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A Pursuit Differential Game Problem on a Closed Convex Subset of a Hilbert Space

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Abstract

In this paper, we study a pursuit differential game problem with finite number of pursuers and one evader on a nonempty closed convex subset of the Hilbert space l_2. Players move according to certain first order ordinary differential equations and control functions of the pursuers and evader are subject to integral constraints. Pursuers win the game if the geometric positions of a pursuer and the evader coincide. We formulate and prove theorems that are concerned with conditions that ensure win for the pursuers. Consequently, winning strategies of the pursuers are constructed. Furthermore, illustrative example is given to demonstrate the result.

Keywords: Pursuit, integral constraint, closed convex set.

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1. Introduction

Differential game has been an area of great interest to many Applied Mathematicians due its numerous applications in solving many real life problems. it was birthed as a result of inter-field research activities in game theory and optimal control.. Thus, many research articles have been devoted to this field and a lot of results were published (see for example, [1-16]).

In some of these research works, players move according to the following differential equations:

$$\begin{cases} \dot{x} = a(t)u, & x(0) = x_0, \\ \dot{y} = b(t)v, & y(0) = y_0 \end{cases} \qquad (1)$$

where $u(t)$ *and* $v(t)$ are control functions of the players which are either subject to integral or geometrical constraints and $a(\cdot)$, $b(\cdot)$ are scalar measurable functions.

The problems considered in [1, 7, 12, 13, 18, 19] involve players' motion described by the differential equations (1), where $a(t) = b(t) = 1$. Whereas, in the problems considered in [6, 8, 9, 10, 14, 17], players move according to the differential equations (1), where $a(t) = b(t) \neq 1$. Problem in which players move according to (1), with $a(t) \neq b(t)$ are investigated in [2]. In this work, control functions of the players are subject to integral constraints. Optimal strategies of the players are constructed and value of the game is found.

In all of the above cited works, only in [1, 8, 10, 18] constraints on the state variables are considered. The paper [1] reports study of pursuit problem on a closed convex subset of R^n with Control functions of players subject to coordinate-wise in-

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tegral constraints. Some conditions under which pursuit can be completed from any position of the players in the given set are obtained. Moreover, strategies for the pursuers are constructed.

The work [8] is concerned with pursuit problem in which players control functions are subject to integral constraints. Players are not allowed to move out of a closed convex subset of R^n. Optimal time of pursuit is found and optimal strategies for the players are constructed. Ibragimov and Satimov in [10] studied pursuit differential game problem on a nonempty convex subset of R^n. In the game, both pursuers and evaders are not allowed to leave the given set and their control functions subject to integral constraints. Sufficient conditions for completion of pursuit are obtained.

A differential pursuit problem of an evader by finite number of pursuers on a closed convex set of l_2, was studied by Leong and Ibragimov in [18]. The authors showed that an evading player cannot avoid an exact contact with any finite number of pursuing players whose individual resources are less than that of this evading players.

In the present paper, pursuit differential game problem with finite number of pursuers and one evader on a nonempty closed convex subset of l_2 is investigated. Control functions of the pursuers and evader are subject to integral constraints. During the game players are to stay within a closed convex subset of l_2. Players move according to (1), where $a(t) \neq b(t)$.

2. Statement of the Problem

Consider the Hilbert space

$$l_2 = \left\{ \alpha = (\alpha_1, \alpha_2, ..., \alpha_k ...,) \mid \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\},$$

with the inner product

$$(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$$

and norm $\|\alpha\| = \left(\sum_{k=1}^{\infty} \alpha_k^2\right)^{1/2}$. Let N be closed convex subset of l_2 and define a ball in l_2, with center at x_0 and radius r by $H(x_0, r) := \{x \in l_2 : \|x - x_0\| \leq r\}$. Let $C(0, \theta; l_2)$ denotes space of continuous functions $f(t) = (f_1(t), f_2(t), \dots)$ defined on the interval $[0, \theta]$ with absolutely continuous coordinates. The following definitions are important in the paper:

Definition 1. *Let (X, μ_1) and (Y, μ_2), where μ_1 and μ_2 are σ-algebra on X and Y respectively, be a measurable spaces . The function $f : (X, \mu_1) \to (Y, \mu_2)$ is Borel measurable if $f^{-1}(E) \in \mu_1$ for all $E \in \mu_2$ [20].*

Definition 2. *Let A be a subset of a Hilbert space X , if every point of the Hilbert space has exactly one projection onto A, then A is a Chebyshev set [3].*

Definition 3. *Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear map, then T is said to be Lipschitz, if there exists a constant $K > 0$ such that, for each $x \in X$ $\|Tx\| \leq K\|x\|$ [4].*

We define a pursuit differential game problem in which countable but finite number of pursuers P_j, $j \in J = \{1, 2, \dots, m\}$ and evader E move according to the following equations:

$$\begin{cases} P_j : \dot{x}_j = a(t)u_j, \ x_j(0) = x_{j0}, \\ E : \dot{y} = \ b(t)v, \quad y(0) = y_0, \end{cases} \tag{2}$$

where $x_j, x_{j0}, u_j, y, y_0, v \in l_2, u_j = (u_{j1}, u_{j2}, \dots)$ is a control parameter of the pursuer P_j and $v = (v_1, v_2, \dots)$ is that of the evader E. Additionally, $a(t)$ and $b(t)$ are scalar measurable functions such that $1 \leq b(t) \leq a(t)$ for all $t \in [0, \theta]$. The positive number θ is denoting duration of the game.

Definition 4. *A Borel measurable function $u_j(\cdot)$; $u_j : [0, \theta] \to H(0, \rho_j)$ such that*

$$\left(\int_0^{\infty} \|u_j(t)\|^2 dt \right)^{1/2} \leq \rho_j, \tag{3}$$

is called ***admissible control of the pursuer*** *P_j.*

Definition 5. *A Borel measurable function $v(\cdot)$; $v : [0, \theta] \to H(0, \sigma)$ such that*

$$\left(\int_0^{\infty} \|v(t)\|^2 dt \right)^{1/2} \leq \sigma, \tag{4}$$

is called ***admissible control of the evader****.*

Definition 6. *A function $U(t, x_j, y, v)$, $U_j : [0, \theta) \times l_2 \times l_2 \times H(0, \sigma) \to H(0, \rho_j)$, such that the system*

$$\begin{cases} \dot{x}_j = U_j(t), \ x_j(0) = x_{0j}, \\ \dot{y} = \ v(t), \ \ y(0) = y_0, \end{cases}$$

has a unique solution $(x_j(\cdot), y(\cdot))$ with $x_j(\cdot), y(\cdot) \in C(0, \theta; l_2)$, for an arbitrary admissible control $v = v(t)$, $0 \leq t \leq \theta$, of the evader E is called strategy of the pursuer P_j. A strategy U_j is said to be admissible if each control formed by this strategy is admissible.

In what follows, we refer the game described by (2) in which the control functions $u(\cdot)$ and $v(\cdot)$ satisfying (3) and (4) respectively, as game $G1$.

Definition 7. *Pursuers win the game $G1$, if there exist pursuer's strategy U_j that ensure the equality $x_j(\theta) = y(\theta)$ for some $j \in J$.*

When the pursuer P_j and evader E use admissible controls $u_j(t) = (u_{j1}(t), u_{j2}(t), \dots)$ and $v(t) = (v_1(t), v_2(t), \dots)$ respectively, then from (2) their corresponding motions is given by

$$x_j(t) = (x_{j1}(t), x_{j2}(t), \dots), \quad y(t) = (y_1(t), y_2(t), \dots),$$

where

$$x_j(t) = (x_{j1}(t), x_{j2}(t), \dots, x_{jk}(t), \dots),\ \ x_{jk}(t) = x_{j0k} + \int_0^t a(t)u_{jk}(s)ds;$$

$$y(t) = (y_1(t), y_2(t), \dots, y_k(t), \dots),\ \ y_k(t) = y_{k0} + \int_0^t b(t)v_k(s)ds.$$

It can easily be shown that $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$.

Problem. *What are the sufficient conditions for pursuer to win the game $G1$?*

3. Results

To present our result, we need the following notations: $\rho^2 := \sum_{j=1}^{m} \rho_j^2$ and $\sigma_j := \frac{\sigma}{\rho}\rho_j,\ for\ \ j \in J$. Then, it is easy to see that $\sigma_j^2 < \rho_j^2$ and $\sigma^2 = \sum_{j=1}^m \sigma_j^2$.
The following Lemma is useful in the presentation of our results

Lemma 8. *Let $1 \le p \le \infty$. If $f, g \in L^p(X,\mu)$ then $f + g \in L^p(X,\mu)$ and $\|f+g\| \le \|f\|_p + \|g\|_p$ [5].*

The theorem below gives sufficient conditions for pursuers to win the game when the players move freely without state constraint in the space l_2.

Theorem 1. . *If $\rho_j^2 < \sigma^2$ for all $j \in J$ and $\sum_{j=1}^{m} \rho_j^2 > \sigma^2$, then for any initial positions of the players, pursuers win the game $G1$.*

Proof:
If $y_0 = x_{j0}$, for some $j \in J$, then the proof is trivial. Therefore, let $y_0 \neq x_{j0}$, for all $j \in J$. We construct the strategies of the pursuers as follows:

In the first phase of the game, we allow only one pursuer to move using a define strategy. Without loss of generality, we allow the first pursuer to move and others to stay static. That is, the pursuers to use the strategies define by

$$\begin{cases} u_1(t) = \frac{y_0 - x_{10}}{a(t)\theta_1} + \frac{b(t)}{a(t)}v(t), & j = 1 \\ u_j(t) = 0, & j \neq 1, \end{cases} \tag{5}$$

where x_{10} and y_0 is the initial position of the first pursuer and the evader respectively; $\theta_1 = \left(\frac{\|y_0 - x_{10}\|}{\rho_1 - \sigma_1}\right)^2$. Now we show that the strategy (5) is admissible whenever $\int_0^{\theta_1} \|v(t)\|^2 dt \le \sigma_1$. Indeed,

$$\begin{aligned}
\left(\int_0^{\theta_1} \|u_1(t)\|^2 dt\right)^{1/2} &= \left(\int_0^{\theta_1} \left\|\frac{y_0 - x_{10}}{a(t)\theta_1} + \frac{b(t)}{a(t)}v(t)\right\|^2 dt\right)^{1/2} \\
&\le \left(\int_0^{\theta_1} \left\|\frac{y_0 - x_{10}}{a(t)\theta_1}\right\|^2 dt\right)^{1/2} + \left(\int_0^{\theta_1} \left\|\frac{b(t)}{a(t)}v(t)\right\|^2 dt\right)^{1/2} \\
&\le \left(\int_0^{\theta_1} \frac{\|y_0 - x_{10}\|^2}{a^2(t)\theta_1{}^2} dt\right)^{1/2} + \left(\int_0^{\theta_1} \|v(t)\|^2 dt\right)^{1/2} \\
&\le \frac{\|y_0 - x_{10}\|}{\theta_1}\left(\int_0^{\theta_1} \frac{1}{a^2(t)} dt\right)^{1/2} + \sigma_1 \\
&\le \frac{\|y_0 - x_{10}\|}{\theta_1}\left(\int_0^{\theta_1} dt\right)^{1/2} + \sigma_1 \\
&= \frac{\|y_0 - x_{10}\|}{\theta_1^{1/2}} + \sigma_1 \\
&= \frac{\|y_0 - x_{10}\|}{\left(\left(\frac{\|y_0 - x_{10}\|}{\rho_1 - \sigma_1}\right)^2\right)^{1/2}} + \sigma_1 \\
&\le \rho_1 - \sigma_1 + \sigma_1 = \rho_1.
\end{aligned}$$

Now if the pursuer P_1 uses the strategy (5) then,

$$\begin{aligned}
x_1(\theta_1) - y(\theta_1) &= x_{10} - y_0 + \int_0^{\theta_1} a(t)u_1(t)dt - \int_0^{\theta_1} b(t)v(t)dt \\
&= x_{10} - y_0 + \int_0^{\theta_1} a(t)\left(\frac{y_0 - x_{10}}{a(t)\theta_1} + \frac{b(t)}{a(t)}v(t)\right)dt \\
&\quad - \int_0^{\theta_1} b(t)v(t)dt \\
&= x_{10} - y_0 + \int_0^{\theta_1} \frac{y_0 - x_{10}}{\theta_1}dt + \int_0^{\theta_1} b(t)v(t)dt \\
&\quad - \int_0^{\theta_1} b(t)v(t)dt \\
&= x_{10} - y_0 + \frac{y_0 - x_{10}}{\theta_1}\theta_1 \\
&= x_{10} - y_0 + y_0 - x_{10} = 0.
\end{aligned}$$

This means that $x_1(\theta_1) = y(\theta_1)$ and implies that pursuers win the game. However, if $x_1(t) \neq y(t)$ for all $t \in [0, \theta_1]$, then energy expanded by the evader in the time interval $[0, \theta_1]$ is more than the energy of the first pursuer. That is

$$\int_0^{\theta_1} \|v(t)\|^2 dt > \rho_1^2.$$

But from the fact that $\rho_1^2 > \sigma_1^2$, then we have

$$\int_0^{\theta_1} \|v(t)\|^2 dt > \sigma_1^2. \tag{6}$$

If the first pursuer cannot win the game for the pursuers then the game will proceed into the second phase. In the second phase, the second pursuer will move using the strategy similar to (5) and the remaining pursuers $P_j,\ j = 1, 3, 4, \dots, m$ remain static. In general, during the k^{th} phase of the game, pursuers $P_j,\ j = 1, 2, ...k-1, k+1, \dots, m$, remain static and only the k^{th} pursuer moves. That is, the strategies of the pursuers in the k^{th} phase of the game are given by

$$\begin{cases} u_k(t) = \frac{y(\tau_{k-1}) - x_{k0}}{a(t)\theta_k} + \frac{b(t)}{a(t)}v(t) & j = k \\ u_j(t) = 0, & j \neq k; \end{cases} \tag{7}$$

where $t \in [\tau_{k-1}, \tau_k], k = 1, 2, \ldots, m, \tau_0 = 0$; $\tau_k = \sum_{j=1}^{k} \theta_j$, and $\theta_k = \left(\frac{\|y(\tau_{k-1}) - x_{k0}\|}{\rho_k - \sigma_k}\right)^2$. Again, for any admissible control of the evader $v(\cdot)$ the constructed strategy (5) is admissible and pursuers win the game.

Now, if in each phase of the game the equation $x_j(t) = y(t)$ does not hold for some $t \in [0, \tau_m]$, then the following inequalities must hold

$$\int_0^{\tau_1} \|v(t)\|^2 dt > \sigma_1^2, \ \int_{\tau_1}^{\tau_2} \|v(t)\|^2 dt > \sigma_2^2, ..., + \int_{\tau_{m-1}}^{\tau_m} \|v(t)\|^2 dt > \sigma_m^2.$$

Consequently,

$$\int_0^{\infty} \|v(t)\|^2 dt \geq \int_0^{\tau_1} \|v(t)\|^2 dt + \int_{\tau_1}^{\tau_2} \|v(t)\|^2 dt + \ldots + \int_{\tau_{m-1}}^{\tau_m} \|v(t)\|^2 dt > \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_m^2 = \sigma^2.$$

This contradicts (4). Therefore, we must have the equality $x_j(t) = y(t)$ holding for some $j \in J$ and for some $t \in [0, \tau_m]$. This completes the proof. Now, consider the game $G1$ in which $x_j, x_0, y, y_0 \in N \subset l_2$ and all player cannot move out of the closed convex set N. Then we have the following theorem:

Theorem 2. *If $\rho_j^2 < \sigma^2$ for all $j \in J$ and $\sum_{j=1}^{m} \rho_j^2 > \sigma^2$, then for any initial positions of the players, pursuers win the game $G1$.*

Proof: For purpose of the proof of this theorem, we introduce m number of dummy pursuers $\bar{P}_j, j = 1, ..., m$ which moves according to the following their equations

$$\bar{P}_j \ : \ \dot{w} = a(t)\bar{u}_j(t), j = 1, ..., m; \ w_j(0) = w_{j0}. \tag{8}$$

where the controls $\bar{u}_j$ is such that

$$\int_0^{\infty} \|\bar{u}_j(t)\|^2 dt \leq \rho_j^2.$$

The dummy pursuers has no restriction in their movements. That is, they can move outside the set N. Therefore, dummy pursuers can win the game if we define the strategy of each of the dummy pursuer $\bar{P}_j, j \in I$ (same with the pursuers' strategies define in the proof of theorem (1) as follows:

$$\bar{u}_j(t) = \frac{y(\tau_{j-1}) - z_{j0}}{a(t)\theta_j} + \frac{b(t)}{a(t)} v(t), \ \ \tau_{j-1} \leq t \leq \tau_j;$$

$$\bar{u}_k(t) = 0, \forall k = 1, 2, ..., j-1, j+1, ..., m.$$

Define by $F_N(x)$ the projection of a point $x \in l_2$ onto N :

$$|x - F_N(x)| = \min_{y \in N} |x - y|.$$

Since N is closed convex subset of l_2, it follows that N is a Chebyshev set and the inequality

$$|F_N(x) - F_N(y)| \leq |x - y| \tag{9}$$

holds for any $x, y \in \ l_2$, that is $F_N(\cdot)$ is Lipschitz continuous with 1. The operator $F_N(\cdot)$ associates each of the absolute continuous function $w_j(t), 0 \leq t \leq \tau_m$, to an absolute continuous function $x_j(t)$. That is, $F_N : l_2 \to N$, such that

$$F_N(w_j(t)) = \begin{cases} x_j(t) & if \ \ w_j(t) \neq N, \\ w_j(t) = x_j(t), & if \ \ w_j(t) \in N, \end{cases} \tag{10}$$

where $t \in [0, \tau_m]$. We define the strategies of the real pursuers by

$$u_j(t) = \begin{cases} \bar{u}_j(t), & if \ \ w_j(t) \in N \\ \frac{1}{a(t)} \dot{F}(w_j(t)), & if \ \ w_j(t) \notin N. \end{cases} \tag{11}$$

We now show that this strategy ensure win for the pursuers and is admissible. By theorem (1) we can infer that the equality $w_i(t^*) = y(t^*)$ holds for some time $t^* \in [0, \tau_m]$, for an index $i \in I$. Since $y(t) \in N$, then using (10) yields $x_i(t^*) = F_N(w_i(t^*)) = w_i(t^*) = y(t^*)$.

Lastly, we show the admissibility of the strategy (11). Indeed, if $w_j(t) \in N$ then

$$\int_0^{\infty} \|u_j(t)\|^2 dt = \int_{\tau_{j-1}}^{\tau_j} \|\bar{u}_j(t)\|^2 dt \leq \rho_j^2.$$

In the other hand, we use the fact that $F_N(w_j(\cdot)) = x_j(\cdot) \in N$; N is closed convex and the inequality (9) to deduce that

$$\|F_N(w_j(t+h)) - F_N(w_j(t)\| \leq \|w_j(t+h)) - w_j(t)\|.$$

In view of this inequality and for $w_j(t) \notin N$, we have

$$\begin{aligned} \int_0^{\infty} \|u_j(t)\|^2 dt &= \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \|\dot{F}(w_j(t))\|^2 dt \\ &= \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \left\| \lim_{h \to 0} \ \frac{F_N(w_j(t+h)) - F_N(w_j(t)}{h} \right\|^2 dt \\ &= \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \lim_{h \to 0} \ \frac{\|F_N(w_j(t+h)) - F_N(w_j(t)\|^2}{h^2} dt \\ &\leq \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \left\| \lim_{h \to 0} \ \frac{w_j(t+h) - w_j(t)}{h} \right\|^2 dt \\ &= \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \|\dot{w}_j(t))\|^2 dt = \int_{\tau_{j-1}}^{\tau_j} \frac{1}{a^2(t)} \|a(t)\bar{u}_j(t)\|^2 dt \\ &= \int_{\tau_{j-1}}^{\tau_j} \|\bar{u}_j(t)\|^2 dt \leq \rho_j^2. \end{aligned}$$

4. Illustrative Example

Consider the game in which five dynamical object(pursuers) versus one dynamical object(evader) confined in a circular arena with radius 3 and whose dynamics described by

$$\begin{cases} P_j : \dot{x}_j = (2+t)u_j,\ x_j(0) = x_{j0}, j = 1,2,3. \\ E : \dot{y} = \ (1+t)v, \quad y(0) = y_0, \end{cases}$$

where $x_{10} = (1,0,0,\dots)$; $x_{20} = (0,1,0.\dots)$; $x_{30} = (0,0,1,\dots)$; $y_0 = (0,0,0,\dots)$; $\rho_1 = 3,\ \ \rho_2 = 4,\ \ \rho_3 = 5, \sigma = 7$. Observe that $1 < (1+t) < (2+t); \rho_j^2 < \sigma^2,\ \ \forall j = 1,...,3$ and $\sum_{j=1}^{3} \rho_j^2 > \sigma^2$. This means that hypothesis of our theorems is satisfied, therefore pursuers win this game.

5. Conclusion

We studied pursuit problem in a closed convex subset of a Hilbert space in which energy of each of pursuer is less than that of the evader. We were able to show that pursuers win the game when the total energy of the pursuers is greater than that of the single evader. Furthermore, pursuers' wining time is at least $\min_{j\in J} \theta_j$ and at most τ_m.

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