Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 4 (2022) 797

Journal of the Nigerian Society of Physical Sciences

A One-Step Block Hybrid Integrator for Solving Fifth Order Korteweg-de Vries Equations

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Abstract

Fifth-order Korteweg-de Vries (KdV) equations, arise in modeling waves phenomena such as the propagation of shallow water waves over a flat surface, gravity-capillary waves and sound waves in plasmas. In this work, a one-step block hybrid linear multistep method was derived using the collocation technique, to solve fifth-order KdV models via the Method of Line (MoL). The consistency, stability and convergence of the method were established. The efficiency of the method can be seen from comparison of the exact solutions of problems and other methods cited from literature.

DOI:10.46481/jnsps.2022.797

Keywords: Korteweg-de Vries (KdV) equations, Fifth-order PDE, Linear multistep, Block Method, Convergence

Article History :
Received: 03 May 2022
Received in revised form: 12 July 2022
Accepted for publication: 25 July 2022
Published: 19 August 2022

Communicated by: T. Latunde

1. Introduction

In this work, the numerical solution of fifth order PDE of the form;

$$\begin{cases} a(x)y_{tt} + b(x)y_{xxxxx} = G(x, t, y, y^2, y_x, y_t, y_{xx}, y_{xxx}), \\ y(x, b) = g_1(x), \quad y(a, t) = g_3(t) \end{cases} \tag{1}$$

defined on the domain $\Omega = \{(x, t) : a < x < b, \quad c < t < d\}$, and the boundary $\partial\Omega$. where a, b, c, d are real numbers.

Formulation of models into PDEs of the type in equation (1) arise in sciences, for example, fifth-order Korteweg-de Vries (KdV) type equations, models different wave phenomena which includes the propagation of shallow water waves over a flat surface, gravity-capillary waves and sound wave propagation in plasmas see [1, 2, 3]. Prominent examples includes the "good" Boussinesq equation, the Kaup-Kupershmidt equation and an extended KdV5 equation, [4]. These equations model phenomenon like laser optics, nonlinear dispersive waves in a wide range of applications, water waves, and plasma: generally, these are PDEs with higher order spatial derivatives, [4, 5, 6]. Some of the notable general (linear and nonlinear) fifth order KdV equations include though not limited to: the Kawahara equation [4]; the Lax equation [7]; the Caudrey-Dodd-Gibbon (C-D-G) equation [8]; the fifth-order KdV equation [9]; the Sawada-Kotera (SK) equation [10]. They take the general form

$$Ay_t + By^2y_x + Cy_xy_{xx} + Dyy_{xxx} + Ey_{xxxxx} = 0 \tag{2}$$

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with appropriate initial-boundary conditions, where A, B, C, D, E are constants. the form (2) is referred to as the generalized fifth-order Korteweg-de Vries (GFKdV), [11, 12].

These GFKdV equations models diverse important physical phenomena. This equation do not just show the movement of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice, but at the same time, it is a significant mathematical model for magneto-sound propagation in plasmas [13] and a chain of coupled nonlinear oscillators [14].

Obtaining the exact solution of the GFKdV equation appears to be unreachable in most cases especially for the nonlinear case, save the special case of solitary waves in [15]. Some methods which are semi numerical and complete numerical in literature that were derived for solving fifth order KdV types include, Modified Variational Iteration Technique [11], Modified Variational Iteration Algorithm-II (MVIA-II) [12], He's Variational Iteration Method, [16], homotopy analysis method, [17], Group analysis method [18], among others.

In this work, we derive a 1-Step Block Hybrid Integrator (1SBHI), assembled in block form using collocation technique, for the solution of fifth-order Korteweg-de Vries (KdV) type equations using the MOLs approach, see [19, 20, 21]. The linear multistep method (LMM) derived using the collocation technique is conveniently used to approximate, with high accuracy, continuous Initial-Boundary Value Problems (IBVPs) of both the ordinary and partial differential equations [20]. It has several advantages among which are the fact that they do not require any staring value to kick-start their implementation, they are self-starting [22, 23]. They also have the advantage of producing smaller global errors (at the end of the range of integration) than those produced by the step-by-step methods due to the presence of accumulated errors at each step in the step-by-step method, see [24, 25, 26, 27, 28].

The Method of Lines (MOLs) approach is a very important tool used for solving Partial Differential Equations (PDEs), where a PDE is transformed into a system of Ordinary Differential Equations (ODEs) whereby appropriate derivatives are replaced by finite difference approximations. In this work, we transform the PDEs into a system of ODEs such that only one of the spatial derivatives is substituted with central difference approximations. The system that results is solved using 1SHBI. In the case of this work, we discretize the space variable t with meshing

$$\Delta t = \frac{d-c}{M}, \quad t_m = c + m\Delta t, \quad m = 0, 1, \ldots M$$

and then define the vector

$$y = [y_{1,1}, y_{1,2}, y_{2,1}, \ldots, y_{n,m-1}, y_{n-1,m}, \ldots, y_{n-1,m-1}]^T$$

and

$$G = [G_{1,1}, G_{1,2}, G_{2,1}, \ldots, G_{n,m-1}, G_{n-1,m}, \ldots, G_{n-1,m-1}]^T$$

where $y_m \approx y(x, t_m)$. The central differences approximation formulae for the first and second derivatives are

$$y_t \approx \frac{y_{m+1} - y_{m-1}}{(2\Delta t)}$$

$$y_{tt} \approx \frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2}$$

The solution $y(x, t)$, of equation (1), where (x, t) is in the rectangle $[a, b]\times[c, d]$, is obtained by solving the system of fifth-order ODEs in the spatial variable x. Here the mesh of x is spaced is:

$$\pi_N = a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b,$$

with the step-size h given as

$$h = \frac{b-a}{N}, \quad x_n = a + nh, \quad n = 0, 1, \ldots, N$$

Then, the equation (1) has the semidiscretized form

$$\frac{dy^5_{n,m}}{dx^5} \approx -\frac{a_{n,m}}{b_{n,m}} + \frac{1}{b_{n,m}}\left(\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + G_{n,m}\right) \quad (3)$$

where

$$G_{n,m} = \left(x_n, t_m, y_{n,m}, y^2_{n,m}, \frac{dy_{n,m}}{dx}, \frac{y_{m+1} - u_{m-1}}{(2\Delta t)}, \frac{dy^2_{n,m}}{dx^2}, \frac{dy^3_{n,m}}{dx^3}\right)$$

and $b_{n,m} \neq 0$ for $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, M$. The equivalent form for equation (3) is the following system of fifth order ODE given by

$$\mathrm{y}^{(5)} = f(x, \mathrm{y}, \mathrm{y}', \mathrm{y}'', \mathrm{y}''', \mathrm{y}^{(4)}), \quad a < x < b \quad (4)$$

subject to the boundary conditions:

$$\mathrm{y}(a) = \mathrm{y}_0, \quad \mathrm{y}'(a) = \mathrm{y}'_0, \qquad \mathrm{y}''(a) = \mathrm{y}''_0,$$
$$\mathrm{y}(b) = \mathrm{y}^1_{n,m}, \quad \mathrm{y}'(b) = \mathrm{y}^2_{n,m}$$

where $f(x, \mathrm{y}, \mathrm{y}', \mathrm{y}'', \mathrm{y}''', \mathrm{y}^{(4)}) = AY + g$ and A is a k by k matrix of constants with $(k = (N-1)(M-1))$ which results from the semi-discretized system of equation (3), expressed in the form the equation (4) and hence solved using 1SBHI. Here g is a vector of constants. Note that j in $\mathrm{y}^j_{n,m}$, for $j = 1, 2$, are only index notation for different $\mathrm{y}_{n,m}$. In what follows, the derivation of the approximate solution of equation (4) is discussed.

It is noteworthy to see that this technique is easy to derive and implement for solving this class of problem defined in equation (1).

2. Derivation of the Method

Suppose the approximate solution $y_n(x)$ of equation (4) have the continuous form

$$y(x) \approx u(x) = \sum_{j=0}^{10} a_j x^j \quad (5)$$

where its fifth derivative is given by

$$y^{(v)}(x) \approx u^{(v)}(x) = \sum_{j=5}^{10} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5} \quad (6)$$

Evaluating equation (5) at the points $x = x_{n+v_j}$, $j = 0(1)4$ where $v_0 = 0$, $v_1 = \frac{1}{6}$, $v_2 = \frac{1}{3}$, $v_3 = \frac{2}{3}$, $v_4 = \frac{5}{6}$ and equation (6) at the points $x = x_{n+v_j}$, $j = 0(1)5$ where $v_5 = 1$, the following system of equations are obtained.

$$u(x_n) = y_n; \quad u'(x_n) = y'_n; \quad u''(x_n) = y''_n; \quad u'''(x_n) = y'''_n;$$
$$u^{(iv)}(x_n) = y_n^{(iv)}; \quad u^{(v)}(x_n) = f_n; \quad u^{(v)}(x_{n+v}) = f_{n+v_j}, j = 1(1)5. \tag{7}$$

Solving the above system simultaneously with the use of CAS in Mathematica 12.0, the coefficients a_j, $\quad j = 0(1)10$ are obtained (though not included here due to their cumbersome terms). They are in turn substituted into equation (5), to obtain the following continuous method:

$$y_{n+v_k} = \sum_{j=0}^{4} \alpha_j^k h^i y_n^{(j)} + h^5 \sum_{j=0}^{5} \beta_j^k f_{n+v_j} \Big|_{k=1(1)5} \tag{8}$$

where α_j and β_j are shown in the Table 1.

The first derivative of equation (8) is given as

$$y'_{n+v_k} = \sum_{i=1}^{4} \alpha_i h^{i-1} y_n^{(i)} + h^4 \sum_{j=0}^{5} \beta_i f_{n+v_i} \Big|_{k=1(1)5} \tag{9}$$

here, α_j and β_j are shown in the Table 2.

The second derivative of equation (8) is given as

$$y''_{n+v_k} = \sum_{i=2}^{4} \alpha_i h^{i-2} y_n^{(i)} + h^3 \sum_{j=0}^{5} \beta_i f_{n+v_i} \Big|_{k=1(1)5} \tag{10}$$

where α_j and β_j are shown in the Table 3

The third derivative of equation (8) is given as

$$y'''_{n+v_k} = \sum_{i=3}^{4} \alpha_i h^{i-3} y_n^{(i)} + h^2 \sum_{j=0}^{5} \beta_i f_{n+v_i} \Big|_{k=1(1)5} \tag{11}$$

here α_j and β_j are shown in the Table 4.

The fourth derivative of equation (8) is given as

$$y_{n+v_k}^{(iv)} = y_n^{(iv)} + h \sum_{j=0}^{5} \beta_i f_{n+v_i} \Big|_{k=1(1)5} \tag{12}$$

where the coefficients α_j and β_j are shown in the Table 5.

2.1. Characteristics of the method

The formula in equation (8) (and its derivatives in equations (9)-(12)) is a continuous schemes associated with the linear differential operator defined by

$$\mathcal{L}[z(x + v_k h); \; h] = \sum_{i=0}^{4} \alpha_j^k h^j z^{(j)}(x) - \sum_{j=0}^{5} h^5 \beta_i z^{(5)}(x + v_j h) \Big|_{k=1(1)5} \tag{13}$$

Expanding equation (13) in Taylor series, the constants $C_i's$ written as linear combination of the derivatives of $z(x)$ up to $p+1$ derivative as

$$\mathcal{L}[z(x); \; h] = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_p h^p z^{(p)}(x) + O(h^{(p+1)}) \tag{14}$$

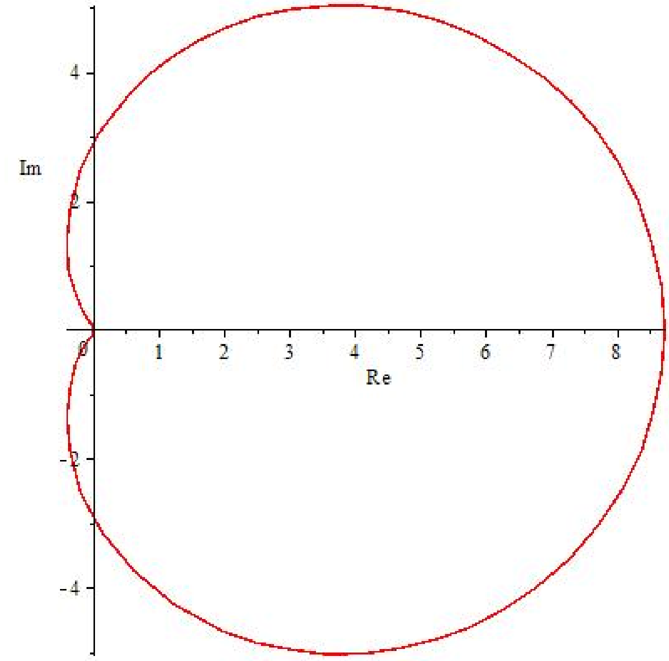

Figure 1. The region of stability

where p is the order of equation (13) and consequently the formula (13). By definition, the LMM (8) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+4} = 0,$ *and* $\; C_{p+5} \neq 0$ in which

$$\mathcal{L}[z(x); \; h] = C_{p+5} h^{p+5} z^{(p+5)}(x) + O(h^{(p+6)}) \tag{15}$$

In this case, C_{p+5} is the principal error constant, see [29].

Definition 2.1. *A linear difference operator $\mathcal{L}[z(x); \; h]$ of order p is consistent if $p > 1$.*

Thus, the order of the formula in equation (8) is $p = 6$ with the following error constants

$$C_{p+5} = (-6.87857 \times 10^{-13}, -2.36466 \times 10^{-11}, -5.40035 \times 10^{-10}, -1.47456 \times 10^{-9}, -3.4408 \times 10^{-9})^T$$

The order of the formulas in equations (9)-(12) is $p = 6$ and their error constants are obtained similarly.

2.2. Zero-stability and convergence

A numerical method in equations (8)-(12) is zero-stable provided as $h \to 0$, the solutions is bounded.

Following [20], the block method in (8)-(12) may be rewritten in a matrix form as

$$A_0 Y_\mu = A_1 Y_{\mu-1} + h^5 (F_\mu + F_{\mu-1}) \tag{16}$$

where where

$$Y_\mu = \left(Y_\mu^0, Y_\mu^1, \ldots, Y_\mu^5\right)^T, \quad Y_{\mu-1} = \left(Y_{\mu-1}^0, Y_{\mu-1}^1, \ldots, Y_{\mu-1}^5\right)^T$$

Table 1. Coefficients of α_j, $j = 0(1)4$, and β_j, $j = 0(1)5$ in equation (8)

k	α_0	α_1	α_2	α_3	α_4	β_0	β_1	β_2	β_3	β_4	β_5
1	1	$\frac{1}{6}$	$\frac{1}{72}$	$\frac{1}{1296}$	$\frac{1}{31104}$	$86197\eta_1$	$58392\eta_1$	$-30815\eta_1$	$14415\eta_1$	$-9112\eta_1$	$1883\eta_1$
2	1	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{1}{162}$	$\frac{1}{1944}$	$7963\eta_2$	$10528\eta_2$	$-4365\eta_2$	$1985\eta_2$	$-1248\eta_2$	$257\eta_2$
3	1	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{4}{81}$	$\frac{2}{243}$	$622\eta_3$	$1392\eta_3$	$-215\eta_3$	$180\eta_3$	$-112\eta_3$	$23\eta_3$
4	1	$\frac{5}{6}$	$\frac{25}{72}$	$\frac{125}{1296}$	$\frac{625}{31104}$	$6685\eta_4$	$16600\eta_4$	$-375\eta_4$	$2375\eta_4$	$-1368\eta_4$	$275\eta_4$
5	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$401\eta_5$	$1056\eta_5$	$105\eta_5$	$195\eta_5$	$-96\eta_5$	$19\eta_5$

$\eta_1^0 = \frac{1}{112870195200}$; $\eta_2^0 = \frac{1}{440899200}$; $\eta_3^0 = \frac{2}{3444525}$; $\eta_4^0 = \frac{625}{4514807808}$; $\eta_5^0 = \frac{1}{201600}$

Table 2. Coefficients of α_j, $j = 1(1)4$, and β_j, $j = 0(1)5$ in equation (9)

k	α_1	α_2	α_3	α_4	β_0	β_1	β_2	β_3	β_4	β_5
1	1	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{162}$	$100795\eta_1^1$	82832_1^1	-42175_1^1	19525_1^1	-12320_1^1	2543_1^1
2	1	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{1}{162}$	$2209\eta_2^1$	$3518\eta_2^1$	$-1300\eta_2^1$	$595\eta_2^1$	$-374\eta_2^1$	$77\eta_2^1$
3	1	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{4}{81}$	$1316\eta_3^1$	$3328\eta_3^1$	$-140\eta_3^1$	$425\eta_3^1$	$-256\eta_3^1$	$52\eta_3^1$
4	1	$\frac{5}{6}$	$\frac{25}{72}$	$\frac{125}{1296}$	$7027\eta_4^1$	$19040\eta_4^1$	$2225\eta_4^1$	$3325\eta_4^1$	$-1712\eta_4^1$	$335\eta_4^1$
5	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$35\eta_5^1$	$98\eta_5^1$	$25\eta_5^1$	$25\eta_5^1$	$-10\eta_5^1$	$2\eta_5^1$

$\eta_1^1 = \frac{1}{4702924800}$; $\eta_2^1 = \frac{1}{9185400}$; $\eta_3^1 = \frac{2}{1148175}$; $\eta_4^1 = \frac{125}{188116992}$; $\eta_5^1 = \frac{1}{4200}$;

Table 3. Coefficients of α_j, $j = 2, 3, 4$, and β_j, $j = 0(1)5$ in equation (10)

k	α_2	α_3	α_4	β_0	β_1	β_2	β_3	β_4	β_5
1	1	$\frac{1}{6}$	$\frac{1}{72}$	$40501\eta_1^2$	$42484\eta_1^2$	$-20455\eta_1^2$	$9335\eta_1^2$	$-5876\eta_1^2$	0
2	1	$\frac{1}{3}$	$\frac{1}{18}$	$1646\eta_2^2$	$3304\eta_2^2$	$-985\eta_2^2$	$470\eta_2^2$	$-296\eta_2^2$	$61\eta_2^2$
3	1	$\frac{1}{3}$	$\frac{1}{9}$	$467\eta_3^2$	$1328\eta_3^2$	$190\eta_3^2$	$205\eta_3^2$	$-112\eta_3^2$	$22\eta_3^2$
4	1	$\frac{5}{6}$	$\frac{25}{72}$	$497\eta_4^2$	$1444\eta_4^2$	$485\eta_4^2$	$395\eta_4^2$	$-164\eta_4^2$	$31\eta_4^2$
5	1	1	$\frac{1}{2}$	$222\eta_5^2$	$648\eta_5^2$	$315\eta_5^2$	$270\eta_5^2$	$-72\eta_5^2$	$17\eta_5^2$

$\eta_1^2 = \frac{1}{87091200}$; $\eta_2^2 = \frac{1}{680400}$; $\eta_3^2 = \frac{1}{42525}$; $\eta_4^2 = \frac{125}{3483648}$; $\eta_5^2 = \frac{1}{8400}$;

Table 4. Coefficients of α_j, $j = 3, 4$, and β_j, $j = 0(1)5$ in equation (11)

k	α_3	α_4	β_0	β_1	β_2	β_3	β_4	β_5
1	1	$\frac{1}{6}$	$5561\eta_1^3$	$37504\eta_1^3$	$-16325\eta_1^3$	$7295\eta_1^3$	$-4576\eta_1^3$	$941\eta_1^3$
2	1	$\frac{1}{3}$	$1843\eta_2^3$	$5032\eta_2^3$	$-830\eta_2^3$	$515\eta_2^3$	$-328\eta_2^3$	$68\eta_2^3$
3	1	$\frac{2}{3}$	$254\eta_3^3$	$776\eta_3^3$	$395\eta_3^3$	$235\eta_3^3$	$-104\eta_3^3$	$19\eta_3^3$
4	1	$\frac{5}{6}$	$269\eta_4^3$	$800\eta_4^3$	$575\eta_4^3$	$475\eta_4^3$	$-128\eta_4^3$	$25\eta_4^3$
5	1	1	$239\eta_4^3$	$696\eta_4^3$	$600\eta_4^3$	$555\eta_4^3$	$-24\eta_4^3$	$34\eta_4^3$

$\eta_1^3 = \frac{1}{3628800}$; $\eta_2^3 = \frac{1}{113400}$; $\eta_3^3 = \frac{2}{14175}$; $\eta_4^3 = \frac{25}{145152}$; $\eta_5^3 = \frac{1}{4200}$;

Table 5. Coefficients of α_4 and β_j, $j = 0(1)5$ in equation (12)

k	α_4	β_0	β_1	β_2	β_3	β_4	β_5
1	1	$4991\eta_1^4$	$12824\eta_1^4$	$-4365\eta_1^4$	$1885\eta_1^4$	$-1176\eta_1^4$	$241\eta_1^4$
2	1	$287\eta_2^4$	$1248\eta_2^4$	$245\eta_2^4$	$45\eta_2^4$	$-32\eta_2^4$	$7\eta_2^4$
3	1	$43\eta_3^4$	$112\eta_3^4$	$180\eta_3^4$	$155\eta_3^4$	$-48\eta_3^4$	$8\eta_3^4$
4	1	$43\eta_4^4$	$120\eta_4^4$	$175\eta_4^4$	$225\eta_4^4$	$8\eta_4^4$	$5\eta_4^4$
5	1	$13\eta_5^4$	$32\eta_5^4$	$55\eta_5^4$	$55\eta_5^4$	$32\eta_5^4$	$13\eta_5^4$

$\eta_1^4 = \frac{1}{86400}$; $\eta_2^4 = \frac{1}{5400}$; $\eta_3^4 = \frac{1}{675}$; $\eta_4^4 = \frac{5}{3456}$; $\eta_5^4 = \frac{1}{200}$;

$$
\begin{aligned}
Y_\mu^0 &= (y_{n+\frac{1}{6}}, y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+\frac{5}{6}}, y_{n+1})^T \\
&\vdots \\
Y_\mu^5 &= (y^{(v)}_{n+\frac{1}{6}}, y^{(v)}_{n+\frac{1}{3}}, y^{(v)}_{n+\frac{2}{3}}, y^{(v)}_{n+\frac{5}{6}}, y^{(v)}_{n+1})^T \\
Y_{\mu-1}^0 &= (y_{n-\frac{5}{6}}, y_{n-\frac{2}{3}}, y_{n-\frac{1}{3}}, y_{n-\frac{1}{6}}, y_n)^T \\
&\vdots \\
Y_{\mu-1}^5 &= (y^{(v)}, y^{(v)}, y^{(v)}, y^{(v)}, y^{(v)})^T
\end{aligned}
$$

Hence, as $h \to 0$ the method in equation (16) becomes

$$A_0 Y_\mu = A_1 Y_{\mu-1} \tag{17}$$

where A_0 is the identity matrix of order 25, $A_0 = I_{25}$, and A_1 is

Table 6. Absolute Error for Problem 1

	$x = 1.0$		$x = 1.5$	
t	1SHBI	mVIA-I	1SHBI	mVIA-I
0.01	0.00000	0.00000	2.14572×10^{-18}	0.00000
0.02	1.11257×10^{-18}	8.88178×10^{-16}	2.22141×10^{-18}	8.88178×10^{-16}
0.03	8.47894×10^{-17}	1.19904×10^{-14}	4.71125×10^{-17}	1.95399×10^{-14}
0.04	5.41381×10^{-17}	8.79296×10^{-14}	3.12412×10^{-17}	1.43884×10^{-13}
0.05	9.12414×10^{-16}	4.19229×10^{-13}	1.49787×10^{-17}	6.90114×10^{-13}

Comparing absolute errors of 1SHBI with the mVIA-I in [11] for Example 1.

Table 7. Error for Example 2

x	y_{ex}	y_{approx}	Error
0.12566	-0.07426755	-0.07426785	2.9720E-7
0.25132	0.05130957	0.05130487	4.7001E-6
0.31415	0.11392287	0.11391146	1.1407E-5
0.43982	0.23757397	0.23753066	4.3304E-5
0.50265	0.29812884	0.29805541	7.3430E-5
0.62831	0.41551891	0.41534181	1.7710E-4
0.75398	0.52644076	0.52607803	3.6273E-4
0.81681	0.57893577	0.57843927	4.9650E-4
0.94247	0.67698465	0.67611561	8.6903E-4
1.00530	0.72216309	0.72104522	1.1178E-3

$0 \le x \le 2\pi$, $t = 1$, $h = 0.0628319$.

Table 8. Maximum Error for Problem 2 at $t = 1$

Method	N	L^1	L^2	L^∞
DGFEM	10	0.11E-1	0.12E-1	0.17
	20	0.39E-2	0.43E-2	0.61E-2
	40	0.12E-3	0.14E-3	0.19E-3
	80	0.36E-5	0.40E-5	0.57E-5
1SHBI	10	1.52E-6	3.23E-6	1.17E-7
	20	2.15E-8	3.24E-6	3.21E-8
	40	5.18E-8	2.79E-9	5.27E-8
	80	2.32E-10	4.71E-10	8.11E-10

DGFEM is Discontinuous Galerkin Finite Element Method developed in [32]. The p-norm in Table 8 is given as

$$L^p = \|y_i - y(x_i)\|_p = \left(\sum_{i=1}^{\inf} |y_i - y(x_i)|^p \right)^{\frac{1}{p}}$$

$$L^\infty = \|y_i - y(x_i)\|_\infty = \max_{1 \le i \le N} \{|y_i - y(x_i)|\}$$

a 25×25 matrix given by

$$\begin{pmatrix} A_{11} & & & & \\ & A_{22} & & & \\ & & A_{33} & & \\ & & & A_{44} & \\ & & & & A_{55} \end{pmatrix}$$

with the $A_{11}, \ldots, A_{55}$ being 5×5 matrices respectively, given

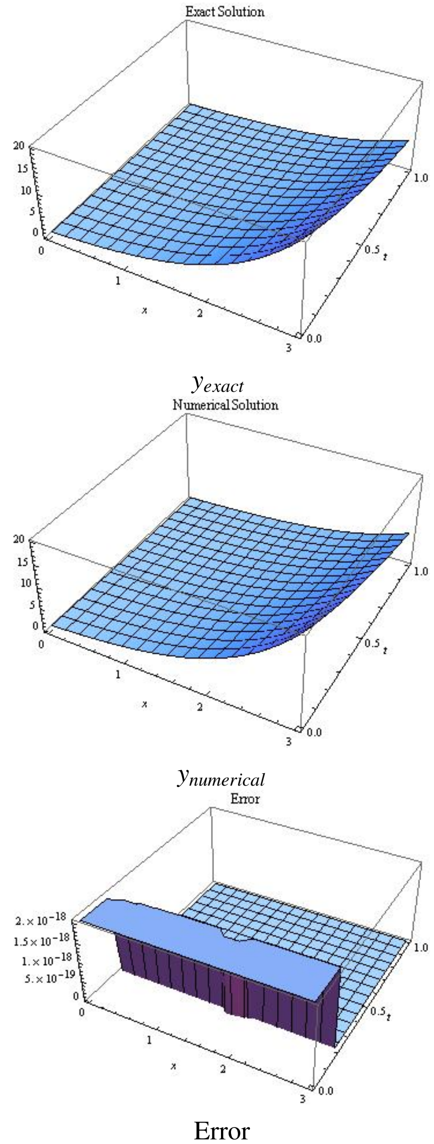

Figure 2. The shaded regions shows the surface plots for the numerical solution, the analytic solution and the residue (error) for Example 1.

by

$$A_{ii} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, 5$$

Table 9. Numerical Results with t = 1 for Example 3

x	Exact	Approximate	Error
0.1	1.999999999998E-10	1.999999999998E-10	0.000000000000000
0.2	1.999999999992E-10	1.999999999991E-10	1.292469707114105E-25
0.3	1.999999999982E-10	1.999999999981E-10	3.618915179919496E-25
0.4	1.999999999968E-10	1.999999999967E-10	6.720842476993354E-25
0.5	1.999999999950E-10	1.999999999949E-10	1.214921524687259E-24
0.6	1.999999999928E-10	1.999999999927E-10	1.757758801675183E-24
0.7	1.999999999902E-10	1.999999999901E-10	2.429843049374518E-24
0.8	1.999999999872E-10	1.999999999871E-10	2.998529720504725E-24
0.9	1.999999999838E-10	1.999999999837E-10	3.127776691216136E-24
1.0	1.999999999800E-10	1.999999999799E-10	2.403993655232236E-24

Table 10. Maximum Error for Example 3 at t = 1

Method	GA with RK	MQ with RK	IMQ with RK	1SHBI
L^2	1.9774E-21	5.9692E-12	1.9774E-21	5.96847E-24
L^∞	1.2705E-21	3.2896E-12	1.2705E-21	3.12778E-24

The matrices A_{ii} have the characteristic polynomial $|A_{jj} - \lambda I_5| = 0$, for $j = 1, \ldots, 5$, that is, $\lambda^4(\lambda - 1) = 0$. The characteristic polynomial has the root $\lambda_r = 0$, for $r = 1, \ldots, 4$ and $\lambda_5 = 1$. The method is zero-stable, since the roots of the characteristic polynomial have one unity while others are zero (see [29]). For convergence, the following theorem apply.

Theorem 2.1. *[30]. A linear multistep method is convergent provided it is consistent (having order $p \geq 1$) and zero-stable.*

The derived method has $p = 6$, and equally zero-stable and hence it is convergent.

2.3. Linear stability analysis

The linear stability for any given $h > 0$ is concern with the behaviour of the underlined problem not just the numerical method. Here, the stability properties for the intending numerical method is analyzed by considering the linear test equation for $\mu > 0$ of the form

$$y^{(v)} = -\mu^5 y \tag{18}$$

the test in equation (18) is appropriate since it has a bounded solution as $\mu \to 0$. Here, μ is the frequency. set $\delta = \mu h$ and then applying to the test equation in (18), and after some manipulations, the following equation results

$$Z_\mu = M(\delta) Z_{\mu-1}, \quad \delta = \mu h \tag{19}$$

where $M(\delta) = \dfrac{A_0 + \delta B_0}{(A_1 - \delta B_1)}$ is the required stability matrix. The amplification matrix in equation (19) results on the dominant eigenvalues given as

$$M(\delta_{dominant}) = \frac{6(-2)^{4/5} 3^{1/5} \left(-1 - 4\delta^{1/6} + 5\delta^{1/3} - 5\delta^{2/3} + 4\delta^{5/6} - \delta\right)^{1/5}}{\left(11 - 56\delta^{1/6} + 285\delta^{1/3} + 285\delta^{2/3} - 56\delta^{5/6} + 11\delta\right)^{1/5}} \tag{20}$$

having the stability region in the Figure 1.

These roots (containing the real and imaginary parts) are plotted and as shown in Figure 1. Here we study the boundedness of the solution under consideration through the eigenvalues of the stability matrix $M(\delta)$ for which these eigenvalues δ is such that $|\delta| < 1$ for the derived method to be stable. If δ is real, then the absolute stability region is reduced to a real interval which is called an interval of stability, see [31]. Figure 1 shows the stability region for the proposed method, with the stability interval (0, 8.7893).

2.4. Implementation

The system formed by the semi-discretization given by equation (4) is solved using the unified block scheme formed by equations (8)-(12) with 5N equations and 5N unknowns solve simultaneously, whose solution provides a set of approximate values of (1) using codes written in Mathematica 12.0, enhanced by the feature NSolve[] for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature FindRoot[]. Following [20], the following algorithm summarizes the computational procedures.

We begin by noting that the solution of the problem in equation (1) is sought in the subintervals $D_N = a = x_0 < x_1 < \ldots < x_N = b$ and $D_M = c = t_0 < t_1 < \ldots < t_M = d$, where $h = \frac{b-a}{N}$, $k = \frac{d-c}{M}$ are constant step-size.

Step 1: Use the block method setting $n = 0$, $m = 0$, to obtain V_1 on the rectangle $[y_{0,m}, y_{1,m}]_{(n,m)\in[a,b]\times[c,d]}$, similarly, for, $n = 1$ so that V_2 is obtained on the rectangle $[y_{1,m}, y_{2,m}]_{(n,m)\in[a,b]\times[c,d]}$, and on the rectangle

$[y_{3,m}, y_{4,m}]_{(n,m)\in[a,b]\times[c,d]} \cdots [y_{n=N-1,m}, y_{n=N,m}]_{(n,m)\in[a,b]\times[c,d]}$ for $m = 0, 1, 2, \ldots, M$ we thus obtain $V_3, V_4, \ldots, V_D$.

Step 2: Solve the unified block given by the system $V_1 \bigcup V_2 \bigcup, \cdots, V_D$ obtained in step 1. **Step 3**: The solution of equation (1) is approximated by the solutions in step 2 as $y_{n,m} = [y(x_1, t); y(x_2, t) \ldots y(x_n, t)]^T$ for $t > 0$, $n = 1, 2, \ldots N$.

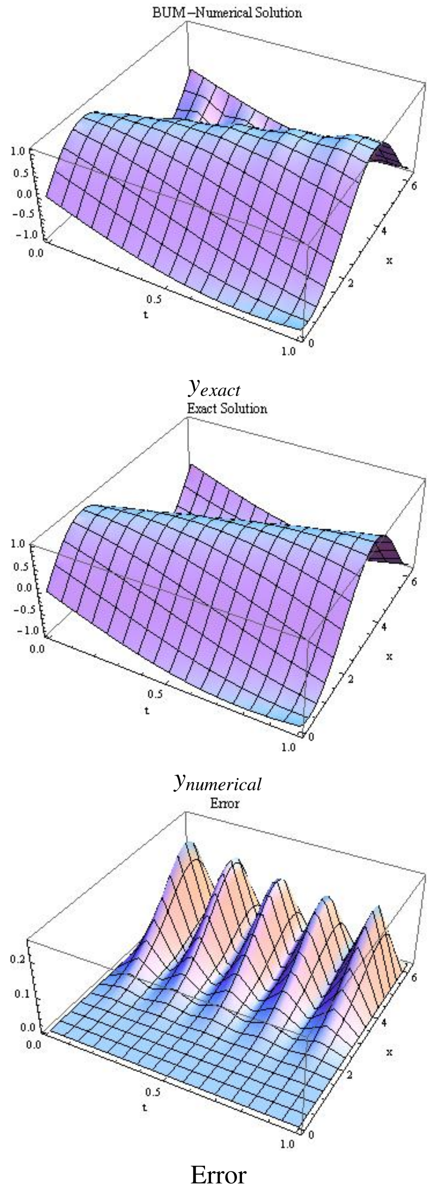

Figure 3. Surface plots showing the numerical solution, the analytic solution and the residue for Example 2.

3. Numerical Examples

In this section, we test the efficiency of the derived method and thus compare only with the exact solution of the given problem extracted from literature.

Example 1.. Consider the fifth order KdV equation,

$$\left.\begin{array}{r} y_t + yy_x - yy_{xxx} + y_{xxxxx} = 0 \\ y(x,0) = e^x, \end{array}\right\} \tag{21}$$

with exact solution $y(x,t) = e^{x-t}$.

After discretization of the "t" variable, we obtain,

$$\frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + y_m \frac{dy_m}{dx} - y_m \frac{d^3 y_m}{dx^3} + \frac{d^5 y_m}{dx^5} = g_m, \quad x \in [0,1] \;\; m = 1, \dots M-1 \tag{22}$$

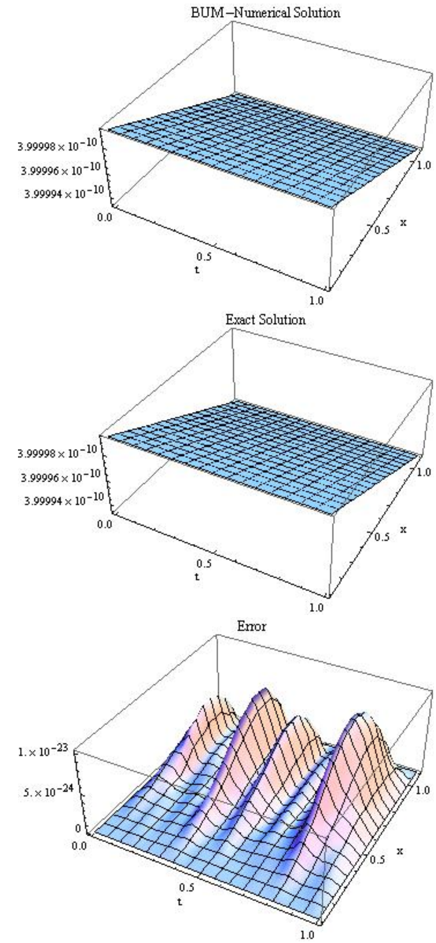

Figure 4. Surface plots for the numerical solution for Example 4

where Δt, t_m, $m = 0, 1, \dots, M$, y, $y_m(x) \approx y(x, t_m)$, and $g_m = 0$, expressed as

$$y^{(v)} = f(x, y, y', y''') = Ay + g \tag{23}$$

A is a k by k matrix.

To show the efficiency of 1SHB1, the above Table 6 shows the absolute error for different values of t the values $x = 1.0$ and 1.5 respectively. It can be observe that the 1SHBI outperformed mVIA-I for the example considered. The Figure 2 shows a good agreement of the numerical and exact solutions.

Example 2.. Consider a time dependent biharmonic fifth order equation discussed in [32].

$$\left.\begin{array}{r} y_t + y_x + y_{xxxxx} = 0, \quad 0 \le x \le 2\pi, \quad t \ge 0 \\ y(x,0) = \sin(x), \\ y(0,t) = y(2\pi, t) \end{array}\right\} \tag{24}$$

The analytic solution for this problem is $y(x,t) = \sin(x - 2t)$.

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + \frac{dy_m}{dx} + \frac{d^5 y_m}{dx^5} = g_m, \quad 0 \le x \le 1, \quad m = 1, \dots M-1 \tag{25}$$

where Δt, t_m, $m = 0, 1, \ldots, M$, y, $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y', y'', y''', y^{(iv)}) = Ay + g \tag{26}$$

A is a $k \times k$ matrix ($k = (N-1)(M-1)$) and $g_m = 0$.
Table 7 compares the errors obtained for Example 2. Figure 3 shows the representation of the numerical solution, analytical solution and residue (errors), for Example 2.
Table 8, for different subinterval N, shows the maximum errors L^∞ and the L^p with $p = 1,\ 2-norm$ obtained, in comparison with the 1SHBI and the method in [32]. This shows the superiority of the 1SHBI.

Example 3.. Consider the nonlinear fifth order KdV equation called Lax's cases of generalized KdV equation discussed [33].

$$\left.\begin{aligned} y_t + 45y^2y_x + 15y_xy_{xx} + 15yy_{xxx} + y_{xxxxx} = 0,\\ 0 \le x \le 1, \quad t \ge 0\\ y(x,0) = 2k^2 \sec h^2(k(x - x_0)) \end{aligned}\right\} \tag{27}$$

The exact solution for this problem is
$y(x,t) = 2k^2 \sec h^2(k(x - 16k4t - x_0))$

Upon discretization of the time variable, we obtain,

$$\begin{aligned} &\frac{y_{m+1} - y_{m-1}}{(2\Delta t)} + 45y_my_m\frac{dy_m}{dx} + 15\frac{dy_m}{dx}\frac{d^2y_m}{dx^2}\\ &+ 15y_m\frac{d^3y_m}{dx^3} + \frac{d^5y_m}{dx^5} = g_m, \quad 0 \le x \le 1, \quad m = 1, \ldots M-1 \end{aligned} \tag{28}$$

where Δt, t_m, $m = 0, 1, \ldots, M$, y, $y_m(x) \approx y(x, t_m)$, g and g_m are expressed in the form

$$y^{(v)} = f(x, y, y', y'', y''', y^{(iv)}) = Ay + g \tag{29}$$

A is as expressed in Example 1 and $g_m = 0$.

In [33], the authors presented a numerical solution of equation (27) using a meshless method of lines. This method uses Meshless, Multiquadric (MQ), Inverse multiquadric (IMQ) and Gaussian (GA) as Radial basis function (RBF) for spatial derivatives and Runge-Kutta method as a time integrator.
The constant step-size used for Table 9 and Table 10 is $h = 0.1$.

Figure 4 shows the graphical representation with surface plot for the numerical solution (shaded region), analytical solution (shaded region) and residue or error (shaded region), for Example 3.

The method presented in [33] for the numerical solution of Example 4 uses a meshless method of lines. This method uses Meshless, Multiquadric (MQ), Inverse multiquadric (IMQ) and Gaussian (GA) as Radial basis function (RBF) for spatial derivatives and Runge-Kutta method as a time integrator.

4. Conclusion

In this work, a one step hybrid block integrator (ISHBI) was derived consisting of continuous linear multistep methods. This was used to solve certain fifth- order PDEs subject to appropriate initial-boundary conditions. To show the robustness of method derived, fifth-order KdV PDEs were solved were transformed into a system of fifth-order ODEs using the method of lines. From the numerical experiments performed, It's evident that the 1SHBI performed well in terms of accuracy as compared to exact solutions of the problems considered in literature.

Acknowledgments

We thank the referees for the positive enlightening comments and suggestions, which have greatly helped us in making improvements to this paper.

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