Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 4 (2022) 848

Journal of the Nigerian Society of Physical Sciences

The Type II Topp-Leone-G Power Series Distribution with Applications on Bladder Cancer

Boikanyo Makubate*, Broderick O. Oluyede, Marang Pearl Matsuokwane, Lesego Gabaitiri, Simbarashe Chamunorwa

Botswana International University of Science and Technology, P. Bag 16, Palapye, Botswana

Abstract

Statistical distributions are important in modeling the real life of an item and therefore proper distributions that provide useful information for sound conclusions and decisions are needed. For that reason, the demand for developing new generalized distributions have become appropriate for data that have both monotonic and non-monotonic hazard rate functions. In this paper, we develop a new distribution called the Type II Topp-Leone-G Power Series (TIITLGPS) distribution by compounding the Type II Topp-Leone-G (TIITLG) distribution with the power series distribution. Statistical properties of the TIITLGPS distribution are obtained. A variety of shapes for the densities and hazard rate are presented of the considered special case. A simulation study to examine the efficiency of the maximum likelihood estimates is also conducted. Finally, the bladder cancer data example is analyzed for illustrative purposes, it is displayed that the introduced distribution provides better fit when compared to other non-nested distributions considered in this work.

DOI:10.46481/jnsps.2022.848

Keywords: Type II Topp-Leone-G Distribution, Power Series Distribution, Maximum Likelihood Estimation.

Article History :
Received: 12 May 2022
Received in revised form: 16 July 2022
Accepted for publication: 20 July 2022
Published: 15 August 2022

Communicated by: B. J. Falaye

1. Introduction

Statistical distributions are of tremendous importance in real lifetime analysis in various fields such as clinical studies, medical studies, biological studies and environmental studies, just to mention few. Several classical distributions have been derived over past years for modeling data. However, data arising from public health and other areas may not fit the classical distributions. Therefore, modifications of the well-established distributions are highly recommended to obtain more flexible new families of distributions. These generalized distributions have proved to be crucial in capturing heavily tailed, and where the hazard rate function is non-monotonic (uni-modal, bathtub, upside bathtub or upside bathtub followed by bathtub) and improving the goodness-of-fit in empirical distribution. Thus, increased demand in generating new families of distributions that provide flexibility in lifetime phenomenon data modeling.

The Topp-Leone (TL) distribution, proposed by Topp and Leone [1] as a lifetime model, is one of the distributions used within the theory and practice of statistics. It has proven to be a useful lifetime model through its applicability in different fields such as medical and actuarial sciences. Ghitany *et al.* [2] studied the TL distribution by providing some reliability measures while Vicaria *et al.* [3] introduced a two-sided generalized version of the TL distribution. Other various families of extended

*Corresponding author tel. no:
Email address: makubateb@biust.ac.bw. (Boikanyo Makubate)

distributions have been introduced. For instance, the Topp-Leone-Gompertz-G family by Oluyede *et al.* [4], the exponentiated Odd Weibull-Topp-Leone-G family by Chamunorwa *et al.* [5] and the type II Generalized Topp-Leone family by Hassan *et al.* [6]. For any baseline distribution, $G(x;\psi)$, the cdf of the Type II Topp-Leone-G (TIITLG)[7, 8] family of distributions is given by

$$F_{TIITLG}(x;b,\psi) = 1 - [1 - G^2(x;\psi)]^b, \tag{1}$$

where b > 0 and ψ is a parameter vector. The survival function associated with 1 is given by

$$S_{TIITLG}(x;b,\psi) = [1 - G^2(x;\psi)]^b. \tag{2}$$

Suppose that a series system has N components at a given time where N is a discrete random variable with a power series distribution (truncated at zero) and probability mass function (pmf)

$$P(N = n) = \frac{a_n\theta^n}{C(\theta)}, \quad n = 1, 2, ...,$$

where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ and $\theta \in (0, s)$ (s can be ∞) is chosen such that $C(\theta)$ is finite and its three derivatives with respect to θ are defined and given by $C'(.), C''(.)$ and $C'''(.)$, respectively. The power series family of distributions [9, 10, 11] includes Binomial, Poisson, Geometric and Logarithmic distributions. Given N, suppose $(X_1, X_2, ...X_N) \sim TIITLG$ are the independent and identically distributed failure times of the N components. Let X=min$(X_1, ..., X_N)$, then the conditional cdf of $X|N = n$ is given by

$$F_{X|N=n}(x) = 1 - [S_{TII-TL-G}(x;\psi)]^n, \quad x > 0. \tag{3}$$

The Type II Topp-Leone-G Power Series (TIITLGPS) class of distributions is defined by the marginal cdf of X. The cdf of $X \sim TIITLGPS$ is given by

$$F_{TIITLGPS}(x;\theta,b,\psi) = 1 - \frac{C(\theta(1 - G^2(x;\psi))^b)}{C(\theta)}. \tag{4}$$

Table 1 shows some sub-classes of the TIITLGPS distribution.

The density function associated with equation (4) is given by

$$f_{TIITLGPS}(x;\theta,b,\psi) = 2b\theta g(x;\psi)G(x;\psi) \times[1 - G^2(x;\psi)]^{b-1}\frac{C'\left(\theta[1 - G^2(x;\psi)]^b\right)}{C(\theta)}, \tag{5}$$

where $C'(\theta) = \sum_{n=1}^{\infty} na_n\theta^{n-1}$. The hazard function (hrf) is given by

$$h_{TIITLGPS}(x;\theta,b,\psi) = \frac{2b\theta g(x;\psi)G(x;\psi)[1 - G^2(x;\psi)]^{b-1}}{C(\theta(1 - G^2(x;\psi))^b)} \times C'\left(\theta[1 - G^2(x;\psi)]^b\right).$$

The quantile function of the TIITLGPS distribution is given by

$$Q(x) = G^{-1}\left[\left(1 - \left[\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]^{1/b}\right)^{1/2}\right], \tag{6}$$

where G^{-1} is the inverse of the baseline distribution.
Note that, the TIITLG family of distribution defined in equation (1) is a limiting special case of the TIITLGPS class of distributions when $\theta \to 0^+$.

Using the binomial expansion, we express the pdf in 5 as

$$f_{TIITLGPS}(x;\theta,b,\psi) = \sum_{m=0}^{\infty} V_m g_m(x;\psi), \tag{7}$$

where

$$g_m(x;\psi) = (2m+2)g(x;\psi)G^{2m+1}(x;\psi) \tag{8}$$

is the Exp-G distribution with power parameter $2m + 2$, and

$$V_m = \sum_{n=0}^{\infty} \frac{(-1)^m 2b(n+1)a_{(n+1)}\theta^{(n+1)}}{C(\theta)(2m+2)}\binom{b(n+1)-1}{m}. \tag{9}$$

Thus, the statistical properties of the TIITLGPS class of distributions including r^{th} moment, incomplete moment, the moment generating function, the r^{th} moment of residual life, the r^{th} moment of reversed residual life, mean deviations, Bonferroni curves and Lorenz curves can be obtained directly from those of the Exp-G family of the distributions. For, more information regarding the properties of Exp-G distributions the reader is refereed to [12-21].
The primary motivation for developing the Type II Topp-Leone-G Power Series (TIITLGPS) family of distributions is the versatility and flexibility derived from compounding continuous distributions in public health field. The proposed distribution exhibit both monotonic, non-monotic hazard rate functions and heavy tailed data sets, which is a common phenomenon with medical data especially cancer data. Figure 1b shows that the special case TIITLWP distribution can capture upside-down bathtub followed by bathtub and uni-modal shapes hazard rate function(s). Furthermore, from Bladder Cancer data modeling example presented, the introduced distribution captures well heavy tailed data and has better fit compared to some of the selected generalizations in this work.
This paper is organized as follows. In Section 2 some mathematical properties of the new model are presented. Parameter estimation via the method of the maximum likelihood is given in Section 3. Some special models of TIITLGPS class of distributions are presented in Section 4. In Section 5, a simulation study is done. Section 6 contain applications to bladder cancer data set. Concluding remarks are given in Section 7.

2. Some mathematical properties of the TIITLGPS

2.1. Moments and moment generating function

This section discusses the rth moment, mth incomplete moment, moment generating function and residual and reversed residual life of the TII-TL-GPS. The r^{th} moment of the TII-TL-GPS can be represented as

$$\mu_r = E(X^r) = \sum_{m=0}^{\infty} V_m \int_0^{\infty} x^r g_m(x;\psi)dx \tag{10}$$

Table 1. Sub-Classes of the TIITLGPS Distribution

Distribution	a_n	C(θ)	cdf
Type II Topp-Leone G Poisson	$(n!)^{-1}$	$e^{\theta}-1$	1 - $\frac{e^{\theta\left([1-G^2(x;\psi)]^b\right)}-1}{e^{\theta}-1}$
Type II Topp-Leone G Geometric	1	$\theta(1-\theta)^{-1}$	1 - $\frac{\left([1-G^2(x;\psi)]^b\right)(1-\theta)}{1-\theta\left(1-G^2(x;\psi)\right)^b}$
Type II Topp-Leone G Logarithmic	n^{-1}	$-\log(1-\theta)$	1 - $\frac{\log\left(1-\theta[1-G^2(x;\psi)]^b\right)}{\log(1-\theta)}$
Type II Topp-Leone G Binomial	$\binom{m}{n}$	$(1+\theta)^m-1$	1 - $\frac{\left(1+\theta[1-G^2(x;\psi)]^b\right)^m-1}{(1+\theta)^m-1}$

where $g_m(x;\psi)$ and V_m are defined in (8) and (9) respectively and 2p+2 is the power parameter.
The m^{th} incomplete moment can be obtained in the following way

$$\mu_m(y) = \sum_{p=0}^{\infty} V_p \int_0^y x^r g_p(x;\psi)dx. \quad (11)$$

The moment generating function of the TII-TL-GPS is represented as:

$$M_x(t) = E(e^{tx}) = \sum_{p=0}^{\infty} V_p \int_0^{\infty} e^{tx} g_p(x;\psi)dx. \quad (12)$$

We state the following two theorems and provide the proofs at *https://drive.google.com/file/d/1wMwiSK60cj0unyRqA–pwFKBbWGaP9ta/view?usp=sharing*.

Theorem 2.1. *Let $X_1, X_2, .., X_m$ be a random sample of size m from the TIITLGPS class of distributions and $X_{1:m} < X_{2:m} < ... < X_{m:m}$ denote the corresponding ordered random sample of size m. The pdf of the i^{th} order statistic, $X_{i:m}$ is given by*

$$f_{i:m}(x) = \sum_{s=0}^{\infty} \eta_s g_s(x;\psi),$$

where $g_s(x;\psi) = (2s+2)g(x;\psi)G^{2s+1}(x;\psi)$ is the Exp-G distribution with power parameter $2s+2$ and

$$\begin{aligned}\eta_s &= \sum_{j,n,k,z=0}^{\infty} \frac{m!}{(i-1)!(m-i)!} \\ &\times \frac{2b(n+1)a_{(n+1)}\theta^{z+n+1}(-1)^{j+k+s}d_{z,k}}{C^{k+1}(\theta)} \\ &\times \binom{m-i}{j}\binom{j+i-1}{k}\binom{b(z+n+1)-1}{s}(2s+2).\end{aligned}$$

Theorem 2.2. *The Rényi entropy for TIITLGPS class of distributions is given by*

$$I_R(\nu) = \frac{1}{1-\nu} log\left(\sum_{s=0}^{\infty} W_s e^{(1-\nu)I_{REG}}\right),$$

where

$$I_{REG} = \frac{1}{1-\upsilon} log\Big(\int_0^{\infty} \left[(\frac{2s}{\nu}+2)g(x;\psi)G^{\frac{2s}{\nu}+1}(x;\psi)\right]^{\nu} dx\Big)$$

is the Rényi entropy of Exp-G distribution with power parameter $(\frac{2s}{\nu}+2)$ and

$$W_s = \sum_{z=0}^{\infty} (2b)^{\nu} d_{z,\nu}\theta^{z+\nu}(-1)^s \binom{b(\nu+z)-\nu}{s} \frac{1}{(\frac{2s}{\nu}+2)^{\nu}}.$$

3. Maximum Likelihood estimation

This section uses the method of maximum likelihood to estimate unknown parameters of the TIITLGPS class of distributions. Let $x_1, x_2, ..., x_n$ be a random sample from a TIITLGPS distribution given in equation (5), then the log-likelihood of the parameter vector $\phi = (b, \theta, \psi)^T$ is given by:

$$\begin{aligned}\ell(\phi) &= n\log 2 + n\log b + n\log\theta + \sum_{i=1}^{n} \log G(x_i;\psi) \\ &+(b-1)\sum_{i=1}^{n}\log[1-G^2(x_i;\psi)] \\ &-n\log C(\theta) + \sum_{i=1}^{n}\log C'\left(\theta[1-G^2(x_i;\psi)]^b\right).\end{aligned}$$

The components of the score function are obtained by finding the partial derivatives (with respect to the parameters θ, b and ψ, respectively). They are given by:

$$\begin{aligned}U_\theta &= \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} \\ &+ \sum_{i=1}^{n} \frac{\left(C''\left(\theta[1-G^2(x_i;\psi)]^b\right)\right)[1-G^2(x_i;\psi)]}{C'\left(\theta[1-G^2(x_i;\psi)]^b\right)},\end{aligned}$$

$$\begin{aligned}U_b &= \frac{n}{b} + \sum_{i=1}^{n}[1-G^2(x_i;\psi)] \\ &+ \sum_{i=1}^{n} \frac{\left(C''\left(\theta[1-G^2(x_i;\psi)]^b\right)\right)}{C'\left(\theta[1-G^2(x_i;\psi)]^b\right)}[1-G^2(x_i;\psi)]^b \\ &\times\log[1-G^2(x_i;\psi)]\end{aligned}$$

and

$$\begin{aligned} U_{\psi} &= (b-1)\sum_{i=1}^{n}\frac{1}{[1-G^2(x_i;\psi)]}\frac{\partial[1-G^2(x;\psi)]}{\partial\psi} \\ &+\sum_{i=1}^{n}\frac{\frac{\partial G(x_i;psi)}{\partial\psi}}{G(x_i;\psi)}+\sum_{i=1}^{n}\frac{\frac{\partial g(x_i;psi)}{\partial\psi}}{g(x_i;\psi)} \\ &+\sum_{i=1}^{n}\frac{\left(C''\left(\theta[1-G^2(x_i;\psi)]^b\right)\right)}{C'\left(\theta[1-G^2(x_i;\psi)]^b\right)} \\ &\times 2b\theta[1-G^2(x_i,\psi)]^{b-1}G(x_i;\psi)\frac{\partial G(x_i;\psi)}{\partial\psi}. \end{aligned}$$

Note that these equations are non-linear and can not be solved analytically, but can be solved numerically using software like R language.

4. Some Special Models

This section considers and presents some special cases of the TIITLGPS class of distributions when the baseline distributions are Weibull and Burr XII distributions.

4.1. Type-II-Topp-Leone-Weibull-Poisson (TIITLWP) Distribution

The cdf and pdf of the TIITLWP distribution are given by

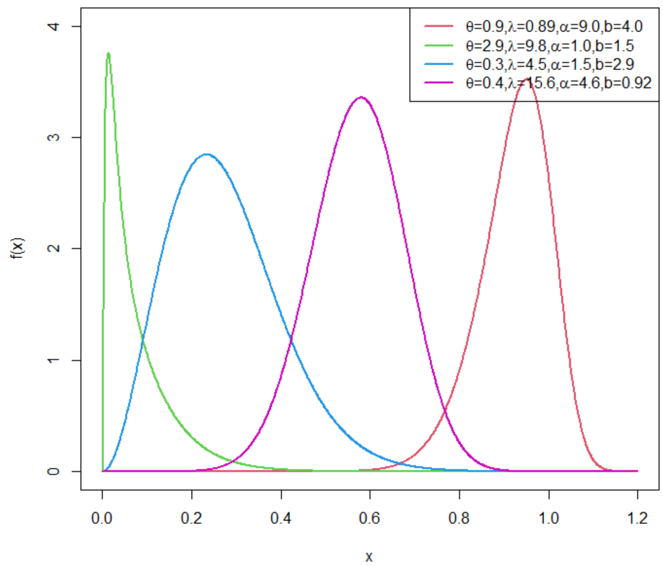
$$F_{TIITLWP}(x;\theta,b,\alpha,\lambda) = 1-\frac{e^{\left(\theta\left[1-\left(1-e^{-\lambda x^{\alpha}}\right)^2\right]^b\right)}-1}{e^{\theta}-1}$$

and

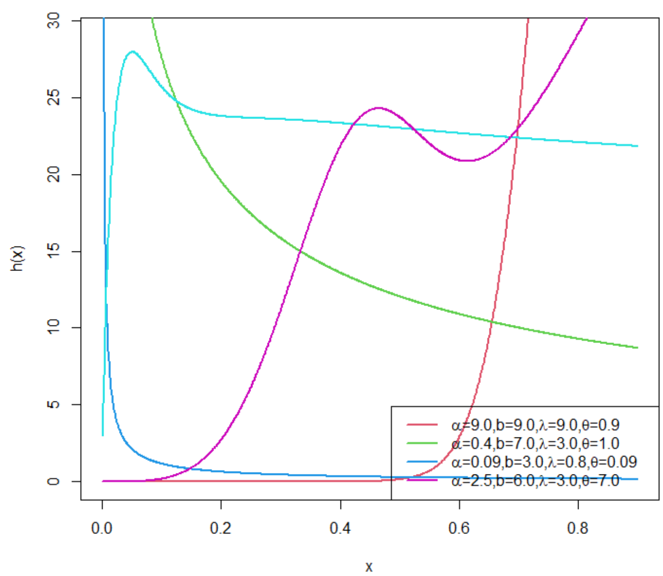
$$\begin{aligned} f_{TIITLWP}(x;\theta,b,\alpha,\lambda) &= 2\alpha b\theta\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}\left(1-e^{-\lambda x^{\alpha}}\right) \\ &\left[1-\left(1-e^{-\lambda x^{\alpha}}\right)^2\right]^{b-1} \\ &\times\frac{e^{\left(\theta\left[1-\left(1-e^{-\lambda x^{\alpha}}\right)^2\right]^b\right)}}{e^{\theta}-1}, \end{aligned}$$

respectively, for $\theta, b, \alpha, \lambda$ and $x > 0$.

Figures 1(a) and 1(b) show the plots of the pdfs and hrfs, respectively, for the TIITLWP distribution for selected parameter values. Plots of the TIITLWP pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and reverse-J shapes. Plots of the hrf of the TIITLWP distribution show different shapes including increasing, decreasing, upside-down bathtub and uni-modal shapes.
The cdf, pdf and plots (pdf and hrfs plots) of Type-II-Topp-Leone-Weibull-Geometric (TIITLWG), Type-II-Topp-Leone-Burr XII-Poisson (TIITLBXIIP), Type-II-Topp-Leone-Burr XII-Geometric (TIITLBXIIG) distributions can be seen at *https://drive.google.com/file/d/1wMwiSK60cj0unyRqA–pwFKBbWGaP9ta/view?usp=sharing*.



(a)



(b)

Figure 1. Pdfs and hrfs plots for the TIITLWP distribution

5. Simulation Study

In this section, we conduct Monte Carlo simulation study to assess the performance of maximum likelihood estimates of the TIITLWP distribution. We generated N=1000 samples of size n= 100, 200, 400 and 800 with the help of the R package. We consider simulations for the following sets of initial parameters values (I: $\alpha = 1.0, b = 0.5, \lambda = 1.5, \theta = 1.0$), (II: $\alpha = 0.5, b = 0.5, \lambda = 0.5, \theta = 1.0$), (III: $\alpha = 1.5, b = 0.5, \lambda = 0.5, \theta = 1.0$), and (IV: $\alpha = 0.5, b =$

Table 2. Parameter estimates and goodness-of-fit statistics for various models fitted for cancer patients data set

	Estimates				Statistics							
Model	α	b	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIITLWP	0.7361	3.0338	0.0749	3.6985	819.4	827.4	827.8	838.9	0.0226	0.1495	0.0371	0.9945
	(0.1340)	(8.2303)	(0.1021)	(2.4628)								
	α	λ	γ	θ								
OWTLLLoGL	1.0089	0.4757	9.7751	1.2905×10^{-9}	826.4	834.4	834.7	845.8	0.0925	0.6058	0.0574	0.7930
	(0.3164)	(0.1060)	(3.1140)	(0.0033)								
	p	α	β	λ								
EGEL	1.0729×10^{-8}	0.8922	1.1774	1.3292	826.2	834.2	834.6	845.6	0.1135	0.6819	0.0707	0.5442
	(0.0095)	(0.6760)	(0.1426)	(10.0710)								
	a	b	α	β								
EPLP	5.8448×10^{-8}	0.5647	0.82441	2.7914	820.9	828.9	829.2	840.3	0.0391	0.2563	0.0428	0.9733
	(0.0196)	(0.1021)	(0.3150)	(1.3114)								
	c	s	α	λ								
EBXIIP	1.0223	1.5133	1.9338	7.8251×10^{-9}	871.7	879.7	880.0	891.1	0.2452	1.600	0.2481	2.872×10^{-7}
	(0.0902)	(0.2426)	(0.2105)	(0.0085)								
	a	b	λ	θ								
BOL-U	1.1799	2.8446	3.2800×10^{5}	7.6115×10^{6}	826.4	834.4	834.7	845.8	0.1186	0.7117	0.0734	0.4955
	(0.1329)	(0.3599)	(3.1630×10^{-6})	(1.3631×10^{-7})								
	β	λ	θ	γ								
ELOLLoGW	6.1173×10^{-5}	0.0378	2.9053	1.0478	828.2	836.2	836.5	847.6	0.1314	0.7864	0.0700	0.5573
	(0.4015)	(0.0061)	(8.4383×10^{-5})	(0.0676)								

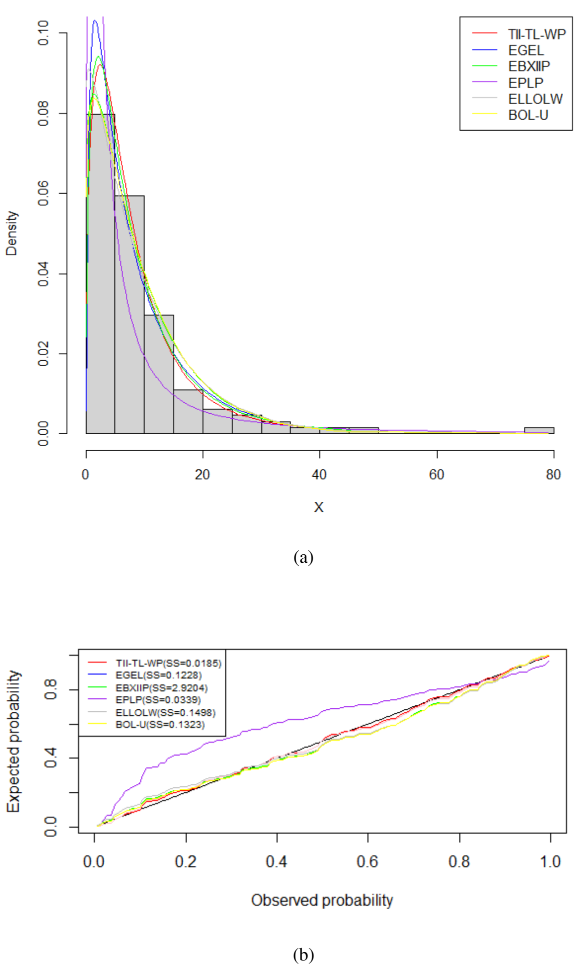


Figure 2. Fitted densities and probability plots for cancer patients data

$1.0, \lambda = 1.5, \theta = 0.5$). Simulation results are shown at *https://drive.google.com/file/d/1wMwiSK60cj0unyRqA–pwFKBbWGaP9ta/view?usp=sharing*.
The results show that as the sample size increases, the estimates approach the true parameter values, since root mean square errors (RMSE) and average bias decays toward zero for all the parameters.

6. Applications and Discussion

In this section, we demonstrate the applicability of the TIITLWP distribution to bladder cancer data set. We compared the TIITLWP distribution to six non-nested models. We use the nlm function in R software to estimate model parameters. We present the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics. We assessed model performance using -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von-Mises (W^*), and Andersen-Darling (A^*) (see Chen and Balakrishnan [23] for details). These statistics are used to verify which model fits the given data well. The smaller the values of these statistics, the better the model. Kolmogorov-Smirnov (K-S) statistic (and its p-value) and sum of squares (SS) from probability plots were also used to assess the fit of the model. The smaller the KS value and the higher the p-value for the K-S statistic the better the model. Tables 2 shows model parameters estimates (standard errors in parentheses) and several goodness-of-fit statistics. Figure 2 shows fitted densities and probability plots (as described by Chambers *et al.* [22]) to demonstrate how well our model fits the selected data sets.

The non-nested models considered in this paper are the exponentiated generalized logarithmic (EGEL) distribution by

Oluyede *et al.* [24], odd Weibull-Topp-Leone-log logistic logarithmic (OWTLLLoGL) distribution by Oluyede *et al.* [25], exponentiated power Lindley Poisson (EPLP) distribution by Pararai *et al.* [21], exponentiated Burr XII Poisson (EBXIIP) distribution by da Silva *et al.* [12], beta odd Lindley-Uniform (BOL-U) distribution by Chipepa *et al.* [26] and exponential Lindley odd log-logistic Weibull (ELOLLoGW) distribution by Korkmaz[15]. The pdfs of the non-nested models can be seen at *https://drive.google.com/file/d/1wMwiSK60cj0unyRqA–pwFKBbWGaP9ta/view?usp=sharing*. For the EBXIIP and ELOLLW distributions we consider $k = 1$ and $\alpha = 1$, respectively.

6.1. Bladder Cancer Patients Data

This data set represents remission times of a random sample of 128 bladder cancer patients. For more information regarding the bladder cancer data set, Lee and Wang [27] is recommended.
From the values of the goodness-of-fit statistics A^*, W^*, K-S and the p-value of the K-S statistic as shown in Table 2, we conclude that the TIITLWP model performs better than the non-nested models considered in this paper. Figures 2(a) and 2(b) show the fitted densities and probability plots for the TIITLWP model. We observe that the TIITLWP model has better fit to extreme tailed data compared to the selected competitive models.

7. Conclusions

The demand of developing new families of distributions from classical ones has been of interest among authors in the past years. A new distribution called the TIITLGPS distribution is developed which combines the TIITL-G with the power series to provide compound TIITL-GPS family of distributions with better performance compared to selected models. We study some of mathematical properties of the new family of distributions, such as ordinary moment, quantile functions, order statistics and Rényi entropy. We discuss maximum likelihood estimates of the model parameters for bladder cancer data. Additionally, the performance of the MLEs of the two selected members is assessed via Monte Carlo simulation studies based on two criteria; bias and RMSE. The study exhibits a good performance when estimating the parameters of the proposed family using the maximum likelihood method. Finally, the special case of the new distribution applied to the bladder cancer data set and a simulation study demonstrate its usefulness and potentiality to analysis of lifetime data. We hope that the proposed distribution will attract wider application in various fields such as health, survival and lifetime data, finance, insurance, among others.

Future Work

- Take into consideration bivariate extensions of proposed model via copulas like the Farlie-Gumbel-Margestern Copulas, Ali-Mikhail-Haq copula, Clayton copula and Rényi entropy copula.
- Apply different parameter estimation techniques such as Bayesian technique, weighted least squares (WLS), maximum product of spacings (MPS) on the proposed models.

Limitations

- We derive mathematical and statistical properties from the exponentiated G distribution cause its a bit complicated to derive them straight from the TIITLGPS pdf.
- The model can not perform well on some data sets.

References

[1] C. W. Topp & F. C. Leone, "A Family of J-shaped Frequency Functions", Journal of the American Statistical Association **50** (1955) 209. DOI: http://dx.doi.org/10.1080/01621459.1955.10501259.

[2] M. E. Ghitany, S. Kotz & M. Xie, "On Some Reliability Measures and their Stochastic Orderings for the Topp-Leone Distribution", Journal of Applied Statistics **32** (2005) 715. DOI: https://doi.org/10.1080/02664760500079613

[3] D. Vicaria, J. R. V. Dorp & S. Kotz, "Two-Sided Generalized Topp and Leone (TS-GTL) Distributions", Journal of Applied Statistics **35** (2008) 1115.

[4] B. Oluyede, S. Chamunorwa, F. Chipepa & M. Alizadeh, "The Topp-Leone-Gompertz-G Family of Distributions with Applications", Journal of Statistics and Management Systems (2022) DOI https://doi.org/10.1080/09720510.2021.1972623.

[5] S. Chamunorwa, B. Oluyede, B. Makubate & F. Chipepa "The exponentiated Odd Weibull-Topp-Leone-G Family of Distributions: Model, Properties and Applications", Pakistan Journal of Statistics **37** (2021) 143-158. Doi: https://www.pakjs.com/wp-content/uploads/2021/03/372-05.pdf.

[6] A. Hassan, M. Elgarhy, & A. Zubair, "Type II Generalized Topp-Leone Family of Distributions: Properties and Applications", Journal of data science: JDS (2019) doi: 10.6339/JDS.201910_17(4).0001.

[7] A. Al-Shomrani, O.Arif, A. Shawky, S. Hanif & M. Q. Shahbaz, "Topp Leone Family of Distributions: Some Properties and Application", Pakistan Journal of Statistics and Operation Research **12** (2016) 443. DOI: 10.18187/pjsor.v12i3.1458.

[8] M. Elgarhy, M. Arslan Nasir, F.Jamal & G. Gamze Ozel, "The Type II Topp-Leone Generated Family of Distributions : Properties and Applications", Journal of Statistics and Management Systems 21 (2018) 1529-1551. DOI: https://doi.org/10.1080/09720510.2018.1516725.

[9] N. L. Johnson, S. Kotz & N. Balakrishnan, *Continuous Distributions*, Volume 1, John Wiley and Sons, New York, NY (1994).

[10] B. Makubate, B. Oluyede & M. Gabanakgosi, "A New Lindley-Burr XII Distribution: Model, Properties and Applications", International Journal of Statistics and Probability **10** (2021) 33. DOI:10.5539/ijsp.v10n4p33

[11] B. Makubate, B. Oluyede & G. Motsewabagale, "Dagum Power Series Class of Distribution with Applications to Lifetime Data", Research and Reviews: Journal of Statistics and Mathematical Sciences, **92** (2019) 76.

[12] R. V. da Silva, F. Gomes-Silva, M. W. A. Ramos, G. M. Cordeiro & E. M. M. Ortega, "The Exponentiated Burr XII Poisson Distribution with Application to Lifetime Data", International Journal of Statistics and Probability **347** (2015) 1399.

[13] R. C. Gupta, P. L. Gupta & R. D. Gupta, "Modeling failure time data by Lehmann alternatives",Comm. Statist. Theory Methods **27** (1998) 887.

[14] R. D.Gupta & D. Kundu, "Generalized exponential distributions", Aust.N.Z.J.Stat. **41** (1999) 173-188.

[15] M. C. Korkmaz, H. M. Yousof & Hamedani, G. G., "The Exponential Lindley Odd Log-Logistic-G Family: Properties, Characterizations and Applications", Journal of Statistical Theory and Applications **17** (2018) 554-571.DOI: https://doi.org/10.2991/jsta.2018.17.3.10

[16] G. S. Mudholkar & D. K. Srivastava, "Exponentiated Weibull family for analysing bathtub failure rate data", IEEE Trans. Reliab. **42** (1993) 299.

[17] G. S. Mudholkar, D. K. Srivastava & Freimer, M., "The Exponentiated Weibull family: A reanalysis of the bus-motor-failure data", Technometrics **37** (1995) 436. DOI: https://doi.org/10.2307/1269735

[18] S. Nadarajah, "The Exponentiated Gumbel distribution with climate application", Environmetrics 17 (2005) 13-23. DOI: https://doi.org/10.1002/env.739

[19] S. Nadarajah & Kotz, S., "The Exponentiated-type distributions", Acta Appl. Math. **92** (2006) 97-111. DOI: https://doi.org/10.1007/s10440-006-9055-0

[20] S. Nadarajah & A. K.Gupta, "The Exponentiated gamma distribution with application to drought data", Calcutta Statist. Assoc. Bull. **59** (2007) 29.

[21] M. Pararai, G. Warahena-Liyanage & B.Oluyede, "Exponentiated Power Lindley Poisson Distribution: Properties and Application", Communications in Statistics-Theory and Methods **46** (2017) 4726, doi: 10.1080/03610926.2015.1076473.

[22] J. Chambers, W. Cleveland, B. Kleiner & P. Tukey, "Graphical Methods of Data Analysis", Chapman and Hall (1983)

[23] G. Chen & N. Balakrishnan, "A General Purpose Approximate Goodness-of-Fit Test", Journal of Quality Technology 27(1995) 154. DOI: https://doi.org/10.1080/00224065.1995.11979578

[24] B. Oluyede, B. Mashabe, A. Fagbamigbe, B. Makubate & D. Wanduku, "The Exponentiated Generalized Power Series: Family of Distributions: Theory, Properties and Applications", Heliyon, **6** (2020) doi: 10.1016/j.heliyon.2020.e04653.

[25] B. Oluyede, F. Chipepa & D. Wanduku, "The Odd Weibull-Topp-Leone-G Power Series Family of Distributions: Model, Properties, and Applications", Journal of Nonlinear Science and Application **14** (2021) 60. doi: 10.22436/jnsa.014.04.06.

[26] F. Chipepa, B. Oluyede & B. Makubate, "A New Generalized Family of Odd Lindley-G Distributions with Application", International Journal of Statistics and Probability **8** (2019) doi:10.5539/ijsp.v8n6p1.

[27] E. T.Lee & J. W. Wang, *Statistical Methods for Survival Data Analysis*, 3rd Edn., John Wiley and Sons, New York, ISBN: 9780471458555 (2003) 534.