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Time-Fractional Differential Equations with an Approximate Solution

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Abstract

This paper shows how to use the fractional Sumudu homotopy perturbation technique (SHP) with the Caputo fractional operator (CF) to solve time fractional linear and nonlinear partial differential equations. The Sumudu transform (ST) and the homotopy perturbation technique (HP) are combined in this approach. In the Caputo definition, the fractional derivative is defined. In general, the method is straightforward to execute and yields good results. There are some examples offered to demonstrate the technique's validity and use.

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1. Introduction

For characterizing nonlocal structures, fractional calculus has arisen as a new mathematical technique. Due to their numerous applications in physics and engineering, fractional differential equations have attracted a lot of attention during the last decade [1-4].

Many authors have investigated fractional calculus theory, and current study is centered on proving the important benefits of fractional calculus over classical calculus [5, 6]. In several instances, the fractional calculus gave the best outcomes than the conventional method.

There is a significant literature dealing with the subject area of approximating solutions to fractional differential equations using various approaches known as perturbation methods. The perturbation techniques have certain disadvantages; for example, the approximate solution requires a succession of smaller parameters, which is challenging because the majority of nonlinear problems do not have any small values. Despite proper selections of smaller parameters might occasionally lead to an optimal solution, in most circumstances, bad choices have major consequences in the solutions. As a result, an analytical technique that does not need a smaller parameter in the equation representing the phenomena is preferred. He [7] was the first to propose the homotopy perturbation technique (HPM). Many writers investigated the HPM to analyze linear and nonlinear equations encountered in numerous scientific and technical sectors.

In general, higher performance of the fractional calculus is demonstrated by reduced error levels created during an estimating procedure. Various approximation and methodologies, like

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the fractional Adomian decomposition method (FADM) [8-10], fractional homotopy method (FHPM) [11, 12, 13], fractional function decomposition method [14, 15], fractional variational iteration method (FVIM) [16-18], fractional reduce differential transform method (FRDTM) [18, 19, 20, 21], fractional differential transform method [22, 23, 24], fractional Laplace variational iteration method [25-32], fractional Laplace homotopy perturbation method (FLHPM) [33], fractional Laplace decomposition method (FLDM) [34, 35], fractional Sumudu homotopy analysis method [36], fractional Sumudu variational iteration method (FVIM) [37, 38], fractional Sumudu decomposition method (FSDM) [39-41], fractional natural decomposition method (FNDM) [42, 43], fractional Sumudu homotopy perturbation method (FSHPM) [44, 45], energy balance method (EBM) [46], power series methods (PSM) [47], have been used in latest years to analyze partial differential equations within Caputo sense.Moreover, many studies provide functional differential equations using various types of transforms, such as the Mellin transform [48]. Also, the Lagrange interpolation is used to estimate the delay argument [49]. Furthermore, the local truncation errors and stability polynomials are calculated [50-52].

Our objective is to develop and use the Sumudu homotopy perturbation methodology, which combines the Sumudu transform with the homotopy perturbation method to solve nonlinear fractional differential equations.

2. Preliminaries

This section goes through some of the fractional calculus definitions and notation that will be utilized in this study [4, 45, 46, 47, 53, 54, 55].

Definition 2.1. *The Riemann Liouville fractional integral operator of order $\delta \geq 0$ of a function $\varphi(\mu) \in C_\vartheta, \vartheta \geq -1$ is given by the form*

$$I^\delta \varphi(\mu) = \begin{cases} \dfrac{1}{\Gamma(\delta)} \int_0^\mu (\mu-\tau)^{\delta-1}\varphi(\tau)d\tau, & \delta > 0, \mu > 0 \\ I^0\varphi(\mu) = \varphi(\mu), & \delta = 0 \end{cases} \tag{1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator $I^\delta$are as follows: For $\varphi \in C_\vartheta, \vartheta \geq -1, \delta, \gamma \geq 0$, then

1. $I^\delta I^\gamma \varphi(\mu) = I^{\delta+\gamma}\varphi(\mu)$
2. $I^\delta I^\gamma \varphi(\mu) = I^\gamma I^\delta \varphi(\mu)$
3. $I^\delta \mu^m = \dfrac{\Gamma(m+1)}{\Gamma(\delta+m+1)}\mu^{\delta+m}$

Definition 2.2. *In the Caputo interpretation, the fractional derivative of $\varphi(\mu)$ is given as*

$$\begin{aligned} D^\delta\varphi(\mu) &= I^{m-\delta}D^m\varphi(\mu) \\ &= \frac{1}{\Gamma(m-\delta)}\int_0^\mu (\mu-\tau)^{m-\delta-1}\varphi^{(m)}(\tau)d\tau, \end{aligned} \tag{2}$$

for $m-1 < \alpha \leq m, m \in N, \mu > 0$ and $\varphi \in C_{-1}^m$.

The fundamental properties of the operator D^δ are given as follows:

1. $D^\delta I^\delta \varphi(\mu) = \varphi(\mu)$
2. $D^\delta I^\delta \varphi(\mu) = \varphi(\mu) - \sum_{k=0}^{n-1}\varphi^{(k)}(0)\dfrac{\mu^k}{k!}$

Definition 2.3. *For $\delta > 0$, the gamma function $\Gamma(\cdot)$, is defined as follows:*

$$\Gamma(\delta) = \int_0^\infty x^{\delta-1}e^{-x}dx \tag{3}$$

Definition 2.4. *By considering E_δ with $\delta > 0$, the definition of the Mittag–Leffler function given as the following:*

$$E_\delta(z) = \sum_{m=0}^{\infty}\frac{z^\delta}{\Gamma(m\delta+1)} \tag{4}$$

Some special cases of the Mittag-Leffler function $E_\delta(z)$

1. $E_0(z) = \frac{1}{1-z}, \quad |z| < 1,$
2. $E_1(z) = e^z,$
3. $E_2(z) = cosh\sqrt{z}, \quad z \in C,$
4. $E_2(-z^2) = cosz, \quad z \in C$

Definition 2.5. *The Sumudu transform is identified based on a collection of functions*

$$A = \{\varphi(\tau) : \exists M, \omega_1, \omega_2 > 0,$$
$$with\ |\varphi(\tau)| \leq Me^{\frac{|\tau|}{\omega_j}},\ if \tau \in (-1)^j \times [0,\infty)\}$$

as determined by the formula

$$\mathbb{S}[\varphi(\omega)] = \mathbb{G}(\omega) = \int_0^\infty e^{-\tau}\varphi(\omega\tau)d\tau,\ \ \omega \in (-\omega_1, \omega_2)$$

Some properties of Sumudu Transform

1. $\mathbb{S}[k] = k$, *k constant*
2. $\mathbb{S}\left[\frac{\tau^{m\delta}}{\Gamma(m\delta+1)}\right] = \omega^{m\delta}$

Definition 2.6. *The Caputo fractional derivative for the Sumudu transform is given as the following*

$$\mathbb{S}\left[D_\tau^{\mu\delta}\varphi(\mu,\tau)\right] = \omega^{-\mu\delta}\mathbb{S}\left[\varphi(\mu,\tau)\right] - \sum_{k=0}^{m-1}\omega^{(-\mu\delta+k)}\varphi^{(k)}(\mu,0),$$
$$m-1 < m\delta < m \tag{5}$$

3. Fractional Sumudu Homotopy Perturbation Method ($\mathbb{FSHPM}$)

Consider this generic fractional nonlinear PDEs:

$$\begin{aligned} &D_\tau^{m\delta}\varphi_i(\mu,\tau) + \mathbb{R}\left[\varphi(\mu,\tau)\right] + \mathbb{N}\left[\varphi(\mu,\tau)\right] \\ &= \eth(\mu,\tau), m-1 < m\delta \leq m \end{aligned} \tag{6}$$

with

$$\varphi(\mu,0) = f(\mu), \tag{7}$$

in which $D_\tau^{m\delta}\varphi(\mu,\tau)$ is the Caputo fractional derivative of the function $\varphi(\mu,\tau)$, $\mathbb{R}$ refers for the linear differential operator, while $\mathbb{N}$ refers for a nonlinear system differential operator, and $ð(\mu,\tau)$ is the origin term. Using the $\mathbb{ST}$ to both sides of (6), we get

$$\mathbb{S}[D_\tau^{\mu\delta}\varphi(\mu,\tau)] + \mathbb{S}[\mathbb{R}[\varphi(\mu,\tau)]] + \mathbb{S}[\mathbb{N}[\varphi(\mu,\tau)]] = \mathbb{S}[ð(\mu,\tau)] \tag{8}$$

We obtain by utilizing the $\mathbb{ST}$'s property

$$\mathbb{S}[\varphi(\mu,\tau)] = -\omega^\delta \sum_{k=0}^{m-1} \varphi^{(k)}(\mu,0) + \omega^\delta \mathbb{S}\Big[ð(\mu,\tau)\Big] - \omega^\delta \mathbb{S}\Big[\mathbb{R}[\varphi(\mu,\tau)] + \mathbb{N}[\varphi(\mu,\tau)]\Big] \tag{9}$$

Operating the inverse Sumudu transform on both sides of (9), we get

$$\varphi(\mu,\tau) = \mathbb{S}^{-1}\Big[\omega^\delta \sum_{k=0}^{m-1} \varphi^{(k)}(\mu,0)\Big] + \mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}[ð(\mu,\tau)]\Big] - \mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}\Big[\mathbb{R}[\varphi(\mu,\tau)] + \mathbb{N}[\varphi(\mu,\tau)]\Big]\Big] \tag{10}$$

We implement the HPM:

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau), \tag{11}$$

Thus the nonlinear term may be decomposed as follows:

$$\mathbb{N}[\varphi(\mu,\tau)] = \sum_{n=0}^{\infty} p^n H_n(\varphi_0,\varphi_1,\ldots,\varphi_n), \tag{12}$$

where

$$H_n(\varphi_0,\varphi_1,\ldots,\varphi_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\Big[\mathbb{N}\Big[\sum_{i=0}^{\infty} p^i \varphi_i(\mu,\tau)\Big]\Big]_{p=0}$$

When we substitute (11) and (12) into (10), we obtain

$$\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) = \mathbb{S}^{-1}\Big[\omega^\delta \sum_{k=0}^{m-1} \varphi^{(k)}(\mu,0)\Big] + \mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}[ð(\mu,\tau)]\Big] - p\mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\mathbb{R}\Big[\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)\Big] + \sum_{n=0}^{\infty} p^n H_n\Big]\Big)$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= \mathbb{S}^{-1}\Big[\omega^\delta \sum_{k=0}^{m-1} \varphi^{(k)}(\mu,0)\Big] + \mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}[ð(\mu,\tau)]\Big] \\
P^1 : \varphi_1(\mu,\tau) &= -\mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}\Big[\mathbb{R}[\varphi_0(\mu,\tau)] + H_0\Big]\Big] \\
P^2 : \varphi_2(\mu,\tau) &= -\mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}\Big[\mathbb{R}[\varphi_1(\mu,\tau)] + H_1\Big]\Big] \\
&\vdots \\
P^n : \varphi_n(\mu,\tau) &= -\mathbb{S}^{-1}\Big[\omega^\delta \mathbb{S}\Big[\mathbb{R}[\varphi_{n-1}(\mu,\tau)] + H_{n-1}\Big]\Big], \\
n &\geq 1
\end{aligned} \tag{13}$$

Consequently, we use truncated series to estimate the analytical result $\varphi(\mu,\tau)$

$$\varphi(\mu,\tau) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) \tag{14}$$

4. Applications

This section will put the proposed method for solving time fractional partial differential equations into application.

4.1. Example

Firstly, examine the time-fractional Cauchy reaction–diffusion equation shown below

$$D_\tau^\delta \varphi(\mu,\tau) = \varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau), \quad 0 < \delta \leq 1 \tag{15}$$

with

$$\varphi(\mu,0) = e^{-\mu} + \mu \tag{16}$$

For the (15), applying the Sumudu transform $\mathbb{ST}$ on both sides of it, we achieve

$$\mathbb{S}[\varphi(\mu,\tau)] = \varphi(\mu,0) + \omega^\delta \mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\Big] \tag{17}$$

Using the inverse ST to (17), we obtain

$$\mathbb{S}[\varphi(\mu,\tau)] = e^{-\mu} + \mu + \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\Big]\Big) \tag{18}$$

According to the $\mathbb{HPM}$, and substituting

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)$$

in (18), we have

$$\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) = e^{-\mu} + \mu + p\mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2}{\partial\mu^2}\Big[\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)\Big] - \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)\Big]\Big) \tag{19}$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= e^{-\mu} + \mu \\
P^1 : \varphi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2 \varphi_0(\mu,\tau)}{\partial\mu^2} - \varphi_0(\mu,\tau)\Big]\Big) \\
&= \mathbb{S}^{-1}(\omega^\delta \mathbb{S}[-\mu]) \\
&= -\frac{\mu\tau^\delta}{\Gamma(\delta+1)}, \\
P^2 : \varphi_2(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2 \varphi_1(\mu,\tau)}{\partial\mu^2} - \varphi_1(\mu,\tau)\Big]\Big) \\
&= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[-\frac{\mu\tau^\delta}{\Gamma(\delta+1)}\Big]\Big) \\
&= -\frac{\mu\tau^{2\delta}}{\Gamma(2\delta+1)},
\end{aligned}$$

$$
\begin{aligned}
P^3 : \varphi_3(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_2(\mu,\tau)}{\partial\mu^2} - \varphi_2(\mu,\tau)\Big]\Big) \\
&= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\big[-\frac{\mu\tau^{2\delta}}{\Gamma(2\delta+1)}\big]\Big) \\
&= -\frac{\mu\tau^{3\delta}}{\Gamma(3\delta+1)}, \\
P^4 : \varphi_4(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_3(\mu,\tau)}{\partial\mu^2} - \varphi_3(\mu,\tau)\Big]\Big) \\
&= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\big[-\frac{\mu\tau^{3\delta}}{\Gamma(3\delta+1)}\big]\Big) \\
&= -\frac{\mu\tau^{4\delta}}{\Gamma(4\delta+1)}, \\
&\vdots \\
P^n : \varphi_n(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_{n-1}(\mu,\tau)}{\partial\mu^2} - \varphi_{n-1}(\mu,\tau)\Big]\Big) \\
&= (-1)^n\frac{\mu\tau^{n\delta}}{\Gamma(n\delta+1)}
\end{aligned}
$$

Hence, the outcome of (15) is provided by

$$\varphi(\mu,\tau) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)$$

$$= e^{-\mu} + \mu\Big[1 - \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \frac{\tau^{3\delta}}{\Gamma(3\delta+1)}\cdots + (-1)^n\frac{\tau^{n\delta}}{\Gamma(n\delta+1)}\Big]$$

$$= e^{-\mu} + \mu E_\delta(\tau^\delta) \tag{20}$$

For $\delta = 1$, the (20) is close to the form $\varphi(\mu,\tau) = e^{-\mu} + \mu e^{-\tau}$, which is the precise solution of (15) for $\delta = 1$. The outcome is same as with $\mathbb{FSDJM}$ [53].

4.2. Example

We determine the most important time fractional Cauchy reaction-diffusion equation:

$$D_\tau^\delta\varphi(\mu,\tau) = \varphi_{\mu\mu}(\mu,\tau) - (1+4\mu^2)\varphi(\mu,\tau), \quad 0 < \delta \le 1 \tag{21}$$

with

$$\varphi(\mu,0) = e^{\mu^2} \tag{22}$$

We obtain by utilizing the $\mathbb{ST}$'s property on both sides of (21), we get

$$\mathbb{S}[\varphi(\mu,\tau)] = \varphi(\mu,0) + \omega^\delta\mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - (1+4\mu^2)\varphi(\mu,\tau)\Big] \tag{23}$$

Using the inverse $\mathbb{ST}$ to (23), we have

$$\mathbb{S}[\varphi(\mu,\tau)] = e^{\mu 2} + \mathbb{S}^{-1}\Big(\omega^\delta\mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - (1+4\mu^2)\varphi(\mu,\tau)\Big]\Big) \tag{24}$$

According to the $\mathbb{HPM}$, and substituting

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)$$

in (24), we have

$$\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau) = e^{\mu^2} + p\mathbb{S}^{-1}\Big(\omega^\delta\mathbb{S}\Big[\frac{\partial^2}{\partial\mu^2}\big[\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)\big] - (1+4\mu^2)\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)\Big]\Big) \tag{25}$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$
\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= e^{\mu^2} \\
P^1 : \varphi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_0(\mu,\tau)}{\partial\mu^2} - (1+4\mu^2)\varphi_0(\mu,\tau)\Big]\Big) = \frac{\tau^\delta}{\Gamma(\delta+1)}e^{\mu^2}, \\
P^2 : \varphi_2(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_1(\mu,\tau)}{\partial\mu^2} - (1+4\mu^2)\varphi_1(\mu,\tau)\Big]\Big) \\
&= \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}e^{\mu^2}, \\
P^3 : \varphi_3(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_2(\mu,\tau)}{\partial\mu^2} - (1+4\mu^2)\varphi_2(\mu,\tau)\Big]\Big) \\
&= \frac{\tau^{3\delta}}{\Gamma(3\delta+1)}e^{\mu^2}, \\
P^4 : \varphi_4(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_3(\mu,\tau)}{\partial\mu^2} - (1+4\mu^2)\varphi_3(\mu,\tau)\Big]\Big) \\
&= \frac{\tau^{4\delta}}{\Gamma(4\delta+1)}e^{\mu^2}, \\
&\vdots \\
P^n : \varphi_n(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_{n-1}(\mu,\tau)}{\partial\mu^2} - (1+4\mu^2)\varphi_{n-1}(\mu,\tau)\Big]\Big) \\
&= \frac{\tau^{n\delta}}{\Gamma(n\delta+1)}e^{\mu^2}
\end{aligned}
$$

Hence, the outcome of (21) is provided by

$$\varphi(\mu,\tau) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n\varphi_n(\mu,\tau)$$

$$= e^{\mu^2}\Big[1 + \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \frac{\tau^{3\delta}}{\Gamma(3\delta+1)}\cdots +$$

$$\frac{\tau^{n\delta}}{\Gamma(n\delta+1)}\Big]$$

$$= e^{\mu^2} E_\delta(\tau^\delta) \tag{26}$$

For $\delta = 1$, the (26) is close to the form $\varphi(\mu,\tau) = e^{\mu^2+\tau}$, which is the precise solution of (21) for $\delta = 1$. The outcome is same as with $\mathbb{FSDJM}$ [53].

4.3. Example

Consider the nonlinear fractional Cauchy reaction–diffusion equation is given as the following

$$D^\delta_\tau \varphi(\mu,\tau) = \varphi_{\mu\mu}(\mu,\tau) - \varphi_\mu(\mu,\tau) + \varphi(\mu,\tau)\varphi_{\mu\mu}(\mu,\tau) \\ -\varphi^2(\mu,\tau) + \varphi(\mu,\tau), \quad 0 < \delta \leq 1 \tag{27}$$

w.r.t initial condition

$$\varphi(\mu,0) = e^\mu \tag{28}$$

Operating the ST on both sides of (27), and employing $\mathbb{ST}$'s differential property, we obtain

$$\mathbb{S}[\varphi(\mu,\tau)] = \varphi(\mu,0) + \omega^\delta \mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - \varphi_\mu(\mu,\tau) \\ +\varphi(\mu,\tau)\varphi_{\mu\mu}(\mu,\tau) - \varphi^2(\mu,\tau) + \varphi(\mu,\tau)\Big] \tag{29}$$

We obtain by applying the inverse Sumudu transform on both sides of (29)

$$\varphi(\mu,\tau) = e^\mu + \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\varphi_{\mu\mu}(\mu,\tau) - \varphi_\mu(\mu,\tau) + \\ \varphi(\mu,\tau)\varphi_{\mu\mu}(\mu,\tau) - \varphi^2(\mu,\tau) + \varphi(\mu,\tau)\Big]\Big) \tag{30}$$

According to the $\mathbb{HPM}$, and substituting

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau)$$

$$\varphi\varphi_{\mu\mu} = \sum_{n=0}^{\infty} p^n H_n$$

$$\varphi^2 = \sum_{n=0}^{\infty} p^n G_n$$

in (30), we have

$$\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) =$$

$$e^\mu + p\mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2}{\partial\mu^2}\Big[\sum_{n=0}^{\infty} p^n\varphi_n\Big] - \frac{\partial}{\partial\mu}\Big[\sum_{n=0}^{\infty} p^n\varphi_n\Big] \\ + \sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n G_n - \sum_{n=0}^{\infty} p^n\varphi_n\Big]\Big) \tag{31}$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= e^\mu, \\
P^1 : \varphi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_0}{\partial\mu^2} - \frac{\partial\varphi_0}{\partial\mu} + H_0 - G_0 \\
&\quad +\varphi_0\Big]\Big) = \frac{\tau^\delta}{\Gamma(\delta+1)} e^\mu, \\
P^2 : \varphi_2(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_1}{\partial\mu^2} - \frac{\partial\varphi_1}{\partial\mu} + H_1 - G_1 \\
&\quad +\varphi_1\Big]\Big) = \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} e^\mu, \\
P^3 : \varphi_3(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_2}{\partial\mu^2} - \frac{\partial\varphi_2}{\partial\mu} + H_2 - G_2 \\
&\quad +\varphi_2\Big]\Big) = \frac{\tau^{3\delta}}{\Gamma(3\delta+1)} e^\mu, \\
P^4 : \varphi_4(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_3}{\partial\mu^2} - \frac{\partial\varphi_3}{\partial\mu} + H_3 - G_3 \\
&\quad +\varphi_3\Big]\Big) = \frac{\tau^{4\delta}}{\Gamma(4\delta+1)} e^\mu,
\end{aligned} \tag{32}$$

$$\vdots$$

$$P^n : \varphi_n(\mu,\tau) = \mathbb{S}^{-1}\Big(\omega^\delta \mathbb{S}\Big[\frac{\partial^2\varphi_{n-1}}{\partial\mu^2} - \frac{\partial\varphi_{n-1}}{\partial\mu} \\ +H_{n-1} - G_{n-1} + \varphi_{n-1}\Big]\Big) \\ = \frac{\tau^{n\delta}}{\Gamma(n\delta+1)} e^\mu$$

Hence, the outcome of (27) is provided by

$$\begin{aligned}
\varphi(\mu,\tau) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) \\
&= e^\mu\Big[1 + \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \cdots + \frac{\tau^{n\delta}}{\Gamma(n\delta+1)}\Big] \\
&= e^\mu E_\delta(\tau^\delta)
\end{aligned} \tag{33}$$

For $\delta = 1$, the (33) is close to the form $\varphi(\mu,\tau) = e^{\mu+\tau}$, which is the precise solution of (27) for $\delta = 1$. The outcome is same as with $\mathbb{FSDJM}$ [53].

4.4. Example

Assume the coupled fractional Burger's equations shown below where $0 < \delta \leq 1, 0 < \gamma \leq 1$

$$D^\delta_\tau \varphi(\mu,\tau) - \varphi_{\mu\mu} - 2\varphi\varphi_\mu + (\varphi\psi)_\mu = 0, \\ D^\delta_\tau \psi(\mu,\tau) - \psi_{\mu\mu} - 2\psi\psi_\mu + (\varphi\psi)_\mu = 0 \tag{34}$$

with

$$\varphi(\mu,0) = e^\mu \\ \psi(\mu,0) = e^\mu \tag{35}$$

Operating the $\mathbb{ST}$ on both sides of (34), and employing $\mathbb{ST}$'s differential property, we obtain

$$\mathbb{S}[\varphi(\mu,\tau)] = \varphi(\mu,0) + \omega^{\delta}\mathbb{S}[\varphi_{\mu\mu} + 2\varphi\varphi_{\mu} - (\varphi\psi)_{\mu}],$$
$$\mathbb{S}[\psi(\mu,\tau)] = \psi(\mu,0) + \omega^{\gamma}\mathbb{S}[\psi_{\mu\mu} + 2\psi\psi_{\mu} - (\varphi\psi)_{\mu}] \quad (36)$$

Implementing with the inverse $\mathbb{ST}$ to (36), we have

$$\varphi(\mu,\tau) = e^{\mu} + \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}[\varphi_{\mu\mu} + 2\varphi\varphi_{\mu} - (\varphi\psi)_{\mu}]\Big),$$
$$\psi(\mu,\tau) = e^{\mu} + \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}[\psi_{\mu\mu} + 2\psi\psi_{\mu} - (\varphi\psi)_{\mu}]\Big) \quad (37)$$

Assume that

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) \quad (38)$$

$$\psi = \sum_{n=0}^{\infty} p^n \psi_n \quad (39)$$

$$\varphi\varphi_{\mu} = \sum_{n=0}^{\infty} p^n H_n \quad (40)$$

$$\psi\psi_{\mu} = \sum_{n=0}^{\infty} p^n K_n \quad (41)$$

$$(\varphi\psi)_{\mu} = \sum_{n=0}^{\infty} p^n G_n \quad (42)$$

By applying the $\mathbb{HPM}$, and substituting (38)-(42) in (37), we get

$$\sum_{n=0}^{\infty} p^n \varphi_n(\mu,\tau) = e^{\mu} + p\mathbb{S}^{-1}\Bigg(\omega^{\delta}\mathbb{S}\Bigg[\frac{\partial^2}{\partial\mu^2}\Big[\sum_{n=0}^{\infty} p^n\varphi_n\Big] + 2\sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n G_n\Bigg]\Bigg),$$

$$\sum_{n=0}^{\infty} p^n \psi_n(\mu,\tau) = e^{\mu} + p\mathbb{S}^{-1}\Bigg(\omega^{\gamma}\mathbb{S}\Bigg[\frac{\partial^2}{\partial\mu^2}\Big[\sum_{n=0}^{\infty} p^n\psi_n\Big] + 2\sum_{n=0}^{\infty} p^n K_n - \sum_{n=0}^{\infty} p^n G_n\Bigg]\Bigg) \quad (43)$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= e^{\mu},\\
: \psi_0(\mu,\tau) &= e^{\mu}\\
P^1 : \varphi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}\Big[\frac{\partial^2\varphi_0}{\partial\mu^2} + 2H_0 - G_0\Big]\Big)\\
: \psi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}\Big[\frac{\partial^2\psi_0}{\partial\mu^2} + 2K_0 - G_0\Big]\Big)\\
P^2 : \varphi_2(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}\Big[\frac{\partial^2\varphi_1}{\partial\mu^2} + 2H_1 - G_1\Big]\Big)\\
: \psi_2(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}\Big[\frac{\partial^2\psi_1}{\partial\mu^2} + 2K_1 - G_1\Big]\Big)\\
P^3 : \varphi_3(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}\Big[\frac{\partial^2\varphi_2}{\partial\mu^2} + 2H_2 - G_2\Big]\Big)\\
: \psi_3(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}\Big[\frac{\partial^2\psi_2}{\partial\mu^2} + 2K_2 - G_2\Big]\Big)\\
&\vdots
\end{aligned}$$

Then, we have

$$\begin{aligned}
P^0 : \varphi_0(\mu,\tau) &= e^{\mu},\\
: \psi_0(\mu,\tau) &= e^{\mu}\\
P^1 : \varphi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}\Big[e^{\mu} + 2e^{2\mu} - 2e^{2\mu}\Big]\Big)\\
: \psi_1(\mu,\tau) &= \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}\Big[e^{\mu} + 2e^{2\mu} - 2e^{2\mu}\Big]\Big)
\end{aligned}$$

$$= \mathbb{S}^{-1}(\omega^{\delta}e^{\mu}) = \frac{\tau^{\delta}}{\Gamma(\delta+1)}e^{\mu}$$
$$= \mathbb{S}^{-1}(\omega^{\gamma}e^{\mu}) = \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}e^{\mu}$$

$$\begin{aligned}
P^2 : \varphi_2(\mu,\tau) = \mathbb{S}^{-1}\Big(\omega^{\delta}\mathbb{S}\Big[&\frac{\tau^{\delta}}{\Gamma(\delta+1)}e^{\mu} + 2\frac{\tau^{\delta}}{\Gamma(\delta+1)}e^{2\mu} - 2\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}e^{2\mu}\Big]\Big)\\
: \psi_2(\mu,\tau) = \mathbb{S}^{-1}\Big(\omega^{\gamma}\mathbb{S}\Big[&\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}e^{\mu} + 2\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}e^{2\mu} - 2\frac{\tau^{\delta}}{\Gamma(\delta+1)}e^{2\mu}\Big]\Big)\\
&= \mathbb{S}^{-1}(\omega^{2\delta}e^{\mu} + 2\omega^{2\delta}e^{2\mu} - 2\omega^{\delta+\gamma}e^{2\mu})\\
&= \mathbb{S}^{-1}(\omega^{2\gamma}e^{\mu} + 2\omega^{2\gamma}e^{2\mu} - 2\omega^{\delta+\gamma}e^{2\mu})\\
&= \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}e^{\mu} + 2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}e^{2\mu} - 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}e^{2\mu}\\
&= \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}e^{\mu} + 2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}e^{2\mu} - 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}e^{2\mu}\\
&\vdots
\end{aligned} \quad (44)$$

Thus, the outcome of (34) is given by

$$\varphi(\mu,\tau) = e^{\mu}\Big[1 - \frac{\tau^{\delta}}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\dots\Big] + e^{2\mu}\Big[2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)} - 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\dots\Big]$$
$$\psi(\mu,\tau) = e^{\mu}\Big[1 - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\dots\Big] + e^{2\mu}\Big[2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} - 2\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\dots\Big]$$

(45)

Setting $\delta = \gamma$ in (45), we obtain

$$\varphi(\mu,\tau) = e^{\mu}\left[1 - \frac{\tau^{\delta}}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\dots\right] = E_{\delta}(-\tau^{\delta})e^{\mu}$$

$$\psi(\mu,\tau) = e^{\mu}\left[1 - \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\dots\right] = E_{\gamma}(-\tau^{\gamma})e^{\mu} \tag{46}$$

The (46) is close to the form $\varphi(\mu,\tau) = \psi(\mu,\tau) = e^{\mu-\tau}$ for $\delta = \gamma = 1$, which is the precise solution of (34) for $\delta = \gamma = 1$.

5. Conclusion

The Sumudu homotopy perturbation approach was effectively used in this work to discover approximate solutions to time-fractional partial differential equations. The analytical approach generates a convergence analysis that fast converges to the optimal solution. The simplicity and high precision of the analytical method are clearly illustrated, solving equations includes linear and nonlinear fractional PDEs and a nonlinear system of fractional PDEs.

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