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Hybrid Block Methods with Constructed Orthogonal Basis for Solution of Third-Order Ordinary Differential Equations

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Abstract

In this work, an orthogonal polynomial with a weight function $w(x) = x^2 + x + 1$ in the interval [-1, 1] was constructed and used as the basis function to develop block methods, using collocation and interpolation approach. An efficient class of continuous and discrete numerical integration schemes of implicit hybrid form for third-order problems were developed and successfully implemented. Three different problems were solved with these schemes, and they performed favourably. The investigation, using the appropriate existing theorems, shows that the methods are consistent, zero-stable, and hence, convergent.

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1. Introduction

There are no analytic solutions for some initial value problems of third-order Ordinary Differential Equations (ODEs) of the form

$$y''' = f(x, y, y', y''); y(a) = \alpha,$$
$$y'(a) = \beta, y''(a) = \gamma, x \in (a, b) \quad (1)$$

On the other hand, numerical methods for solving such involves reducing the equation to systems of first order differential equations, this technique increases the dimension of the real problem, thus involves more computations, [1]. Also, linear multistep methods (LMMs) implemented by the predictor-corrector mode have been found to be prone to error propagation and are also known to be very expensive to implement in terms of the number of function evaluations per step. The predictors often have a lower order of accuracy than the correctors, especially when all the grid and off-grid points are used for collocation and interpolation. This disadvantage led to the development of block methods from LMMs. Apart from being self-starting, a block method does not require the development of a predictor separately. The development of block methods for solving IVPs in ODEs is the central concern of this work. Many researchers

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have used this block method to integrate IVPs, among them are [2], [3], [4], and [5]. Some researchers also constructed orthogonal polynomials using different weight functions; see [6]. These constructed polynomials were employed as basis function for the derivation of numerical schemes. They include; [7] who used orthogonal polynomials with weight function $w(x) = x^2$ for interval $[-1, 1]$. [8] employed orthogonal polynomials with weight function $w(x) = x^2$ in the interval [0, 1], while [9] engaged orthogonal polynomials with weight function $w(x) = x$ for interval [0, 1]. Virtually all the a forementioned contributions are for first order differential equations. [10] constructed an orthogonal basis in the interval [-1, 1] with respect to the weight function $w(x) = 1 + \frac{x}{2}$, to solve the initial value problem of a third-order differential equation. [11] constructed a block method to integrate third-order ODE, [12] developed harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems, while [13] formulated a block algorithm for general third-order ordinary differential equations. Several methods have been formulated using different polynomials and targeting varying orders of ODEs (see [14], [15]).

The need to further explore this area of research with the view to solving higher order differential equations effectively is the major focus of this work. To this end, orthogonal polynomials were constructed to serve as basis functions for the development of hybrid block methods for the class of problem (1). The polynomial is characterized with $w(x) = x^2+x+1$ in the interval [-1, 1].

2. Formulation of the Orthogonal Polynomial

We formulate here, the basis function for the numerical scheme to be developed by constructing an orthogonal polynomial $Q_n(x)$ of degree n. As established in literature, orthogonality of two polynomials is attained if their inner product varnishes in the interval within which they are defined [9].

i. e. the inner product yields:

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = h_n\delta_{mn}$$

for

$$\delta_{mn} = \begin{cases} 0, m\neq n \\ 1, m=1 \end{cases}$$

where $w(x)$ is a continuous and positive weight function in the range [a,b], and the moments

$$\mu = \int_a^b w(x)x^n dx, \quad n = 0, 1, 2, ...$$

exist.

The integral

$$\langle\phi_m, \phi_n\rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx$$

.

is the inner product of polynomials ϕ_m and ϕ_n.

When it comes to orthogonality,

$$\langle\phi_m, \phi_n\rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad for\ m \neq n$$

For the construction of a befitting approximating polynomial, consider the orthogonal polynomial class $\{\phi n(x)\}$, which is defined as

$$\phi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r$$

which requires that:

$$\phi_n(1) = 1$$

and

$$< \phi_m(x), \phi_n(x) >= 0 \qquad (m \neq n)$$

.

Let $w(x) = 1 + x + x^2$ and $[a, b] = [-1, 1]$.

Then, for $n = 0$, we have

$$\phi_0(x) = C_0^{(0)} \quad \text{and} \quad \phi_0(1) = 1 = C_0^{(0)}$$

Hence, $\phi_0(x) = 1$

For $n = 1$, we have two equations

$$\phi_1(1) = C_0^{(1)} + C_1^{(1)} = 1$$

$$< \phi_0(x), \phi_1(x) >= \frac{8}{3}C_0^{(1)} + \frac{2}{3}C_1^{(1)} = 0$$

These two equations yield:

$$\phi_1(x) = -\frac{1}{3} + \frac{4}{3}x$$

For $n = 2$, we solve

$$C_0^{(1)} + C_1^{(2)} + C_2^{(2)} = 1 = \phi_2(1)$$

$$\frac{8}{3}C_0^{(1)} + \frac{2}{3}C_1^{(2)} + \frac{16}{5}C_2^{(2)} = 0 =< \phi_0(x), \phi_2(x) >$$

$$\frac{6}{5}C_1^{(2)} + \frac{8}{45}C_2^{(2)} = 0 =< \phi_1(x), \phi_2(x) >$$

which gives the polynomial

$$\phi_2(x) = -\frac{49}{66} - \frac{10}{33}x + \frac{45}{22}x^2$$

Following this approach, we have a class of orthogonal polynomials given as follows:

$$
\begin{aligned}
Q_0(x) &= 1\\
Q_1(x) &= \frac{4}{3}x - \frac{1}{3}\\
Q_2(x) &= \frac{45}{22}x^2 - \frac{10}{33} - \frac{49}{66}\\
Q_3(x) &= \frac{1484}{419}x^3 - \frac{483}{838}x^2 - \frac{930}{419}x + \frac{213}{838}\\
Q_4(x) &= \frac{49427}{7880}x^4 - \frac{994}{985}x^3 - \frac{4347}{788}x^2 + \frac{2002}{2995}x + \frac{13609}{23640}\\
Q_5(x) &= \frac{169587}{14882}x^5 - \frac{108625}{59528}x^4 - \frac{13735}{1063}x^3 + \frac{50445}{29764}x^2 + \frac{127805}{44646}x - \frac{5285}{25512}\\
Q_6(x) &= \frac{21640593}{1029296}x^6 - \frac{865683}{257324}x^5 - \frac{29995065}{1029296}x^4 + \frac{511695}{128662}x^3 + \frac{112595805}{11322256}x^2\\
&\quad - \frac{2700945}{2830564}x - \frac{5509685}{11322256}\\
Q_7(x) &= \frac{912217735}{23254947}x^7 - \frac{2332825495}{372079152}x^6 - \frac{663966303}{10335532}x^5 + \frac{3345316975}{372079152}x^4\\
&\quad + \frac{1383496345}{46509894}x^3 - \frac{134149785}{41342128}x^2 - \frac{315107135}{93019788}x + \frac{66696665}{372079152}\\
Q_8(x) &= \frac{206238477159}{2794077824}x^8 - \frac{4632559789}{392917194}x^7 - \frac{876515334695}{6286675104}x^6 + \frac{867587721}{43657466}x^5\\
&\quad + \frac{1026917752861}{12573350208}x^4 - \frac{3774934163}{392917194}x^3 - \frac{10578992655}{698519456}x^2 + \frac{462916223}{392917194}x + \frac{10792421983}{25146700416}\\
Q_9(x) &= \frac{7592947736959}{54306345568}x^9 - \frac{4849132400307}{21722582272}x^8 - \frac{4061887568739}{13576586392}x^7\\
&\quad + \frac{2347732919259}{54306345568}x^6 + \frac{5760948177621}{27153172784}x^5 - \frac{2848740151257}{108612691136}x^4 - \frac{750682757643}{13576586392}x^3\\
&\quad + \frac{278681373243}{54306345568}x^2 + \frac{208683420559}{54306345568}x - \frac{34811913843}{217225382272}
\end{aligned}
$$

3. Numerical Block Algorithms for Third-Order Problems

The analytical solution of (1) is approximated via experimental solution of the form:

$$Y(x) = \sum_{j=0}^{v+w-1} a_j \phi_j(x) \tag{2}$$

where $x \in [a, b]$, v and w are the number of collocation and interpolation points respectively. The function $\phi_j(x)$ is the j^{th} degree orthogonal polynomial valid in the range of integration of $[a, b]$. The third derivative of (2) is given by

$$y'''(x) = \sum_{j=3}^{v+w-1} a_j \phi_j'''(x) = f(x, y, y', y'') \tag{3}$$

To estimate the solution of the problem given in (1), interpolation must be done at least three times. As a result, equation (2) is interpolated at (x_{n+w}) points, and equation (3) is collocated at (x_{n+v}) points, yielding a system of equations to be solved using the Gaussian elimination method. We will use a hybrid approach to apply this concept.

3.1. Development of One-step Method with $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{4}}$ as the Off-step Point

The new off-step points are $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{4}}$. Interpolating (2)at $x = x_{n+w}$, $w = 0, \frac{1}{4}$ and $\frac{1}{2}$ and collocating (3) at $x = x_{n+v}$, $v = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1 ; yields the system of equations:

$$\begin{bmatrix} 1 & \frac{-5}{3} & \frac{53}{33} & \frac{-689}{419} & \frac{983}{591} & \frac{-519}{310} & \frac{631}{375} & \frac{-10287}{6091} \\ 1 & -1 & \frac{-7}{88} & \frac{823}{1059} & \frac{-400}{641} & \frac{-159}{2278} & \frac{593}{1002} & \frac{-201}{409} \\ 1 & \frac{-1}{3} & \frac{-49}{66} & \frac{213}{838} & \frac{1297}{2253} & \frac{-301}{1453} & \frac{-382}{785} & \frac{197}{1099} \\ 0 & 0 & 0 & \frac{35701}{210} & \frac{-246792}{197} & \frac{140399}{27} & \frac{-112087}{7} & \frac{203929}{5} \\ 0 & 0 & 0 & \frac{35701}{210} & \frac{-24072}{37} & \frac{30440}{33} & \frac{8344}{135} & \frac{-38642}{17} \\ 0 & 0 & 0 & \frac{35701}{210} & \frac{-2761}{57} & \frac{-17986}{29} & \frac{11263}{59} & \frac{39979}{28} \\ 0 & 0 & 0 & \frac{35701}{210} & \frac{21595}{39} & \frac{25171}{44} & \frac{-30707}{63} & \frac{-26670}{13} \\ 0 & 0 & 0 & \frac{35701}{210} & \frac{9247}{8} & \frac{17997}{4} & \frac{105317}{8} & \frac{64401}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{4}} \\ h^3 f_{n+\frac{1}{2}} \\ h^3 f_{n+\frac{3}{4}} \\ h^3 f_{n+1} \end{bmatrix} \tag{4}$$

The solution of (4) gives the values:

$$\begin{aligned} a_0 = \frac{21}{20}y_n + \frac{51}{20}y_{n+\frac{1}{2}} - \frac{13}{44645}y_{n+\frac{1}{4}} + \frac{4}{44645}f_n h^3 + \frac{23}{2378}f_{n+\frac{1}{2}}h^3 \\ + \frac{23}{3087}h^3 f_{n+\frac{1}{4}} + \frac{35}{36069}h^3 f_{n+\frac{3}{4}} + \frac{3}{64369}h^3 f_{n+1} \\ a_1 = \frac{35}{36}y_n + \frac{89}{36}y_{n+\frac{1}{2}} - \frac{31}{9}y_{n+\frac{1}{4}} + \frac{67}{928956}h^3 f_n + \frac{93}{8017}h^3 f_{n+\frac{1}{2}} \\ + \frac{325}{39192}h^3 f_{n+\frac{1}{4}} + \frac{89}{62116}h^3 f_{n+\frac{3}{4}} + \frac{54}{971405}h^3 f_{n+1} \\ a_2 = \frac{44}{45}y_n + \frac{44}{45}y_{n+\frac{1}{2}} - \frac{88}{45}y_{n+\frac{1}{4}} + \frac{15}{148586}h^3 f_n + \frac{126}{14065}h^3 f_{n+\frac{1}{2}} \\ + \frac{70}{11661}h^3 f_{n+\frac{1}{4}} + \frac{6}{3259}h^3 f_{n+\frac{3}{4}} + \frac{6}{187829}h^3 f_{n+1} \\ a_3 = h^3\left(\frac{3}{290044}f_n + \frac{71}{21932}f_{n+\frac{1}{2}} + \frac{55}{50968}f_{n+\frac{1}{4}} + \frac{32}{20807}f_{n+\frac{3}{4}} + \frac{8}{456815}f_{n+1}\right) \\ a_4 = h^3\left(\frac{-1}{96160}f_n - \frac{12}{71539}f_{n+\frac{1}{2}} - \frac{22}{30633}f_{n+\frac{1}{4}} + \frac{33}{37892}f_{n+\frac{3}{4}} + \frac{4}{157477}f_{n+1}\right) \\ a_5 = h^3\left(\frac{1}{56502}f_n - \frac{20}{35299}f_{n+\frac{1}{2}} + \frac{14}{50845}f_{n+\frac{1}{4}} + \frac{13}{55005}f_{n+\frac{3}{4}} + \frac{7}{188168}f_{n+1}\right) \\ a_6 = h^3\left(\frac{-7}{233230}f_n + \frac{4}{220991}f_{n+\frac{1}{2}} + \frac{8}{148167}f_{n+\frac{1}{4}} - \frac{13}{166396}f_{n+\frac{3}{4}} + \frac{4}{110967}f_{n+1}\right) \\ a_7 = h^3\left(\frac{5}{494258}f_n + \frac{15}{247129}f_{n+\frac{1}{2}} - \frac{10}{247129}f_{n+\frac{1}{4}} - \frac{10}{247129}f_{n+\frac{3}{4}} + \frac{5}{494258}f_{n+1}\right) \end{aligned} \tag{5}$$

Substituting (5) in (3) yields the continuous implicit one-step method:

$$\begin{aligned} \bar{y}(x) = \sum_{j=0}^{1}\alpha_j y_{n+j} + \alpha_{\frac{1}{4}}(x)y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(x)y_{n+\frac{3}{4}} \\ +h\left(\sum_{j=0}^{1}\beta_j(x)f_{n+j} + \beta_{\frac{1}{4}}(x)f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}\right) \end{aligned} \tag{6}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients. In (6) the parameters, $\alpha_j(x)$ and $\beta_j(x)$, are obtained as :

$$\alpha_0(t) = 2t^2 + t$$

$$\alpha_{\frac{1}{4}}(t) = -4t^2 - 4t$$

$$\alpha_{\frac{1}{2}}(t) = 2t^2 + 3t + 1$$

$$\beta_0(t) = h^3\left(\frac{t^7}{2520} - \frac{t^6}{720} + \frac{t^5}{180} - \frac{t^4}{144} - \frac{t^3}{1962243502453721500 0} - \frac{t^2}{23040}\right.$$

$$+\frac{13t}{161280}+\frac{1}{903542973820909160000}\Bigg)$$

$$\beta_{\frac{1}{4}}(t)=h^3\left(-\frac{t^7}{630}+\frac{t^6}{720}+\frac{t^5}{180}-\frac{t^4}{144}-\frac{t^3}{1193128086456805900000}+\frac{199}{7103}t^2\right.$$
$$\left.+\frac{47}{6501}t-\frac{1}{3432349569107785300 0}\right)$$

$$\beta_{\frac{1}{2}}(t)=h^3\left(\frac{t^7}{420}+\frac{t^6}{665476954094069860 0}-\frac{t^5}{96}+\frac{t^4}{1318120659808086000}+\frac{t^3}{48}+\frac{21}{1280}t^2\right.$$
$$\left.+\frac{44}{12193}-\frac{1}{917808302492149500}\right)$$

$$\beta_{\frac{3}{4}}(t)=-h^3\left(\frac{t^7}{630}+\frac{t^6}{720}-\frac{t^5}{180}-\frac{t^4}{144}+\frac{t^3}{4590274775345433600}+\frac{5t^2}{2304}\right.$$
$$\left.+\frac{47t}{80640}-\frac{1}{1347944257981110300 0}\right)$$

$$\beta_1(t)=h^3\left(\frac{t^7}{2520}+\frac{t^6}{1440}-\frac{t^5}{2880}-\frac{t^4}{1152}-\frac{t^3}{1479328912500027000 0}+\frac{7t^2}{23040}\right.$$
$$\left.+\frac{7t}{86843}+\frac{1}{1154976066398231600 00}\right) \tag{7}$$

where $t=\frac{2x-2x_n-h}{h}$.

By evaluating (6) at $x_{n+\frac{3}{4}}$ and x_{n+1}, the main methods are obtained as:

$$y_{n+\frac{3}{4}}=y_n-3y_{n+\frac{1}{4}}+3y_{n+\frac{1}{2}}+h^3\left(\frac{1}{15360}f_n+\frac{21}{2560}f_{n+\frac{1}{2}}-\frac{1}{3840}f_{n+\frac{3}{4}}+\frac{1}{15360}f_1\right) \tag{8}$$

$$y_{n+1}=3y_n-8y_{n+\frac{1}{4}}+6y_{n+\frac{1}{2}}+h^3\left(\frac{1}{3840}f_n+\frac{43}{1920}f_{n+\frac{1}{4}}+\frac{21}{640}f_{n+\frac{1}{2}}+\frac{13}{1920}f_{n+\frac{3}{4}}+\frac{1}{3820}f_1\right) \tag{9}$$

Differentiating (6) yields the continuous coefficients:

$$\alpha_0'(t)=\frac{8t+2}{h}$$
$$\alpha_{\frac{1}{4}}(t)=-\frac{16t+8}{h}$$
$$\alpha_{\frac{1}{2}}'(t)=\frac{8t+6}{h}$$
$$\beta_0'(t)=h^2\left(-\frac{t^6}{180}+\frac{t^5}{120}+\frac{t^4}{288}-\frac{t^3}{144}+\frac{t^2}{5537761027558691530 0}+\frac{t}{5760}-\frac{13}{86040}\right)$$
$$\beta_{\frac{1}{4}}'(t)=h^2\left(-\frac{t^6}{45}+\frac{t^5}{60}+\frac{t^4}{18}-\frac{t^3}{18}-\frac{t^2}{2305645383593181400}+\frac{193t}{2880}+\frac{94}{6501}\right)$$
$$\beta_{\frac{1}{2}}'(t)=h^2\left(\frac{t^6}{30}+\frac{t^5}{495329736356353410}-\frac{5t^4}{48}+\frac{t^3}{164765082476010720}+\frac{t^2}{8}+\frac{21t}{320}+\frac{97}{13440}\right)$$
$$\beta_{\frac{3}{4}}'(t)=h^2\left(\frac{t^6}{45}+\frac{t^5}{60}-\frac{t^4}{18}-\frac{t^3}{18}+\frac{t^2}{686884395823310340}+\frac{5t}{576}+\frac{47}{40320}\right)$$
$$\beta_1'(t)=h^2\left(\frac{t^6}{180}+\frac{t^5}{120}-\frac{t^4}{288}-\frac{t^3}{144}-\frac{t^2}{2514196338285239800}+\frac{7t}{5760}+\frac{13}{80640}\right) \tag{10}$$

The second derivatives of the continuous functions are given as

$$\alpha_(''t)=\frac{16}{h^2}$$

$$\alpha''_{\frac{1}{4}}(t) = -\frac{32}{h^2}$$
$$\alpha''_{\frac{1}{2}}(t) = \frac{16}{h^2}$$
$$\beta''_0(t) = -h\left(-\frac{t^5}{15} + \frac{t^4}{12} + \frac{t^3}{36} - \frac{t^2}{24} + \frac{t}{1384440256896728800} + 1\right)$$
$$\beta''_{\frac{1}{4}}(t) = -h\left(\frac{4t^5}{15} - \frac{t^4}{6} - \frac{4t^3}{9} + \frac{t^2}{3} + \frac{t}{576411345898295360} - \frac{193}{1440}\right)$$
$$\beta''_{\frac{1}{2}}(t) = h\left(\frac{2t^5}{5} + \frac{t^4}{40538871860771056} - \frac{t^3}{12000000000} + \frac{t^2}{29029771673218900} + \frac{t}{2}\frac{21}{160}\right)$$

$$\beta''_{\frac{3}{4}}(t) = h\left(-\frac{4t^5}{15} + \frac{t^4}{6} - \frac{4t^3}{9} - \frac{t^2}{3} + \frac{t}{171721098955827550} - \frac{5}{288}\right)$$
$$\beta''_1(t) = h\left(-\frac{t^5}{15} - \frac{t^4}{12} + \frac{t^3}{36} + \frac{t^2}{24} + \frac{t}{628549084571309950} - \frac{7}{2880}\right) \tag{11}$$

The additional methods to be coupled with the main method are obtained by evaluating the first and second derivatives of (6) at $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}$ and x_{n+1} respectively to give:

$$\begin{aligned} &hy'_n + 6y_n - 8y_{n+\frac{1}{4}} + 2y_{n+\frac{1}{2}} \\ &= h^3\left(\frac{76}{19963}f_n + \frac{245}{12457}f_{n+\frac{1}{4}} - \frac{19}{4480}f_{n+\frac{1}{2}} + \frac{29}{14801}f_{n+\frac{3}{4}} - \frac{13}{36149}f_{n+1}\right) \end{aligned} \tag{12}$$

$$\begin{aligned} &hy'_{n+\frac{1}{4}} + 2y_n + 0y_{n+\frac{1}{4}} - 2y_{n+\frac{1}{2}} \\ &= h^3\left(-\frac{76}{19963}f_n - \frac{62}{6527}f_{n+\frac{1}{4}} - \frac{3}{8960}f_{n+\frac{1}{2}} - \frac{1}{8064}f_{n+\frac{3}{4}} - \frac{1}{32256}f_{n+1}\right) \end{aligned} \tag{13}$$

$$\begin{aligned} &hy'_{n+\frac{1}{2}} - 2y_n + 8y_{n+\frac{1}{4}} - 6y_{n+\frac{1}{2}} \\ &= h^3\left(\frac{13}{80640}f_n + \frac{94}{6501}f_{n+\frac{1}{4}} + \frac{97}{13440}f_{n+\frac{1}{2}} - \frac{47}{40320}f_{n+\frac{3}{4}} + \frac{13}{80640}f_{n+1}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} &hy'_{n+\frac{3}{4}} - 6y_n + 16y_{n+\frac{1}{4}} - 10y_{n+\frac{1}{2}} \\ &= h^3\left(\frac{15}{27182}f_n + \frac{209}{4679}f_{n+\frac{1}{4}} + \frac{117}{1792}f_{n+\frac{1}{2}} + \frac{58}{14347}f_{n+\frac{3}{4}} + \frac{h}{32256}f_{n+1}\right) \end{aligned} \tag{15}$$

$$\begin{aligned} &hy'_{n+1} - 10y_n + 24y_{n+\frac{1}{4}} - 14y_{n+\frac{1}{2}} \\ &= h^3\left(-\frac{11}{16128}f_n + \frac{273}{3596}f_{n+\frac{1}{4}} + \frac{324}{2551}f_{n+\frac{1}{2}} + \frac{220}{3527}f_{n+\frac{3}{4}} + \frac{104}{21449}f_{n+1}\right) \end{aligned} \tag{16}$$

$$\begin{aligned} &h^2y''_n - 16y_n + 32y_{n+\frac{1}{4}} - 16y_{n+\frac{1}{2}} \\ &= h^3\left(-\frac{233}{2880}f_n - \frac{101}{480}f_{n+\frac{1}{4}} + \frac{31}{480}f_{n+\frac{1}{2}} - \frac{41}{1440}f_{n+\frac{3}{4}} + \frac{1}{192}f_{n+1}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} &h^2y''_{n+\frac{1}{4}} - 16y_n + 32y_{n+\frac{1}{4}} - 16y_{n+\frac{1}{2}} \\ &= h^3\left(\frac{1}{60}f_n + \frac{1}{72}f_{n+\frac{1}{4}} - \frac{13}{480}f_{n+\frac{1}{2}} + \frac{1}{120}f_{n+\frac{3}{4}} - \frac{1}{720}f_{n+1}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} &h^2y''_{n+\frac{1}{2}} - 16y_n + 32y_{n+\frac{1}{4}} - 16y_{n+\frac{1}{2}} \\ &= h^3\left(-\frac{1}{2880}f_n + \frac{193}{1440}f_{n+\frac{1}{4}} + \frac{21}{160}f_{n+\frac{1}{2}} - \frac{5}{288}f_{n+\frac{3}{4}} + \frac{7}{2880}f_{n+1}\right) \end{aligned} \tag{19}$$

$$
\begin{aligned}
&h^2 y''_{n+1} - 16y_n + 32y_{n+\frac{1}{4}} - 16y_{n+\frac{1}{2}} \\
&= h^3\left(-\frac{1}{320}f_n + \frac{209}{1440}f_{n+\frac{1}{4}} + \frac{19}{96}f_{n+\frac{1}{2}} + \frac{157}{480}f_{n+\frac{3}{4}} + \frac{239}{280}f_{n+1}\right)
\end{aligned} \tag{20}
$$

Equations (8), (9) and (12)-(20) are solved using [22] block formula defined as

$$
Ay_m = hBF(y_m) + E(y_n) + hDf_n \tag{21}
$$

$A = (a_{ij}), B = (b_{ij})$, column vectors $D = (e_1 ... e_r)^T$,$D = (d_1 ... d_r)^T$, $y_m = (y_{n+1} ... y_{n+r})^T$ and $f(y_m) = (f_{n+1}, ..., f_{n+r})^T$
Thus, A, B, D and E are obtained as follows:

$$
A = \begin{bmatrix}
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & -10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
24 & -14 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
B = \begin{bmatrix}
\frac{29}{3840} & \frac{21}{2560} & \frac{-1}{3840} & \frac{1}{15360} \\
\frac{43}{1920} & \frac{21}{640} & \frac{13}{1920} & \frac{1}{3840} \\
\frac{245}{12457} & \frac{-19}{4480} & \frac{29}{14801} & \frac{-13}{36149} \\
\frac{-62}{6527} & \frac{-3}{8960} & \frac{-1}{8064} & \frac{1}{32256} \\
\frac{94}{6501} & \frac{97}{13440} & \frac{-47}{40320} & \frac{13}{80640} \\
\frac{209}{4679} & \frac{117}{1792} & \frac{58}{14347} & \frac{1}{32256} \\
\frac{273}{3596} & \frac{324}{2551} & \frac{220}{3527} & \frac{104}{21449} \\
\frac{-101}{480} & \frac{31}{480} & \frac{-41}{1440} & \frac{1}{192} \\
\frac{1}{72} & \frac{-13}{480} & \frac{1}{120} & \frac{-1}{720} \\
\frac{193}{1440} & \frac{21}{160} & \frac{-5}{288} & \frac{7}{2880} \\
\frac{13}{120} & \frac{139}{480} & \frac{37}{360} & \frac{-1}{240} \\
\frac{209}{1440} & \frac{19}{96} & \frac{157}{480} & \frac{239}{2880}
\end{bmatrix},
$$

$$
D = \begin{bmatrix}
\frac{1}{15360} \\ \frac{1}{3840} \\ \frac{76}{19963} \\ \frac{-27}{55121} \\ \frac{13}{80640} \\ \frac{15}{27182} \\ \frac{11}{16128} \\ \frac{-233}{2880} \\ \frac{1}{160} \\ \frac{-1}{2880} \\ \frac{1}{288} \\ \frac{-1}{320}
\end{bmatrix},
E = \begin{bmatrix}
1 & 0 & 0 \\
3 & 0 & 0 \\
-6 & -1 & 0 \\
-2 & 0 & 0 \\
2 & 0 & 0 \\
6 & 0 & 0 \\
10 & 0 & 0 \\
16 & 0 & -1 \\
16 & 0 & 0 \\
16 & 0 & 0 \\
16 & 0 & 0 \\
16 & 0 & 0
\end{bmatrix}.
$$

Substituting A, B, D and E into (21) yields the explicit schemes:

$$
y_{n+\frac{1}{4}} = y_n + \frac{1}{4}y'_n + \frac{h^2}{32}y''_n + h^3\left(\frac{116}{73583}f_n + \frac{83}{50042} - \frac{60}{62633}f_{n+\frac{1}{2}} + \frac{15}{37507}f_{n+\frac{3}{4}} - \frac{17}{233341}f_{n+1}\right) \tag{22}
$$

$$
y_{n+\frac{1}{2}} = y_n + \frac{1}{2}y'_n + \frac{h^2}{8}y''_n + h^3\left(\frac{347}{42269}f_n + \frac{83}{5040} - \frac{1}{168}f_{n+\frac{1}{2}} + \frac{13}{5040}f_{n+\frac{3}{4}} - \frac{19}{40320}f_{n+1}\right) \tag{23}
$$

$$
y_{n+\frac{3}{4}} = y_n + \frac{3}{4}y'_n + \frac{9h^2}{32}y''_n + h^3\left(\frac{409}{70578}f_n + \frac{164}{3155} - \frac{49}{7227}f_{n+\frac{1}{2}} + \frac{45}{7168}f_{n+\frac{3}{4}} - \frac{16}{14159}f_{n+1}\right) \tag{24}
$$

$$
y_{n+1} = y_n + y'_n + \frac{h^2}{2}y''_n + h^3\left(\frac{31}{840}f_n + \frac{34}{315} + \frac{1}{210}f_{n+\frac{1}{2}} + \frac{2}{105}f_{n+\frac{3}{4}} - \frac{1}{504}f_{n+1}\right) \tag{25}
$$

$$
y'_{n+\frac{1}{4}} = y'_n + \frac{h}{4}y''_n + h^2\left(\frac{222}{1393}f_n + \frac{3}{128}f_{n+\frac{1}{4}} - \frac{47}{3840}f_{n+\frac{1}{2}} + \frac{29}{5760}f_{n+\frac{3}{4}} - \frac{7}{7680}f_{n+1}\right) \tag{26}
$$

$$
y'_{n+\frac{1}{2}} = y'_n + \frac{h}{2}y''_n + h^2\left(\frac{53}{1440}f_n + \frac{1}{10}f_{n+\frac{1}{4}} - \frac{1}{48}f_{n+\frac{1}{2}} + \frac{1}{90}f_{n+\frac{3}{4}} - \frac{1}{480}f_{n+1}\right) \tag{27}
$$

$$
y'_{n+\frac{3}{4}} = y'_n + \frac{3h}{4}y''_n + h^2\left(\frac{147}{2560}f_n + \frac{117}{640}f_{n+\frac{1}{4}} + \frac{27}{1280}f_{n+\frac{1}{2}} + \frac{3}{128}f_{n+\frac{3}{4}} - \frac{9}{2560}f_{n+1}\right) \tag{28}
$$

$$y'_{n+1} = y'_n + hy''_n + h^2\left(\frac{7}{90}f_n + \frac{4}{15}f_{n+\frac{1}{4}} + \frac{50793}{761894}f_{n+\frac{1}{2}} + \frac{4}{45}f_{n+\frac{3}{4}} + 0\right) \tag{29}$$

$$y''_{n+\frac{1}{4}} = y''_n + h\left(\frac{251}{2880}f_n + \frac{323}{1440}f_{n+\frac{1}{4}} - \frac{11}{120}f_{n+\frac{1}{2}} + \frac{53}{1440}f_{n+\frac{3}{4}} - \frac{19}{2880}f_{n+1}\right) \tag{30}$$

$$y''_{n+\frac{1}{2}} = y''_n + h\left(\frac{29}{360}f_n + \frac{31}{90}f_{n+\frac{1}{4}} + \frac{1}{15}f_{n+\frac{1}{2}} + \frac{1}{90}f_{n+\frac{3}{4}} - \frac{1}{360}f_{n+1}\right) \tag{31}$$

$$y''_{n+\frac{3}{4}} = y''_n + h\left(\frac{27}{320}f_n + \frac{51}{160}f_{n+\frac{1}{4}} + \frac{9}{40}f_{n+\frac{1}{2}} + \frac{21}{160}f_{n+\frac{3}{4}} - \frac{3}{320}f_{n+1}\right) \tag{32}$$

$$y''_{n+1} = y''_n + h\left(\frac{7}{90}f_n + \frac{16}{45}f_{n+\frac{1}{4}} + \frac{2}{15}f_{n+\frac{1}{2}} + \frac{16}{45}f_{n+\frac{3}{4}}\frac{7}{90}f_{n+1}\right) \tag{33}$$

3.2. Development of Two-steps Method with $x_{n+\frac{1}{3}}$ and $x_{n+\frac{2}{3}}$ as the Off-step Point

In this section, we introduce $x_{n+\frac{1}{3}}$ and $x_{n+\frac{2}{3}}$ as the new off-step point. Interpolating (2) at $x = x_{n+s}$, $s = 0, \frac{1}{3}, \frac{2}{3}$ and collocating (3) at $x = x_{n+r}$, $r = 0, \frac{1}{3}, \frac{2}{3}, 1$ and 2 yields the system:

$$\begin{bmatrix} 1 & \frac{-5}{3} & \frac{53}{33} & \frac{-689}{419} & \frac{983}{591} & \frac{-519}{310} & \frac{631}{375} & \frac{-10287}{6091} \\ 1 & \frac{-11}{9} & \frac{73}{198} & \frac{236}{551} & \frac{-1033}{1308} & \frac{841}{1390} & \frac{-153}{1999} & \frac{-1835}{4279} \\ 1 & \frac{-7}{9} & \frac{-41}{99} & \frac{4586}{5741} & \frac{-199}{1342} & \frac{-397}{704} & \frac{852}{1805} & \frac{281}{1441} \\ 0 & 0 & 0 & \frac{8904}{419} & \frac{-30849}{197} & \frac{138449}{213} & \frac{45884}{9} & \\ 0 & 0 & 0 & \frac{8904}{419} & \frac{-7449}{70} & \frac{13033}{51} & \frac{-37143}{107} & \\ 0 & 0 & 0 & \frac{8904}{419} & \frac{-4555}{81} & \frac{6221}{477} & \frac{12701}{90} & \frac{-455}{44} \\ 0 & 0 & 0 & \frac{8904}{419} & \frac{-3203}{529} & \frac{-8993}{116} & \frac{3985}{167} & \frac{11958}{67} \\ 0 & 0 & 0 & \frac{8904}{419} & \frac{13726}{95} & \frac{17997}{32} & \frac{31266}{19} & \frac{68426}{17} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{\frac{2}{3}} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{3}} \\ h^3 fn + \frac{2}{3} \\ h^3 f_{n+1} \\ h^3 f_{n+2} \end{bmatrix} \tag{34}$$

Solving this system gives the values of the unknown parameters a_j, $j = 0(1)7$ as:

$$\begin{aligned} a_0 &= \frac{157}{40}y_n - \frac{48y}{5}y_{n+\frac{1}{3}} + \frac{267}{40}y_{n+\frac{2}{3}} \\ &+h^3\left(\frac{253}{42525}f_n + \frac{356}{3685}f_{n+1} + \frac{31}{17539}f_{n+2} + \frac{187}{1851}f_{n+\frac{1}{3}} + \frac{f_{n+\frac{2}{3}}}{2479}\right) \\ a_1 &= \frac{31}{8}y_n - 10y_{n+\frac{1}{3}} + \frac{49}{8}y_{n+\frac{2}{3}} \\ &-h^3\left(\frac{241}{28690}f_n - \frac{406}{3287}f_{n+1} - \frac{85}{34601}f_{n+2} - \frac{449}{3939}f_{n+\frac{1}{3}} - \frac{247}{6237}f_{n+\frac{2}{3}}\right) \\ a_2 &= \frac{11}{5}y_n - \frac{22}{5}y_{n+\frac{1}{3}} + \frac{11}{5}y_{n+\frac{2}{3}} \\ &-h^3\left(\frac{146}{15675}f_n - \frac{680}{5903}f_{n+1} - \frac{46}{16677}f_{n+2} + -\frac{166}{1999}f_{n+\frac{1}{3}} + \frac{71}{4953}f_{n+\frac{2}{3}}\right) \\ a_3 &= \frac{264}{4033} - \frac{79}{11205}f_{n+1} + \frac{8}{3683}f_{n+2} + \frac{253}{7124}f_{n+\frac{1}{3}} - \frac{79}{1611}f_{n+\frac{2}{3}} \\ a_4 &= \frac{332}{15051} - \frac{41}{11313}f_{n+1} + \frac{48}{35893}f_{n+2} + \frac{381}{25870}f_{n+\frac{1}{3}} - \frac{220}{6377}f_{n+\frac{2}{3}} \\ a_5 &= \frac{37}{25260} - \frac{31}{31135}f_{n+1} + \frac{1}{1636}f_{n+2} + \frac{63}{1067}f_{n+\frac{1}{3}} - \frac{135}{19337}f_{n+\frac{2}{3}} \\ a_6 &= \frac{11}{135050} - \frac{19}{9761}f_{n+1} - \frac{11}{58979}f_{n+2} - \frac{13}{9535}f_{n+\frac{1}{3}} + \frac{34}{11177}f_{n+\frac{2}{3}} \\ a_7 &= \frac{47}{172075} - \frac{41}{75054}f_{n+1} + \frac{10}{366177}f_{n+2} - \frac{121}{123056}f_{n+\frac{1}{3}} + \frac{15}{122039}f_{n+\frac{2}{3}} \end{aligned} \tag{35}$$

Substituting (35) in (2) yields a continuous implicit two-steps method in the form:

$$y(x) = \sum_{j=0}^{2}\alpha_j(x)y_{n+j} + \alpha_{\frac{1}{3}}(x)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(x)y_{n+\frac{2}{3}} + h\left(\sum_{j=0}^{1}\beta_j(x)f_{n+j} + \beta_{\frac{1}{3}}(x)f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}\right) \tag{36}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients, and the parameters $\alpha_j(x)$ and $\beta_j(x)$ are given by:

$$
\begin{aligned}
\alpha_0(t) &= \frac{9t^2}{2} + \frac{9t}{2} + 1 \\
\alpha_{\frac{1}{3}}(t) &= -9t^2 - 12t - 3 \\
\alpha_{\frac{2}{3}}(t) &= \frac{9t^2}{2} + \frac{15t}{2} + 3 \\
\beta_0(t) &= f_n h^3\left(\frac{3}{280}t^7 - \frac{1}{1811776004788461600}t^6 - \frac{7}{240}t^5 - \frac{1}{48}t^4 + \frac{1}{523827403997699330}t^3 \right. \\
&\left. + \frac{7}{2160}t^2 + \frac{9}{9349}t + \frac{1}{9720}\right) \\
\beta_{\frac{1}{3}}(t) &= f_{n+\frac{1}{3}} h^3\left(-\frac{27}{700}t^7 - \frac{9}{40}t^6 \frac{27}{200}t^5 + \frac{9}{80}t^4 - \frac{1}{119422617446588190}t^3 + \frac{61}{900}t^2 \right. \\
&\left. + \frac{601}{7560}t + \frac{49}{2700}\right) \\
\beta_{\frac{2}{3}}(t) &= f_{n+\frac{2}{3}} h^3\left(\frac{27}{560}t^7 + \frac{9}{160}t^6 - \frac{27}{160}t^5 - \frac{9}{32}t^4 + \frac{1}{321376444486800900}t^3 \right. \\
&\left. + \frac{29}{144}t^2 + \frac{701}{6048}t + \frac{41}{2160}\right) \\
\beta 1(t) &= f_{n+1} h^3\left(-\frac{3}{140}t^7 - \frac{3}{80}t^6 + \frac{7}{120}t^5 + \frac{3}{16}t^4 + \frac{1}{6}t^3 + \frac{11}{180}t^2 + \frac{258}{35179}t - \frac{1}{4860}\right) \\
\beta_2(t) &= f_{n+2} h^3\left(\frac{3}{2800}t^7 + \frac{3}{800}t^6 + \frac{11}{2400}t^5 + \frac{1}{480}t^4 + \frac{1}{216263684830381290 0}t^3 \right. \\
&\left. - \frac{1}{5400}t^2 + \frac{1}{272160}t + \frac{1}{97200}\right)
\end{aligned} \tag{37}
$$

where $t = \frac{x - x_n - h}{h}$.

By evaluating (36) at x_{n+1} and x_{n+2} the main methods are obtained as:

$$y_{n+1} = y_n - 3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^3\left(\frac{1}{9720}f_n + \frac{49}{2700}f_{n+\frac{1}{3}} + \frac{41}{2160}f_{n+\frac{2}{3}} - \frac{1}{4860}f_{n+1} + \frac{1}{97200}f_{n+2}\right) \tag{38}$$

$$y_{n+2} = 10y_n - 24y_{n+\frac{1}{3}} + 15y_{n+\frac{2}{3}} + h^3\left(-\frac{17}{486}f_n + \frac{19}{54}f_{n+\frac{1}{3}} - \frac{1}{108}f_{n+\frac{2}{3}} + \frac{205}{486}f_{n+1} + \frac{11}{972}f_{n+2}\right) \tag{39}$$

The first derivatives of continuous functions (36) are given as:

$$
\begin{aligned}
\alpha_0'(t) &= 9t + \frac{9}{2h} \\
\alpha_{\frac{1}{3}}'(t) &= -18t + \frac{12}{h} \\
\alpha_{\frac{2}{3}}'(t) &= 9t + \frac{15}{2h} \\
\beta_0'(t) &= h^2\left(\frac{3}{40}t^6 - \frac{1}{353797023374953600}t^5 - \frac{7}{48}t^4 - \frac{1}{12}t^3 + \frac{1}{150940880005095680}t^2 + \right. \\
&\left. \frac{7}{1080}t + \frac{9}{9349}\right) \\
\beta_{\frac{1}{3}}'(t) &= h^2\left(-\frac{27}{100}t^6 - \frac{27}{200}t^5 + \frac{27}{40}t^4 + \frac{9}{20}t^3 - \frac{1}{36491974884133584}t^2 + \frac{61}{450}t + \frac{601}{7560}\right) \\
\beta_{\frac{2}{3}}'(t) &= h^2\left(\frac{27}{80}t^6 + \frac{27}{80}t^5 - \frac{27}{32}t^4 - \frac{9}{8}t^3 + \frac{1}{90911263767416096}t^2 + \frac{29}{72}t + \frac{701}{6048}\right)
\end{aligned}
$$

$$\beta'_{n+1}(t) = h^2\left(-\frac{3}{20}t^6 - \frac{9}{40}t^5 + \frac{7}{24}t^4 + \frac{3}{4}t^3 + \frac{1}{2}t^2 + \frac{11}{90}t + \frac{258}{35179}\right)$$

$$\beta'_{n+2}(t) = h^2\left(\frac{3}{400}t^6 + \frac{9}{400}t^5 + \frac{11}{480}t^4 + \frac{1}{120}t^3 + \frac{1}{758390757276836610}t^2 - \frac{1}{2700}t + \frac{1}{272160}\right) \tag{40}$$

The second derivatives are given as follows:

$$\begin{aligned}
a''_0(t) &= \frac{9}{h^2}\\
a''_{\frac{1}{3}}(t) &= -\frac{18}{h^2}\\
a''_{\frac{2}{3}}(t) &= \frac{9}{h^2}\\
\beta''_0(t) &= h\left(\frac{9}{20}t^5 - \frac{1}{109969890506378530}t^4 - \frac{7}{12}t^3 - \frac{1}{4}t^2 + \frac{1}{75470440002547824}t + \frac{7}{1080}\right)\\
\beta''_{\frac{1}{3}}(t) &= -h\left(\frac{81}{50}t^5 + \frac{27}{40}t^4 - \frac{27}{10}t^3 - \frac{27}{20}t^2 + \frac{1}{18245987442066792}t - \frac{61}{450}\right)\\
\beta''_{\frac{2}{3}}(t) &= h\left(\frac{81}{40}t^5 + \frac{27}{16}t^4 - \frac{27}{8}t^3 - \frac{27}{8}t^2 + \frac{1}{45455631883708040}t + \frac{29}{72}\right)\\
\beta''_1(t) &= h\left(-\frac{9}{10}t^5 - \frac{9}{80}t^4 + \frac{7}{6}t^3 + \frac{9}{4}t^2 + t + \frac{11}{90}\right)\\
\beta''_2(t) &= h\left(\frac{9}{200}t^5 + \frac{9}{80}t^4 + \frac{11}{120}t^3 + \frac{1}{40}t^2 + \frac{1}{379195378638418240}t - \frac{1}{2700}\right)
\end{aligned}$$

We obtain the additional methods to be be coupled with the main methods by evaluating the first and second derivatives of (36) at $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}$ and x_{n+2} respectively to get:

$$\begin{aligned}
&hy'_n + \frac{9}{2}y_n - 6y_{n+\frac{1}{3}} + \frac{3}{2}y_{n+\frac{2}{3}}\\
&= h^3\left(\frac{95}{13608}f_n + \frac{119}{3506}f_{n+\frac{1}{3}} - \frac{17}{3024}f_{n+\frac{2}{3}} + \frac{35}{19681}f_{n+}1 - \frac{14}{328469}f_{n+2}\right)
\end{aligned} \tag{41}$$

$$\begin{aligned}
&hy'_{n+\frac{1}{3}} + \frac{3}{2}y_n - \frac{3}{2}y_{n+\frac{2}{3}}\\
&= h^3\left(\frac{43}{48359}f_n - \frac{127}{7560}f_{n+\frac{1}{3}} - \frac{23}{30240}f_{n+\frac{2}{3}} - \frac{1}{13608}f_{n+1} + \frac{1}{272160}f_{n+2}\right)
\end{aligned} \tag{42}$$

$$\begin{aligned}
&hy'_{n+\frac{2}{3}} - \frac{3}{2}y_n + 6y_{n+\frac{1}{3}} - \frac{9}{2}y_{n+\frac{2}{3}}\\
&= h^3\left(\frac{13}{68040}f_n + \frac{186}{7109}f_{n+\frac{1}{3}}\frac{181}{15120}f_{n+\frac{2}{3}} - \frac{89}{68040}f_{n+1} + \frac{2}{1046770}f_{n+2}\right)
\end{aligned} \tag{43}$$

$$\begin{aligned}
&hy'_{n+1} - \frac{9}{2}y_n + 12y_{n+\frac{1}{3}} - \frac{15}{2}y_{n+\frac{2}{3}}\\
&= h^3\left(\frac{9}{9349}f_n + \frac{601}{7560}f_{n+\frac{1}{3}} + \frac{701}{6048}f_{n+\frac{2}{3}} + \frac{258}{35179}f_{n+1} + \frac{193}{3170}f_{n+2}\right)
\end{aligned} \tag{44}$$

$$\begin{aligned}
&hy'_{n+2} - \frac{27}{2}y_n + 30y_{n+\frac{1}{3}} - \frac{33}{2}y_{n+\frac{2}{3}}\\
&= h^3\left(-\frac{385}{26240}f_n + \frac{6979}{1047}f_{n+\frac{1}{3}} - \frac{1175}{1516}f_{n+\frac{2}{3}} + \frac{652}{503}f_{n+1} + \frac{1}{2160}f_{n+2}\right)
\end{aligned} \tag{45}$$

$$\begin{aligned}
&h^2y''_{n+\frac{1}{3}} - 9y_n + 18y_{n+\frac{1}{3}} - 9y_{n+\frac{2}{3}}\\
&= h^3\left(\frac{29}{3240}f_n + \frac{7}{450}f_{n+\frac{1}{3}} - \frac{11}{360}f_{n+\frac{2}{3}} + \frac{1}{162}f_{n+1} - \frac{1}{8100}f_{n+2}\right)
\end{aligned} \tag{46}$$

$$h^2y''_{n+\frac{2}{3}} - 9y_n + 18y_{n+\frac{1}{3}} - 9y_{n+\frac{2}{3}} = h^3\left(-\frac{1}{648}f_n + \frac{331}{1800}f_{n+\frac{1}{3}} + \frac{119}{720}f_{n+\frac{2}{3}} - \frac{47}{3240}f_{n+1} + \frac{489}{32400}f_{n+2}\right) \tag{47}$$

$$h^2y''_{n+1} - 9y_n + 18y_{n+\frac{1}{3}} - 9y_{n+\frac{2}{3}} = h^3\left(\frac{7}{1080}f_n + \frac{61}{450}f_{n+\frac{1}{3}} + \frac{29}{72}f_{n+\frac{2}{3}} + \frac{11}{90}f_{n+1} - \frac{1}{2700}f_{n+2}\right) \tag{48}$$

$$h^2y''_{n+2} - 9y_n + 18y_{n+\frac{1}{3}} - 9y_{n+\frac{2}{3}} = h^3\left(-\frac{407}{1080}f_n + \frac{1261}{667}f_{n+\frac{1}{3}} - \frac{1897}{720}f_{n+\frac{2}{3}} + \frac{181}{72}f_{n+1} + \frac{489}{1786}f_{n+2}\right) \tag{49}$$

Equations (38) -(39) and (42) - (50) are solved using equation (21). A, B, D and E are obtained from the equations as:

$$A = \begin{bmatrix} 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & -15 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-3}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & \frac{-9}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & \frac{-15}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 30 & \frac{-33}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 18 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 18 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 18 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 18 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 18 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{49}{2700} & \frac{41}{2160} & \frac{-1}{4860} & \frac{1}{97200} \\ \frac{19}{54} & \frac{-1}{108} & \frac{205}{486} & \frac{11}{972} \\ \frac{119}{3506} & \frac{-17}{3024} & \frac{35}{19681} & \frac{-14}{328469} \\ \frac{-127}{7560} & \frac{-23}{30240} & \frac{-1}{13608} & \frac{1}{272160} \\ \frac{186}{7109} & \frac{181}{15120} & \frac{-89}{68040} & \frac{2}{104677} \\ \frac{601}{7560} & \frac{701}{6048} & \frac{258}{35179} & \frac{1}{272160} \\ \frac{979}{1047} & \frac{-1175}{1516} & \frac{652}{503} & \frac{193}{3170} \\ \frac{-97}{360} & \frac{47}{720} & \frac{-7}{360} & \frac{1}{2160} \\ \frac{7}{450} & \frac{-11}{360} & \frac{1}{162} & \frac{-1}{8100} \\ \frac{331}{1800} & \frac{119}{720} & \frac{-47}{3240} & \frac{7}{32400} \\ \frac{61}{640} & \frac{29}{72} & \frac{11}{90} & \frac{-1}{2700} \\ \frac{1261}{667} & \frac{-1897}{720} & \frac{181}{72} & \frac{489}{1786} \end{bmatrix} D = \begin{bmatrix} \frac{1}{9720} \\ \frac{-17}{486} \\ \frac{95}{13608} \\ \frac{-43}{48359} \\ \frac{13}{68040} \\ \frac{9}{9349} \\ \frac{-385}{2624} \\ \frac{-119}{1080} \\ \frac{29}{3240} \\ \frac{1}{648} \\ \frac{7}{1080} \\ \frac{-407}{1080} \end{bmatrix} E = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 0 & 0 \\ \frac{-9}{2} & -1 & 0 \\ \frac{-3}{2} & 0 & 0 \\ \frac{3}{2} & 0 & 0 \\ \frac{9}{2} & 0 & 0 \\ \frac{27}{2} & 0 & 0 \\ 9 & 0 & -1 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$$

Inputting A, B, D and E into (21) yields:

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3}y'_n + \frac{1}{18}y''_n + h^3\left(\frac{69}{18185}f_n + \frac{259}{70858}f_{n+\frac{1}{3}} - \frac{53}{30240}f_{n+\frac{2}{3}} + \frac{11}{22566}f_{n+1} - \frac{2}{173719}f_{n+2}\right) \tag{50}$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}y'_n + \frac{2}{9}y''_n + h^3\left(\frac{233}{11749}f_n + \frac{176}{4725}f_{n+\frac{1}{3}} - \frac{61}{5670}f_{n+\frac{2}{3}} + \frac{16}{5103}f_{n+1} - \frac{2}{255150}f_{n+2}\right) \tag{51}$$

$$y_{n+1} = y_n + y'_n + \frac{1}{2}y''_n + h^3\left(\frac{27}{560}f_n + \frac{333}{2800}f_{n+\frac{1}{3}} - \frac{9}{1120}f_{n+\frac{2}{3}} + \frac{13}{1680}f_{n+1} - \frac{1}{5600}f_{n+2}\right) \tag{52}$$

$$y_{n+2} = y_n + 2y'_n + 2y''_n + h^3\left(\frac{6}{35}f_n + \frac{144}{175}f_{n+\frac{1}{3}} - \frac{9}{70}f_{n+\frac{2}{3}} + \frac{16}{35}f_{n+1} + \frac{11}{1050}f_{n+2}\right) \tag{53}$$

$$y'_{n+\frac{1}{3}} = +y'_n + \frac{1}{3}y''_n + h^2\left(\frac{187}{6480}f_n + \frac{211}{5400}f_{n+\frac{1}{3}} - \frac{73}{4320}f_{n+\frac{2}{3}} + \frac{1}{216}f_{n+1} - \frac{7}{64800}f_{n+2}\right) \tag{54}$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{2}{3}y''_n + h^2\left(\frac{1}{15}f_n + \frac{116}{675}f_{n+\frac{1}{3}} - \frac{7}{270}f_{n+\frac{2}{3}} + \frac{4}{405}f_{n+1} - \frac{1}{4050}f_{n+2}\right) \tag{55}$$

$$y'_{n+1} = y'_n + y''_n + h^2\left(\frac{5}{48}f_n + \frac{63}{200}f_{n+\frac{1}{3}} + \frac{9}{160}f_{n+\frac{2}{3}} + \frac{1}{40}f_{n+1} - \frac{1}{2400}f_{n+2}\right) \tag{56}$$

$$y'_{n+\frac{1}{3}} = y'_n + 2y''_n + h^2\left(\frac{1}{15}f_n + \frac{36}{25}f_{n+\frac{1}{3}} - \frac{9}{10}f_{n+\frac{2}{3}} + \frac{4}{3}f_{n+1} + \frac{3}{50}f_{n+2}\right) \tag{57}$$

$$y''_n = y''_n + h\left(\frac{193}{1620}f_n + \frac{57}{200}f_{n+\frac{1}{3}} - \frac{23}{240}f_{n+\frac{2}{3}} + \frac{83}{3240}f_{n+1} - \frac{19}{32400}f_{n+2}\right) \tag{58}$$

$$y''_{n+\frac{1}{3}} = y''_n + h\left(\frac{181}{1620}f_n + \frac{34}{75}f_{n+\frac{1}{3}} + \frac{1}{10}f_{n+\frac{2}{3}} + \frac{2}{405}f_{n+1} - \frac{1}{4050}f_{n+2}\right) \tag{59}$$

$$y''_{n+1} = y''_n + h\left(\frac{7}{60}f_n + \frac{81}{200}f_{n+\frac{1}{3}} + \frac{27}{80}f_{n+\frac{2}{3}} + \frac{17}{120}f_{n+1} - \frac{1}{1200}f_{n+2}\right) \tag{60}$$

$$y''_{n+2} = y''_n - h\left(\frac{4}{15}f_n + \frac{54}{25}f_{n+\frac{1}{3}} - \frac{27}{10}f_{n+\frac{2}{3}} + \frac{35}{15}f_{n+1} + \frac{41}{150}f_{n+2}\right) \tag{61}$$

4. Analysis of the Methods

In this section, a detailed examination of the performance of the discussed methods is carried out. These involve basic properties of the numerical integration scheme for ODEs, which include the order, error constant, zero stability, and consistency. The main methods derived are discrete schemes belonging to the class of LMMs of the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^3\sum_{j=0}^{k}\beta_j f_{n+j} \tag{62}$$

Following [16] and [18], the Local Truncation Error(LTE) associated with (63) is defined by difference operator;

$$L[y(x):h] = \sum_{j=0}^{k}[\alpha_j y(x_n + jh) - h^3\beta_j f(x_n + jh)] \tag{63}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$. Expanding (63) in Taylor's series about the point x, we obtain the expression

$$L[y(x):h] = c_o y(x) + c_1 h y^{'}(x) + ...c_{p+3}h^{p+3}y^{p+3}(x) \tag{64}$$

where the $c_o, c_1, c_2...c_p...c_{p+3}$ are obtained as follows

$$c_0 = \sum_{j=0}^{k}\alpha_j \tag{65}$$

$$c_1 = \sum_{j=1}^{k}j\alpha_j \tag{66}$$

$$c_3 = \frac{1}{3!}\sum_{j=1}^{k}j^3\alpha_j \tag{67}$$

$$c_q = \frac{1}{q!}[\sum_{j=1}^{k}j^q\alpha_j - q(q-1)(q-2)(q-3)\sum_{j=1}^{k}\beta_j j^{q-3}] \tag{68}$$

In the sense of [18], equation (63) is of order p if $c_o = c_1 = c_2 = c_2 = ...c_p = c_{p+1} = c_{p+2} = 0$ and $c_{p+3} \neq 0$. The $c_{p+3} \neq 0$ is called the error constant and $c_{p+3}h^{p+3}y^{p+3}(x_n)$ is the Principal Local truncation error at the point x_n. equations (8),(9) and (36) all have order p= 8 with error constants C_{p+3} = -$\frac{1493}{845571686400}$, $\frac{1}{1410533428}$, and $\frac{-8}{97197}$ respectively,

4.1. Consistency of the Methods

The LMM is said to be consistent if it has order $p \geq 1$ and the first and second characteristic polynomials, which are defined, respectively, as

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{69}$$

and

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{70}$$

where z is the principal root, satisfy the following conditions:

$$\sum_{j=0}^{k} \alpha_j = 0 \tag{71}$$

$$\rho(1) = \rho'(1) = 0 \tag{72}$$

and

$$\rho'''(1) = 3!\sigma(1) \qquad (Henrichi,\ \ 1962) \tag{73}$$

The schemes (8), (9), and (36) are of order $\rho = 8 > 1$ and they have been investigated to satisfy conditions (I)-(III). Hence, the schemes are consistent.

4.2. Zero Stability of the Methods

The LMM is said to be Zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if and only if every root of modulus one has multiplicity not greater than the order of the differential equation. To analyze the zero-stability of the methods we present equations (22) - (25) and (51)-(54) in block form as:

$$A^0 y_m = hBf(y_m) + A' y_n hDf_n$$

where h is a fixed mesh size within a block. In line with this, the zero stability of equations (24)-(25) gives:

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{83}{5004} & \frac{-60}{62633} & \frac{15}{3750} & \frac{-17}{233341} \\ \frac{83}{5040} & \frac{-1}{168} & \frac{13}{5040} & \frac{-19}{40320} \\ \frac{164}{3155} & \frac{-49}{7227} & \frac{45}{7168} & \frac{-16}{14159} \\ \frac{34}{315} & \frac{1}{210} & \frac{2}{105} & \frac{-1}{504} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \frac{116}{73583} \\ 0 & 0 & 0 & \frac{347}{42269} \\ 0 & 0 & 0 & \frac{1409}{70578} \\ 0 & 0 & 0 & \frac{31}{840} \end{bmatrix}$$

$$\rho(R) = det\left(RA^0 - A'\right) \tag{74}$$

$$\begin{pmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{pmatrix}$$

$$e(R) = R^4 - R^3 = 0 \quad R^3(R-1) = 0$$

$$R^3 = 0 \quad \text{or} \quad R - 1 = 0$$

$$R = 0 \quad \text{or} \quad R = 1$$

Substituting A^0 and A' in equation (75) and solving for R, the values of R are obtained as 0 and 1. The block method equations (24)-(33) are zero-stable, since from (75), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1$ and for those roots with $|R_j| = 1$,the multiplicity does not exceed 3.

The zero stability of equation(51)- (54)

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{259}{70858} & \frac{-53}{30240} & \frac{11}{22566} & \frac{-2}{173719} \\ \frac{176}{4725} & \frac{-61}{670} & \frac{16}{5103} & \frac{-19}{255150} \\ \frac{333}{2800} & \frac{-9}{1120} & \frac{13}{1680} & \frac{-1}{5600} \\ \frac{144}{175} & \frac{-9}{70} & \frac{16}{35} & \frac{11}{1050} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \frac{69}{18185} \\ 0 & 0 & 0 & \frac{233}{11749} \\ 0 & 0 & 0 & \frac{27}{560} \\ 0 & 0 & 0 & \frac{6}{35} \end{bmatrix}$$

$$\rho(R) = det\left(RA^0 - A'\right) \tag{75}$$

$$\begin{pmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{pmatrix}$$

$$e(R) = R^4 - R^3 = 0 \quad R^3(R-1) = 0$$

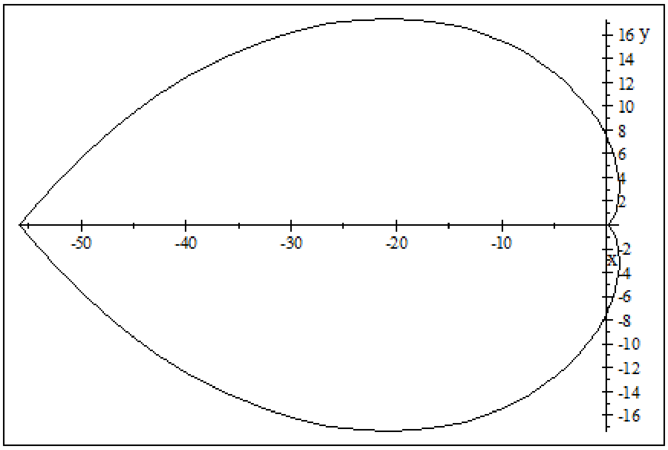

Figure 1. Region of Absolute Stability for One-Step with Off-step Points $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$

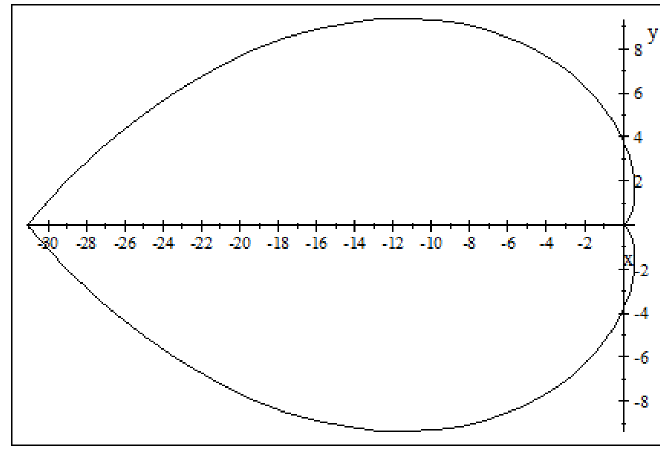

Figure 2. Region of Absolute Stability for Two-steps with Off-step Points $\frac{1}{3}$ and $\frac{2}{3}$

$$R^3 = 0 \quad \text{or} \quad R - 1 = 0$$

$$R = 0 \quad \text{or} \quad R = 1$$

Substituting A^0 and A' in the equation (76) and solving for R, the values of R are obtained as 0 *and* 1. The block method equation (51)-(54) are zero-stable, since from (76), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 3.

4.3. Convergence of the Methods

By the theorem of Dahlguist in [17], the necessary and sufficient condition for an LMM to be convergent, is that, it is consistent and zero-stable. The methods satisfy the two conditions stated in (70)-(74), thus the methods are convergent.

4.4. Region of Absolute Stability (RAS)

For the one-step with off-step Points $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$, we have

$$y_{n+\frac{3}{4}} + 3y_{n+\frac{1}{4}} - 3y_{n+\frac{1}{2}} - y_n = \frac{h^3}{15360}\left(f_n + 126f_{n+\frac{1}{2}} - 4f_{n+\frac{3}{4}} + f_{n+1}\right)$$

$$\bar{h}(z) = \frac{15360\left(z^{\frac{3}{4}} - 1 + 3z^{\frac{1}{4}} - 3z^{\frac{1}{2}}\right)}{z + 126z^{\frac{1}{2}} - 4z^{\frac{3}{4}} + 1}$$

$$\bar{h}(\theta) = \frac{15360\left(e^{i\frac{3}{4}\theta} - 3e^{i\frac{1}{2}\theta} + 3e^{i\frac{1}{4}\theta}\right)}{126e^{i\frac{1}{2}\theta} - 4e^{i\frac{3}{4}\theta} + e^{i\theta} + 1}$$

The RAS is shown in Figure 1

For the Two-steps with Off-step Points $\frac{1}{3}$ and $\frac{2}{3}$, we have

$$y_{n+1} + 3y_{n+\frac{1}{3}} - 3y_{n+\frac{2}{3}} - y_n = \frac{h^3}{97200}\left(10f_n + 1764f_{n+\frac{1}{3}} + 1845f_{n+\frac{2}{3}} - 20f_{n+1} + f_{n+2}\right)$$

$$\bar{h}(z) = \frac{97200(z + 3z^{\frac{1}{3}} - 3z^{\frac{2}{3}} - 1)}{10 + 1764z^{\frac{1}{3}} + 1845z^{\frac{2}{3}} - 20z + z^2}$$

$$\bar{h}(\theta) = \frac{97200(e^{i\theta} + 3e^{i\frac{1}{3}\theta} - 3e^{i\frac{2}{3}\theta} - 1)}{10 + 1764e^{i\frac{1}{3}\theta} + 1845e^{i\frac{2}{3}\theta} - 20e^{i\theta} + e^{i2\theta}}$$

The RAS is shown in the figure below

5. Application of the Methods

Problem 1

Consider the nonlinear Genesio equation that governs chaotic system [12].

$$x'''(t) + Ax''(t) + Bx'(t) = x^2(t) - Cx(t)$$

$$x(0) = 0.2, \quad x'(0) = -0.3, \quad x''(0) = 0.1, \quad t \in [0, 1]$$

where $A = 1.2$, $B = 2.29$ and $C = 6$ are positive constants that satisfy the condition: $AB < C$ for the existence of the solution.

Problem 2

We consider the problem of thin film flow earlier studied by [11] and sourced from [19]:

$$y''' = y^{-k}, \quad y(0) = y'(0) = y''(0) = 1$$

for case $k = 2, 3$.
We choose $h = 0.1$ for its solution.

Problem 3

Here, the constant coefficient homogeneous problem sourced from [13] is considered.

$$y''' + y' = 0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2$$

whose analytic solution is $y(x) = 2(1 - \cos x) + \sin x$ is solved with step size $h = 0.1$.

6. Discussion of results

The tables shown above present the numerical solutions of the proposed methods. Problem one, considered by [12], has

Table 1. Results for Problem 1

	Analytical Solution	One-step with $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two-steps with $v = \frac{1}{3}, \frac{2}{3}$
0.1	0.170440346269364	0.170440347007720	0.170440345967497
0.2	0.141582173138664	0.141582172011890	0.141582152176304
0.3	0.113282963581607	0.113282961614997	0.113282616429441
0.4	0.085554524922736	0.085554528032768	0.085553513248216
0.5	0.058543682864593	0.058543687069665	0.058541676426054
0.6	0.032510877478247	0.032510880258777	0.032507616958020
0.7	0.007806854082744	0.007806857068552	0.007802314622899
0.8	-0.015152336804258	-0.015152330175052	-0.015158060443852
0.9	-0.035911645118586	-0.035911630900664	-0.035918123711900
10	-0.054004107797261	-0.054004083268490	-0.054010808999158

Table 2. Results for Problem 2

	Analytical Solution	One-step with $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two-steps with $v = \frac{1}{3}, \frac{2}{3}$	[11]
0.2	1.22121001337746	1.221210004612670	1.221210004446210	1.221216123
0.4	1.48883473296637	1.488834780302660	1.488834776064700	1.488884596
0.6	1.80736134919721	1.807361398646870	1.807361383887600	1.807527232
0.8	2.17981922624938	2.179819235535390	2.179819203068140	2.180203564
10	2.60827491835218	2.608274869934920	2.608274812039350	2.609006355

Table 3. Results for Problem 3

	Exact solution	One-step with $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two-steps with $v = \frac{1}{3}, \frac{2}{3}$
0.1	0.109825086090778	0.109825086090808	0.109825086091021
0.2	0.238536175112581	0.238536175016773	0.238536175109910
0.3	0.384847228410130	0.384847228107163	0.384847228371941
0.4	0.547296354302880	0.547296353668227	0.547296354184755
0.5	0.724260414823453	0.724260413721916	0.724260414556519
0.6	0.913971243575675	0.913971241864159	0.913971243082311
0.7	1.114533312668710	1.114533310199220	1.114533311853540
0.8	1.323942672205190	1.323942668827970	1.323942670968180
0.9	1.540106973086150	1.540106968652950	1.540106971317480
1.0	1.760866373071620	1.760866367438980	1.760866370661570

Table 4. Error for Problem 1

	One-step with $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two-steps with $v = \frac{1}{3}, \frac{2}{3}$
0.1	$7.383560000 \times 10^{-10}$	$3.01867000 \times 10^{-10}$
0.2	$1.126774000 \times 10^{-9}$	$2.09623600 \times 10^{-8}$
0.3	$1.96661000 \times 10^{-9}$	$3.471521660 \times 10^{-7}$
0.4	3.1100320×10^{-9}	$1.011674520 \times 10^{-6}$
0.5	$4.20507200 \times 10^{-9}$	$2.006438539 \times 10^{-6}$
0.6	$2.78053000 \times 10^{-9}$	$3.260520227 \times 10^{-6}$
0.7	$2.98580800 \times 10^{-9}$	$4.539459845 \times 10^{-6}$
0.8	$6.62920600 \times 10^{-9}$	$5.723639594 \times 10^{-6}$
0.9	$1.42179220 \times 10^{-8}$	$6.478593314 \times 10^{-6}$
1.0	$2.45287710 \times 10^{-8}$	$6.701201897 \times 10^{-6}$

no exact solution and, the solution of [12] is not available for comparison. However, exact numerical solutions were generated via MAPLE 18 [23], and the solutions of the two proposed methods and their errors are compared in Tables 1 and 4 respectively. Tables 2 and 3 present the solutions to Problems 2 and 3, respectively, and in Tables 5 and 6, error comparisons with existing methods are displayed. It is obvious that the proposed methods performed favourably.

7. Conclusion

Two unique hybrid algorithms have been proposed as initial value problem solvers. These self-starting algorithms effectively and accurately handle third-order IVPs. On investigation, the schemes were found to be convergent and accurate. The approaches are highly recommended for solving third-order IVPs

Table 5. Error for Problem 2

	One-step with $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two steps $withv = \frac{1}{3}, \frac{2}{3}$	Error in [13] Order p = 4
0.2	8.7647900×10^{-9}	8.9312500×10^{-9}	1.070000×10^{-6}
0.4	$4.73362900 \times 10^{-8}$	$4.30983000 \times 10^{-8}$	4.130000×10^{-7}
0.6	4.9449660×10^{-8}	$3.46903000 \times 10^{-8}$	8.5100000×10^{-7}
0.8	9.2860100×10^{-9}	$2.31812400 \times 10^{-8}$	1.7100000×10^{-6}
1.0	4.8417260×10^{-8}	$1.063128300 \times 10^{-7}$	3.8600000×10^{-6}

Table 6. Error for Problem 3

	One-step $v = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	Two-steps with $v = \frac{1}{3}, \frac{2}{3}$	Error in [13]
0.1	$-3.000000 \times 10^{-14}$	$-1.3100000 \times 10^{-14}$	$1.60880000 \times 10^{-9}$
0.2	$9.580800000 \times 10^{-11}$	$1.44400000 \times 10^{-12}$	$1.03870000 \times 10^{-8}$
0.3	$3.029670000 \times 10^{-10}$	$2.07030000 \times 10^{-11}$	$2.95720000 \times 10^{-8}$
0.4	$6.346530000 \times 10^{-10}$	$6.4005000 \times 10^{-11}$	$2.31470000 \times 10^{-7}$
0.5	$1.101537000 \times 10^{-9}$	$1.430840000 \times 10^{-10}$	$4.54200000 \times 10^{-7}$
0.6	$1.711516000 \times 10^{-9}$	$2.621400000 \times 10^{-10}$	$1.47460000 \times 10^{-6}$
0.7	$2.469490000 \times 10^{-9}$	$4.259500000 \times 10^{-10}$	$2.87340000 \times 10^{-6}$
0.8	$3.37722000 \times 10^{-9}$	$6.357000000 \times 10^{-10}$	$4.68260000 \times 10^{-6}$
0.9	$4.4332000000 \times 10^{-9}$	$8.89120000 \times 10^{-10}$	$6.92170000 \times 10^{-6}$
1.0	$5.6326400000 \times 10^{-9}$	$1.184590000 \times 10^{-9}$	$9.59740000 \times 10^{-6}$

as a numerical scheme. The development of multi-order IVP solvers will be considered in a future study.

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