborates with proof of Adogbe & Omole, and Adogbe *et al.* [28, 29] that any numerical method belonging to a class of LMM must satisfies the fundamental and adequate conditions. It follows that for such class of method to be convergent it must be consistent and zero-stable. consequently, the HBM satisfies the conditions for consistency and zero-stability, So it is convergent.

4. Implementation Tactics

In this part, we present the comprehensive procedure for the implementation of the new proposed method tagged Higher-order Block Method (HBM). The HBM is implemented in block method together with the aid of Newton-Raphson approach via a Mathematica 11.0 code which uses f-solve for linear and find-root for non-linear to simultaneously generate the solution at the initial point to the terminal point while adjusting for boundary conditions.

Meanwhile, each block integrators in (10), (11) and (12) forms a system of equations which is applied along with the Newton's method. The starting values in the application of the Newton's Raphson method which are considered as the approximations provided by the Taylor series expansion formulas

$$\begin{aligned} y_{n+i} &= y_n + ihy'_n + \left(\frac{(ih)^2}{2}\right) y''_n + \left(\frac{(ih)^3}{6}\right) v_n + \left(\frac{(ih)^4}{24}\right) w_n, \\ & i = 0\,(1)\,..., k \\ y'_{\text{n+i}} &= y'_n + ihy''_n + \left(\frac{(ih)^2}{2}\right) v_n + \left(\frac{(ih)^3}{6}\right) w_n, \\ & i = 0\,(1)\,..., k \\ y''_{\text{n+i}} &= y''_n + ihv_n + \left(\frac{(ih)^2}{2}\right) w_n, \\ & i = 0\,(1)\,..., k. \end{aligned} \tag{19}$$

The $w_{n+i}, i = 0\,(1)\,..., k$ appearing in (19) connotes the fourth derivative at $x_{n+i}, i = 0\,(1)\,..., k$. In other to get a closed form solution of (20) which is expanded in (19), it is important to calculate the values of $w_{n+i}, i = 0\,(1)\,..., k$. For more clarification,

$$y^{(iv)}(x) = \left(\frac{dv(x, y, y', y'', y''')}{dx}\right), i = 0\,(1)\,..., k. \tag{20}$$

$$y^{(iv)}(x) = \frac{dv}{dx} + \frac{dv}{dy} y' + \frac{dv}{dy'} y'' + \frac{dv}{dy''} y''' + v\frac{dv}{dy'''} v, i = 0\,(1)\,..., k. \tag{21}$$

Consequently, the solution of (1) is simultaneously generate on the entire interval of integration by using the main method (10) for $n = 0, 1, ..., N_2$ to obtain $3N-2$ equations and the additional method (11) and (12) which are employed to complete the set of equations needed to simultaneously solve 3N by $3N$ system of equations for solving (1) directly.

5. The Numerical Experiments

It is very important to test the accuracy and usefulness of the HBM by applying the HBM to solve modeled problems. Three standard third-order BVPs varying from linear to non-linear was computed and the results were presented and discussed extensively. All computations and programming were carried out using Maple 18.0 and Mathematica 11.0.

5.1. Test problem 1

First test problem, third-order linear boundary value problem solved by Ahmed [30].

$$y''' - xy + (x^2 - 2x^2 - 5x - 3)e^x, y(0) = 0, y'(0) = 1, y'(1) = -e \tag{22}$$

with theoretical solution as

$$y(x) = x(1-x)e^x \tag{23}$$

Table 1. Numerical results of HBM, AE in HBM and AE in [30] for Problem 1 using N = 10 or h = 0.1

x	y-Exact solution	y-Computed solution	AE in HBM	AE in [30]
0.1	0.09946538262680829	0.09946538262680911	8.18789×10^{-16}	1.36200×10^{-10}
0.2	0.19542444130562720	0.19542444130563258	5.38458×10^{-15}	1.90000×10^{-12}
0.3	0.28347034959096070	0.28347034959097905	1.83742×10^{-14}	1.10000×10^{-12}
0.4	0.35803792743390490	0.35803792743392630	2.14273×10^{-14}	7.00000×10^{-12}
0.5	0.41218031767503205	0.41218031767503670	4.66294×10^{-15}	1.00000×10^{-11}
0.6	0.43730851209372210	0.43730851209368804	3.40838×10^{-14}	1.70000×10^{-12}
0.7	0.42288806856880007	0.42288806856871670	8.33777×10^{-14}	8.80000×10^{-11}
0.8	0.35608654855879485	0.35608654855866470	1.30174×10^{-13}	9.42000×10^{-11}
0.9	0.22136428000412547	0.22136428000395672	1.68754×10^{-13}	1.36800×10^{-10}
1.0	0.00000000000000000	1.83070×10^{-13}	1.83070×10^{-13}	0.00000×10^{-00}

Table 2. Comparison of Maximum absolute error in HBM with other existing methods for Problem 1

References	Maximum Absolute Error
Current method-HBM	1.8307×10^{-13}
[30]	1.3700×10^{-10}
[31]	5.3000×10^{-07}
[32]	1.8400×10^{-06}
[33]	2.3700×10^{-07}
[34]	8.1200×10^{-04}
[35]	1.6400×10^{-02}
[36]	2.6400×10^{-07}
[37]	8.2900×10^{-09}

Table 3. Numerical results of HBM and Absolute error for Problem 2 taking N = 100 or h = 0.01

x	y-Exact solution	y-Computed solution	AE in HBM
0.01	−0.009998833350833248	−0.009998833350833078	1.70003×10^{-16}
0.02	−0.019990667226655746	−0.019990667226655066	6.80012×10^{-16}
0.03	−0.029968504252313413	−0.029968504252311883	1.53003×10^{-15}
0.04	−0.039925351251935550	−0.039925351251932820	2.72699×10^{-15}
0.05	−0.049854221347501636	−0.049854221347497375	4.26048×10^{-15}
0.06	−0.059748136056118590	−0.059748136056112450	6.14092×10^{-15}
0.07	−0.069600127385578860	−0.069600127385570460	8.39606×10^{-15}
0.08	−0.079403239927770000	−0.079403239927759020	1.09773×10^{-14}
0.09	−0.089150532949507160	−0.089150532949493310	1.38500×10^{-14}
0.10	−0.098835082480359870	−0.098835082480342770	1.70974×10^{-14}

Table 4. Numerical results of HBM and Absolute error for Problem 2 taking N = 10 or h = 0.1

x	y-Exact solution	y-Computed solution	AE in HBM
0.1	−0.098835082480359870	−0.09883508248034277	1.70974×10^{-14}
0.2	−0.190722557563258760	−0.19072255756318998	6.87783×10^{-14}
0.3	−0.268923388061819000	−0.26892338806166494	1.54043×10^{-13}
0.4	−0.327111407539266430	−0.32711140753900390	2.62512×10^{-13}
0.5	−0.359569153953152255	−0.35956915395277234	3.79918×10^{-13}
0.6	−0.361371182972822670	−0.36137118297233617	4.86500×10^{-13}
0.7	−0.328551020491222400	−0.32855102049065060	5.71820×10^{-13}
0.8	−0.258248192723828200	−0.25824819272318880	6.39433×10^{-13}
0.9	−0.148832112829221850	−0.14883211282853925	6.82593×10^{-13}
1.0	0.000000000000000000	6.98759×10^{-13}	6.98759×10^{-13}

In Table 1, The theoretical solution, approximate solution, absolute error in HBM and the absolute error in other existing method were presented. On the other hand, Table 2 shows the comparison of Maximum absolute error in HBM against nu-

Table 5. Comparison of Maximum absolute error in HBM with other existing methods for Problem 2

References	Maximum Absolute Error
Current method-HBM	6.98759×10^{-13}
[34]	8.55940×10^{-05}
[35]	8.88390×10^{-03}
[36]	2.15720×10^{-08}

Table 6. Numerical results of HBM and Absolute error Problem 3 with using (N = 10) or (h = 0.1)

x	y-Exact solution	y-Computed solution	AE in HBM
0.1	0.09531017980432493	0.09531017980433536	1.04222×10^{-14}
0.2	0.18232155679395460	0.18232155679398550	3.09197×10^{-14}
0.3	0.26236426446749106	0.26236426446754463	5.35683×10^{-14}
0.4	0.33647223662121290	0.33647223662128860	7.57172×10^{-14}
0.5	0.40546510810816440	0.40546510810826103	9.66449×10^{-14}
0.6	0.47000362924573563	0.47000362924585110	1.15463×10^{-13}
0.7	0.53062825106217040	0.53062825106230150	1.31117×10^{-13}
0.8	0.58778666490211910	0.58778666490226180	1.42775×10^{-13}
0.9	0.64185388617239470	0.64185388617254560	1.50879×10^{-13}
1.0	0.69314718055994530	0.69314718056009890	1.53655×10^{-13}

Table 7. Comparison of the numerical errors in HBM and other existing methods for Problem 3 with N = 10

x	AE in HBM	AE in [37]	AE in [38]
0.1	1.04222×10^{-14}	2.22000×10^{-06}	2.947641×10^{-10}
0.2	3.09197×10^{-14}	4.67000×10^{-06}	2.084566×10^{-11}
0.3	5.35683×10^{-14}	1.890000×10^{-06}	4.147355×10^{-11}
0.4	7.57172×10^{-14}	3.560000×10^{-06}	2.208859×10^{-12}
0.5	9.66449×10^{-14}	7.430000×10^{-07}	2.777215×10^{-11}
0.6	1.15463×10^{-13}	1.220000×10^{-06}	2.647594×10^{-11}
0.7	1.31117×10^{-13}	2.780000×10^{-06}	1.190750×10^{-11}
0.8	1.42775×10^{-13}	3.74000×10^{-06}	1.816331×10^{-12}
0.9	1.50879×10^{-13}	4.11000×10^{-06}	7.666364×10^{-10}
1.0	1.53655×10^{-13}	0.00000×10^{-00}	0.00000×10^{-00}

merous methods proposed by various authors in the literatures. It is very clear that the efficiency and accuracy of HBM is established comprehensively.

5.2. *Test problem 2*

Consider the third-order boundary value problem solved by Abdullah *et al*. [34].

$$y''' - y + (7 - x^2)cosx + (x^2 - 6x - 1)sinx, \quad y(0) = 0, y'(0) = -1, y'(1) = 2sin1 \tag{24}$$

with the analytical solution given as

$$y(x) = (x^2 - 1)sin(x) \tag{25}$$

Here, the interpretation of Tables 3, 4 and 5 are made. The computational results of problem 2 using the proposed method named HBM with $h = 0.01$ is shown, while in Table 2, we presents the computational results of problem 2 using HBM with $h = 0.1$. It could be observed that as the value of h decreases, the accuracies also increases whereas as the h increases, the accuracies of the method decreases as demonstrated in Tables 3 and 4. furthermore, we computed the maximum absolute error of the HBM for problem 2 and compared with other numerical methods in the cited literature. This is illustrated in the Table 5. The HBM gives a minimal error when compared with other existing method.

5.3. *Test problem 3*

Lastly, a non-linear third-order problem studied by Akram *et al.* [37] and, Hossain *et al.* [38] is taking into consideration.

$$y''' + 2e^{3y} = -4(1 + x)^{-3}, \quad y(0) = 0, y'(0) = 1, y'(1) = In(2) \tag{26}$$

with the exact solution.

$$y(x) = In(1 + x) \tag{27}$$

Finally, we considered a non-linear third-order bvp in other to determine the strength and advantage of the HBM. In Table 6,

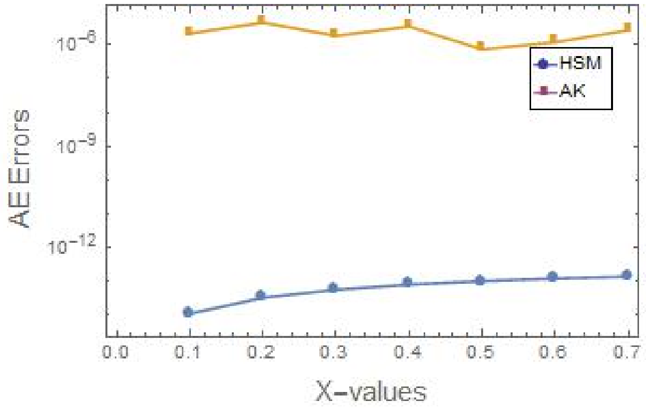

Figure 1. Comparison of Absolute error in HBM with Absolute error in *Akram et al.* [37]

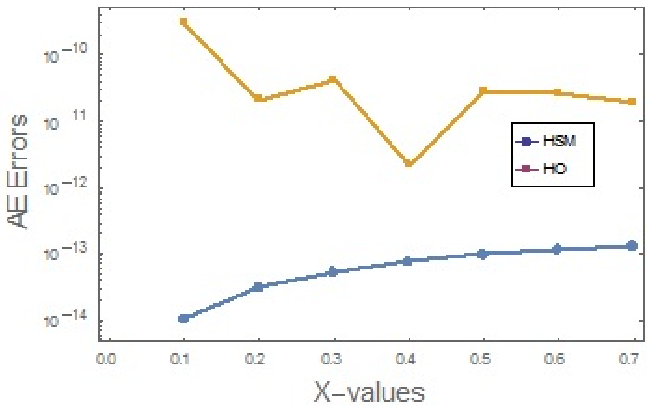

Figure 2. Comparison of Absolute error in HBM with Absolute error in *Hossain et al.* [38]

The computational results and absolute error on HBM is presented while Table 7 presented the comparison of absolute error of the non-linear problem using HBM and compared with other similar methods in the cited work. In addition, the comparison of errors in curves for the non-linear problem were also presented in Figure 1 and 2. without any iota of doubt, we have been able to demonstrate the convergence, efficiency and accuracy of the new method namely HBM over other techniques.

6. Conclusion

This study has successfully presented a construction of new numerical method (HBM), analyse and implemented for solving numerous type of third-order BVPs directly without utilizing the conventional method such as shooting method, reduction to first-order system of equations, finite difference method. In the derivation, the method make use of power series basic function due to its great stability and convergence attributes. The multi-collocation points was introduced for the purpose of improving the order of the work, which also enhances good accuracy, the points of interpolation and collocation were also chosen strategically in order to obtain a desired results. The method was applied to solve linear and non-linear boundary value problems in other to test its efficiency and accuracy. The results was also compared with numerous methods in the literatures as shown in Tables 1, 3, 4 and 6. The results shows that the HBM is more efficient and accurate than other methods in the literature. The method is of order 11 with a smaller error terms. The discussion of HBM were discussed in details in chapter three. both the absolute error and maximum absolute error were also tested in other to ascertain the uniqueness and reliability of the HBM. Hence we conclude that HBM is a best candidate in solving a class of such problems.

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