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On Bivariate Nadarajah-Haghighi Distribution derived from Farlie-Gumbel-Morgenstern Copula in the Presence of Covariates

Yakubu Aliyu[a,*], Umar Usman[b]

[a]*Department of Statistics, Ahmadu Bello University Zaria-Nigeria*
[b]*Department of Mathematics, Usmanu Danfodiyo University Sokoto-Nigeria*

Abstract

An important alternative distribution to the Weibull, generalized exponential and gamma distributions that is used in survival analysis is the Nadarajah-Haghighi exponential distribution. Similar to the Weibull, generalized exponential and gamma distributions, the Nadarajah-Haghighi exponential distribution is an extension of the well known exponential distribution. In this paper, a copula function commonly used to model very weak linear dependence was used to introduced a bivariate Nadarajah-Haghighi distribution. The joint survival function, joint probability density function and joint cumulative distribution were given in closed form. Bayesian method of estimation was used to estimate the model parameters considering the presence of right censoring and covariates. Posterior summaries of interest were obtained via standard Markov Monte Carlo (*MCMC*) technique. Two real data sets were used to illustrate the importance and flexibility of the bivariate model in comparison with some competing models. It was observed that, the bivariate Nadarajah-Haghighi distribution provides a better flt than bivariate exponential, bivariate Weibull, bivariate generalized exponential and bivariate modified Weibull distributions.

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1. Introduction

Exponential distribution is a well known distribution due to the constant hazard rate and memory less property it exhibits. However, in survival/ reliability studies, choosing the exponential distribution may be inappropriate since its hazard rate does not show monotone and/ or non-monotone failure rate behaviours [1]. To solve this problem, many generalizations of the exponential distribution have been developed by researchers so as to add some flexibility to the exponential distribution. These include the generalized exponential distribution developed by [2], Beta-exponential by [3], Nadarajah-Haghighi by [4]. These generalizations are among the generalizations that have received the most attention in the literature as compared with other extensions of the exponential distribution. Other generalizations of the exponential distribution include the Weibull-Burr III by [5], the generalized modified Weibull by [6], log-beta Weibull by [7], two parameter Burr X by [8] and Weibull Kumaraswamy

*Corresponding author tel. no: +2347062898441
Email address: aliyuyakubu40@mail.com (Yakubu Aliyu)

distribution by [9] to mention but a few.

An extension of the exponential distribution which serves as an alternative to the Weibull, generalized exponential and gamma distributions called the Nadarajah-Haghighi exponential distribution was introduced by [4]. The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the univariate Nadarajah-Haghighi exponential (*NH*) distribution with parameters θ and ϕ are respectively given by:

$$f(t/\theta,\phi) = \theta\phi(1+\theta t)^{\phi-1} exp\left(1-(1+\theta t)^{\phi}\right) \quad (1)$$

and

$$F(t/\theta,\phi) = 1-exp\left(1-(1+\theta t)^{\phi}\right) \quad (2)$$

where $\theta, \phi > 0$ are scale and shape parameters respectively. The *pdf* in (1) reduces to the exponential distribution when $\phi = 1$. The shape of the *NH* density can be decreasing and unimodal, while that of the hazard rate function can be monotonically increasing, monotonically decreasing or constant. The survival and hazard rate functions of the *NH* distribution are respectively given by:

$$S(t/\theta,\phi) = exp\left(1-(1+\theta t)^{\phi}\right) \quad (3)$$

and

$$h(t/\theta,\phi) = \theta\phi(1+\theta t)^{\phi-1} \quad (4)$$

The distribution has been extended by different researchers in different directions. For instance, [10] extended the distribution to the unit NH distribution which could be used to model effectively data related to rates and proportions with excess of ones. [11] extended it to the Poisson NH distribution that can be used to model reliability systems and [11] explored different classical methods of estimation to estimate the parameters of the Poisson NH distribution. [12] extended it to a three parameter discrete NH distribution that could serves as an alternative to the Poisson, negative binomial and zero inflated Poisson distribution. It was also extended to the Nadarajah-Haghighi mixture cure rate model by [13], a model that could be used to model effectively information from a population of a mixture of two type of individuals: susceptible and cured individuals.

However, in a situation where we are studying two lifetimes T_1 and T_2 associated to each unit (individual), these extensions could not be used. To solve this problem, some existing bivariate distributions in the literature such as bivariate Weibull by [14, 15, 16, 17, 18], bivariate modified weibull by [19], bivariate exponentiated discrete Weibull distribution by [20], bivariate generalized exponential by [21, 22, 23, 24, 25, 26], bivariate generalized Rayleigh by [27], bivariate inverse weibull distribution by [28, 29, 30, 31, 32, 33], bivariate frechet by [34], bivariate inverted Topp-Leone distribution by [35], bivariate discrete NH by [36] and bivariate NH by [37] distributions could be used. The bivariate discrete NH distribution by [36] could only be used in modelling bivariate discrete lifetimes while the bivariate NH distribution by [37] could be used to modeled bivariate lifetimes with positive and negative dependence structure.

In this paper, we generalized the exponential distribution to the bivariate exponential distribution considering the Nadarajah-Haghighi exponential distribution using the Farlie - Gumbel - Morgestern copula function. Hence, the paper aimed to introduced a bivariate Nadarajah-Haghighi exponential distribution that could be used for describing bivariate data that have weak correlation between variables in lifetime data. The distribution will extends the univariate exponential and Nadarajah-Haghighi exponential distributions, and serves as a good alternative to several bivariate distributions such as: bivariate exponential, bivariate generalized exponential and bivariate weibull distributions for modelling real-valued data. An important motivation of the article is to develop a guideline for estimating the parameters of the introduced distribution in the presence of censoring and covariates, which may be of deep interest to statisticians and practitioners. Some mathematical and statistical properties of the distribution was discussed. Bayesian method of estimation was employed in estimating the parameters of the distribution and finally, we evaluate the performance of the estimators by analyzing some real life data sets. The rest of the paper is organized as follows: in section 2, we derive the survival function, the probability density function and cumulative distribution function of the bivariate Nadarajah-Haghighi distribution, also in this section, parameters of the distribution were estimated using the Bayesian method of estimation procedure. Application of the introduced methodology is given in section 3 and we finally conclude in section 4.

2. Bivariate Nadarajah-Haghighi Distribution

In this section, the bivariate Nadarajah-Haghighi distribution was developed and study some of its statistical properties.

2.1. The Model

Copula functions are used in connecting the joint distribution function of two or more univariate distributions. The copula function is said to be bivariate when it connects the joint distribution function of only two univariate distributions. Let $S(t_k)$ be the univariate survival function for the random variable $T_k, k = 1, 2$, the joint survival function $S(t_1, t_2)$ is defined as:

$$S(t_1,t_2) = C_\lambda(S(t_1)S(t_2)) \quad (5)$$

for $t_1, t_2 > 0$ where λ is a measure of dependence between the random variables T_1 and T_2, and C is a copula function. The Farlie-Gumbel-Morgenstern copula (*FGM*) was first proposed by [38] and later by [39] and [40]. The joint survival function considering the FGM copula for random variables T_1 and T_2 is given by:

$$S(t_1,t_2) = S(t_1)S(t_2)[1+\lambda(1-S(t_1))(1-S(t_2))] \quad (6)$$

where the dependence parameter lies between ± 1 inclusive. The joint survival function in (6) reduces to the survival

function of the product copula function that is $S(t_1, t_2) = S(t_1) S(t_2)$ when the dependence parameter takes the value of zero. In this case, T_1 and T_2 are said to be independent. The dependence parameter λ is related to the kendall and spearman rank correlation coefficients by:

$$\tau(\lambda) = \frac{2\lambda}{9}$$

and

$$\rho(\lambda) = \frac{\lambda}{3}$$

respectively. Hence, the *FGM* copula is only appropriate in modelling weak dependences. The *FGM* copula is useful especially when dependence between the two marginal is modest in magnitude [41].

Assume T_1 and T_2 be two lifetimes associated to the same individual/ device with a dependence structure given by FGM copula function. The density functions of the marginal distributions for the lifetimes T_1 and T_2 are given by:

$$f_1(t_1) = \theta_1\phi_1(1+\theta_1 t_1)^{\phi_1-1} exp\left(1-(1+\theta_1 t_1)^{\phi_1}\right) \quad (7)$$

and

$$f_2(t_2) = \theta_2\phi_2(1+\theta_2 t_2)^{\phi_2-1} exp\left(1-(1+\theta_2 t_2)^{\phi_2}\right) \quad (8)$$

respectively, while the survival functions are given by:

$$S_1(t_1) = exp\left(1-(1+\theta_1 t_1)^{\phi_1}\right) \quad (9)$$

and

$$S_2(t_2) = exp\left(1-(1+\theta_2 t_2)^{\phi_2}\right) \quad (10)$$

respectively. The joint survival function based on the FGM copula given by (6) using the marginal survival functions in (9) and (10) is given by:

$$S(t_1, t_2) = exp\left(2-(1+\theta_1 t_1)^{\phi_1}-(1+\theta_2 t_2)^{\phi_2}\right)$$
$$\left[1+\lambda\left(1-exp\left(1-(1+\theta_1 t_1)^{\phi_1}\right)\right)\left(1-exp\left(1-(1+\theta_2 t_2)^{\phi_2}\right)\right)\right] \quad (11)$$

To obtain the joint density function for the random variables T_1 and T_2, we obtain the second derivative of $S(t_1, t_2)$ with respect to t_1 and t_2. That is, $f(t_1, t_2) = \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2}$ which yields:

$$f(t_1, t_2) = \theta_1\theta_2\phi_1\phi_2 pq_1q_2\left[1+\lambda(1+4p-2p_1-2p_2)\right] \quad (12)$$

where $p = exp\left(2-(1+\theta_1 t_1)^{\phi_1}-(1+\theta_2 t_2)^{\phi_2}\right)$, $p_1 = exp\left(1-(1+\theta_1 t_1)^{\phi_1}\right)$, $p_2 = exp\left(1-(1+\theta_2 t_2)^{\phi_2}\right)$, $q_1 = (1+\theta_1 t_1)^{\phi_1-1}$ and $q_2 = (1+\theta_2 t_2)^{\phi_2-1}$

The joint cumulative distribution function is easily obtain as:

$$F(t_1, t_2) = p(1+\lambda(1-p_1)(1-p_2)) - p_1 - p_2 + 1 \quad (13)$$

The marginal distribution functions are given by $F(t_1) = 1 - S(t_1)$ and $F(t_2) = 1 - S(t_2)$. The first partial derivatives with respect to t_{1i} and t_{2i} are given by:

$$\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{1i}} = -\theta_1\phi_1 p_1 p\left[1+\lambda(1-p_1)(1-p_2)\right] + \lambda\theta_1\phi_1 pq_1 p_1(1-p_2) \quad (14)$$

and

$$\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{2i}} = -\theta_2\phi_2 q_2 p\left[1+\lambda(1-p_1)(1-p_2)\right] + p(1-p_1)p_2 \quad (15)$$

respectively. The survival function in (11) reduce to the survival function of the product bivariate Nadarajah-Haghighi distribution when the dependent parameter assumed the value zero. It also reduce to the *FGM* bivariate exponential distribution when $\phi_1 = \phi_2 = 1$ and to the product bivariate exponential distribution when $\phi_1 = \phi_2 = 1$ and $\lambda = 0$.

2.2. Estimation

In this section, the problem of estimating the parameters of the Farlie-Gumbel-Mogenstern bivariate Nadarajah-Haghighi (*FGMBNH*) distribution based on random samples of size n was addressed using the Bayesian estimation procedure.

Let (T_{11}, T_{21}), (T_{12}, T_{22}), $\cdots$, (T_{1n}, T_{2n}) be bivariate random sample of size n from the *FGMBNH* distribution, let $\mathbf{w} = (\theta_1, \theta_2, \phi_1, \phi_2, \lambda)'$ be the vector of parameters. Then, the likelihood function $L(w)$ when the lifetimes (T_{11}, T_{21}), (T_{12}, T_{22}), $\cdots$, (T_{1n}, T_{2n}) is assumed to be non-censored can be expressed as:

$$L(\mathbf{w}) = \prod_{i=1}^{n} f(t_{1i}, t_{2i}) \quad (16)$$

substituting equation (12) and taking natural logarithm gives:

$$\ell(\Theta) = nlog(\theta_1) + nlog(\theta_2) + nlog(\phi_1) + nlog(\phi_2) +$$
$$2n - \sum_{i=1}^{n}\left[(1+\theta_1 t_{1i})^{\phi_1} + (1+\theta_2 t_{2i})^{\phi_2}\right] + (\phi_1 - 1)$$
$$\sum_{i=1}^{n} log(1+\theta_1 t_{1i}) + (\phi_2 - 1)\sum_{i=1}^{n} log(1+\theta_2 t_{2i}) +$$
$$\sum_{i=1}^{n}\left[1+\lambda\left[1+4exp\left(2-(1+\theta_1 t_{1i})^{\phi_1}-(1+\theta_2 t_{2i})^{\phi_2}\right)-\right.\right.$$
$$\left.\left.2exp\left(1-(1+\theta_1 t_{1i})^{\phi_1}\right)-2exp\left(1-(1+\theta_2 t_{2i})^{\phi_2}\right)\right]\right] \quad (17)$$

On the other hand, assume the lifetimes T_1 or T_2 or both T_1 and T_2 may be right censored. Assume also that, the censoring is independent of the time to the event of interest in the study. Let $(T_{11}, T_{21}), (T_{12}, T_{22}), \cdots, (T_{1n}, T_{2n})$ be a random sample from the *FGMBNH* distribution with parameter $\mathbf{w}$ where $\mathbf{w} = (\theta_1, \theta_2, \phi_1, \phi_2, \lambda)'$ is a parameter space. Then, each *ith* observation $i = 1, 2, \cdots, n$ fall in one of the following groups:

- U_1 : both t_{1i} and t_{2i} are uncensored observations.
- U_2 : t_{1i} is uncensored and t_{2i} is censored observation.
- U_3 : t_{1i} is censored and t_{2i} is uncensored observation.
- U_4 : both t_{1i} and t_{2i} are censored observations.

Then, the likelihood based on these conditions can be expressed as:

$$L = \prod_{i\in U_1}\left[\frac{\partial^2 S\,(t_{1i},t_{2i})}{\partial t_{1i}\partial t_{2i}}\right]\prod_{i\in U_2}\left[\frac{-\partial S\,(t_{1i},t_{2i})}{\partial t_{1i}}\right]\times \prod_{i\in U_3}\left[\frac{\partial S\,(t_{1i},t_{2i})}{\partial t_{2i}}\right]\prod_{i\in U_4} S\,(t_{1i},t_{2i}) \quad (18)$$

Assume τ_{1i} and τ_{2i} be indicator variables, such that

$$\begin{cases} \tau_{ki}=1 & \text{when } t_{ki} \text{ is an uncensored observation} \\ \tau_{ki}=0 & \text{when } t_{ki} \text{ is censored observation} \end{cases}$$

Then, the likelihood function in equation (18) is written as:

$$L = \prod_{i=1}^{n}\left[\frac{\partial^2 S\,(t_{1i},t_{2i})}{\partial t_{1i}\partial t_{2i}}\right]^{\tau_{1i}\tau_{2i}}\left[\frac{-\partial S\,(t_{1i},t_{2i})}{\partial t_{1i}}\right]^{\tau_{1i}(1-\tau_{2i})}\times \left[\frac{-\partial S\,(t_{1i},t_{2i})}{\partial t_{2i}}\right]^{(1-\tau_{1i})\tau_{2i}}[S\,(t_{1i},t_{2i})]^{(1-\tau_{1i})(1-\tau_{2i})} \quad (19)$$

observe that, the likelihood function in expression (19) reduces to the likelihood function of the non-censored observation in expression (16) when $\tau_{1i}=\tau_{2i}=1$.

Furthermore, in the presence of m covariates $x_1, x_2, \cdots, x_m$ affecting the parameters of the *FGMBNH* distribution, a link function is assumed for the parameters ϕ_1, ϕ_2, θ_1 and θ_2. That is,

$$\phi_1 = exp\,(\phi_{10}+\phi_{11}x_1+\phi_{12}x_2+\cdots+\phi_{1m}x_m) \quad (20)$$

$$\phi_2 = exp\,(\phi_{20}+\phi_{21}x_1+\phi_{k2}x_2+\cdots+\phi_{2m}x_m) \quad (21)$$

$$\theta_1 = exp\,(\theta_{10}+\theta_{11}x_1+\theta_{12}x_2+\cdots+\theta_{1m}x_m) \quad (22)$$

and

$$\theta_2 = exp\,(\theta_{20}+\theta_{21}x_1+\theta_{22}x_2+\cdots+\theta_{2m}x_m) \quad (23)$$

Therefore, to obtained inferences from the *FGMBNH* distribution in the presence of covariate(s), values of ϕ_1, ϕ_2, θ_1 and θ_2 in equations (20), (21), (22) and (23) respectively are appropriately substituted in the model.

In Bayesian method of estimation, the joint posterior distribution of the model parameters is proportional to the product of the joint prior distribution of the parameters and the likelihood function for $\mathbf{w}$ given by equation (16) when the lifetimes T_{1i} and T_{2i} are uncensored and by equation (19) when the lifetimes T_{1i} and T_{2i} are censored. Thus, assume the prior distributions

$$\pi_{11}(\theta_1) \propto \theta_1^{b_1-1}e^{a_1\theta_1} \qquad \theta_1>0$$

$$\pi_{12}(\theta_2) \propto \theta_2^{b_2-1}e^{a_2\theta_2} \qquad \theta_2>0$$

$$\pi_{21}(\phi_1) \propto \phi_1^{d_1-1}e^{c_1\phi_1} \qquad \phi_1>0$$

and

$$\pi_{22}(\phi_2) \propto \phi_2^{d_2-1}e^{c_2\phi_2} \qquad \phi_2>0$$

for the parameters θ_1, θ_2, ϕ_1 and ϕ_2 respectively, while the prior distribution of the dependence parameter (λ) was assumed to be uniform distribution. That is $\pi_3(\lambda) \sim Unif(p,q)$, where a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p and q are known hyper-parameters. The hyper-parameters a_1, a_2, b_1 b_2, c_1, c_2, d_1 and d_2 are assumed to be non-negative while the parameters p and q are assumed to be between ± 1. Lets further assumed prior independence and let the joint prior distribution for the parameters θ_1, θ_2, ϕ_1, ϕ_2 and λ be

$$\Pi(\mathbf{w}) = \pi_{11}(\theta_1)\pi_{12}(\theta_2)\pi_{21}(\phi_1)\pi_{22}(\phi_2)\pi_3(\lambda) \quad (24)$$

Moreover, assumed normal prior distribution for the covariate parameters. That is $\phi_{jk} \sim N(\mu,\sigma^2)$ and $\theta_{jk} \sim N(\mu,\sigma^2)$ for $j=1,2$ and $k=1,2,\cdots m$, where μ and σ^2 are known hyper-parameters. Hence, the joint posterior density function for $\mathbf{w}$ is given by:

$$\ell(\mathbf{w}/\mathbf{x}) \propto L \times \Pi(\mathbf{w}) \quad (25)$$

where $\mathbf{x} = (t_1, t_2, \tau_1, \tau_2)'$ is the vector of observed lifetimes, L is the likelihood function given by equations (16) and (19), while $\Pi(\mathbf{w})$ is the product of the product of the prior probability density functions. The full conditional posterior distributions are obtained by deriving the posterior distribution of each parameter given the data and all other parameters of the model. Posterior summaries of interest were obtained by using Markov Chain Monte Carlo (*MCMC*) technique. In our applications, 220,000 Gibbs samples for each model parameter was generated. However, to minimized the effect of initial values, the first 20,000 simulated samples were discarded as burn-in. Furthermore, each $20th$ simulated sample was stored so as to avoid auto-correlation between successive samples. The Bayesian estimates for the parameters were obtained using the medians of the respective posterior distributions since some simulated distributions were quite skewed. Credible intervals were also determine for each model parameter from the $2.5th$ and $97.5th$ centiles of the posterior distribution of each model parameter. Auto-correlation and trace plots were used in assessing the convergence of simulated samples.

Different formulations were assessed using the Deviance Information Criteria (*DIC*). The *DIC* a generalization of the Akaike Information Criteria for the Bayesian analysis, is obtained from the samples generated from the *MCMC* simulation [42]. According to [43], the *DIC* is computed as: $DIC = D(\hat{\mathbf{w}}) + 2n_p = 2\bar{D} - D(\hat{\mathbf{w}})$, where $D(\hat{\mathbf{w}})$ is the deviance evaluated using the mean of the model parameters which is obtained from the *MCMC* samples, $\bar{D}$ is the posterior mean of the deviance and n_p is the effective number of the model parameters which is computed as $n_p = \bar{D} - D(\hat{\mathbf{w}})$. Better model fits are indicated by lower *DIC* values. All Bayesian parameter estimates, their 95% credible Interval (*CrI*) and the *DIC* for each formulation were obtained using the OpenBUGS software version 3.2.3.

3. Applications

In this section, infections in kidney patients data from [44] and Tobacco-stained-fingers data set from [45] were used in demonstrating the applicability of the introduced methodology. The *FGMBNH* model is compared with its special case (product bivariate Nadarajah-Haghighi (*PBNH*) model) and that of some competing bivariate distributions.

3.1. Kidney data

The kidney data show the recurrence times to infection at point of insertion of catheter using portable dialysis equipment. Two recurrence times were recorded for each patient together with censoring indicator (Infection occurs =1 and censored=0) and the risk variable values: age, sex (male=1, female=2) and disease type. Assume T_1 and T_2 refers to first and second recurrence time respectively. The data was first fitted to the *FGMBNH* distribution in the presence of right censoring not including covariate and compared its performance with the fits of *FGM* bivariate exponential (*FGMBE*), *FGM* bivariate Weibull (*FGMBW*), *FGM* bivariate generalized exponential (*FGMBGE*) and *FGM* bivariate modified weibull (*FGMBMW*) distributions. It is also fitted to the special case of these bivariate distributions. That is, the product bivariate Nadarajah-Haghighi (*PBNH*), product bivariate exponential (*PBE*), product bivariate Weibull (*PBW*), product bivariate generalized exponential (*PBGE*) and product bivariate modified Weibull (*PBMW*) distributions. In addition, the data is then fitted to the *FGMBNH* model in the presence of right censoring and covariates.

In analyzing this data set, the procedure discussed in section 2.2 was followed. To be specific with the prior distributions, we assumed $Gamma(1,\ 1)$ for $\theta_1, \theta_2, \phi_1$ and ϕ_2 while we assume $U(-1,\ 1)$ for the dependence parameter(λ). Furthermore, $N(0, 10)$ was assumed for the covariate parameters.

Table 1. posterior summaries considering *FGM* copula - Kidney data

model	parameter	median	sd	MC error	95% *CrI*	DIC
FGM BNH	λ	0.5556	0.3546	0.0059	(-0.3138, 0.9781)	682.5
	ϕ_1	0.4526	0.1496	0.0039	(0.2727, 0.8525)	
	ϕ_2	0.6546	0.2932	0.0079	(0.3430, 1.5020)	
	θ_1	0.0351	0.0330	0.0007	(0.0094, 0.1226)	
	θ_2	0.0152	0.0139	0.0003	(0.0039, 0.0535)	
FGM BE	λ	0.5225	0.3345	0.0067	(-0.2720,0.9676)	686.5
	θ_1	0.0077	0.0013	0.0000	(0.0053,0.0106)	
	θ_2	0.0074	0.0015	0.0000	(0.0050,0.0106)	
FGM BW	β_1	0.7454	0.1022	0.0024	(0.5560,0.9589)	686.4
	β_2	0.8874	0.1225	0.0028	(0.6599,1.1390)	
	λ	0.5608	0.3536	0.0073	(-0.3120,0.9780)	
	θ_1	0.0292	0.0190	0.0005	(0.0090,0.0797)	
	θ_2	0.0134	0.0108	0.0002	(0.0035,0.0433)	
FGM BGE	β_1	0.7479	0.1595	0.0047	(0.4950,1.1240)	687.8
	β_2	1.0470	0.2368	0.0072	(0.6696,1.5840)	
	λ	0.5318	0.346	0.0105	(-0.2887,0.9743)	
	θ_1	0.0062	0.0016	0.0000	(0.0036,0.0098)	
	θ_2	0.0079	0.0020	0.0000	(0.0045,0.0122)	
FGM BMW	α_1	0.0427	0.0211	0.0016	(0.0134,0.0942)	689.3
	α_2	0.0217	0.0189	0.0013	(0.0047,0.0770)	
	β_1	0.6246	0.1068	0.0084	(0.4233,0.8568)	
	β_2	0.7354	0.1594	0.0128	(0.4085,1.0780)	
	λ	0.0009	0.0007	0.0000	(0.0001,0.0026)	
	ϕ_1	0.0011	0.0009	0.0001	(0.0001,0.0035)	
	ϕ_2	0.6511	0.3084	0.0263	(-0.1261,0.9863)	

Table 2. posterior summaries considering product copula - Kidney data

model	parameter	median	sd	MC error	95% *CrI*	DIC
PB NH	ϕ_1	0.4545	0.1598	0.0043	(0.2727, 0.8799)	683.2
	ϕ_2	0.6627	0.2860	0.0072	(0.3551, 1.4520)	
	θ_1	0.0346	0.0308	0.0005	(0.0092, 0.1237)	
	θ_2	0.0149	0.0124	0.0002	(0.0041, 0.0513)	
PBE	θ_1	0.0077	0.0013	0.0000	(0.0054,0.0106)	687.4
	θ_2	0.0076	0.0015	0.0000	(0.0051,0.0110)	
PBW	β_1	0.7487	0.0972	0.0034	(0.5675,0.9525)	686.9
	β_2	0.8998	0.1298	0.0047	(0.6717,1.1770)	
	θ_1	0.0286	0.0172	0.0005	(0.0094,0.0754)	
	θ_2	0.0126	0.0104	0.0003	(0.0029,0.0416)	
PB GE	β_1	0.7498	0.1574	0.0039	(0.4968,1.1180)	688.9
	β_2	1.0510	0.2453	0.0058	(0.6632,1.6240)	
	θ_1	0.0063	0.0016	0.0000	(0.0036,0.0098)	
	θ_2	0.0080	0.0020	0.0000	(0.0045,0.0124)	
PB MW	α_1	0.0378	0.0247	0.0018	(0.0113,0.1058)	688.7
	α_2	0.0185	0.0170	0.0011	(0.0056,0.0695)	
	β_1	0.6436	0.1259	0.0096	(0.4083,0.9010)	
	β_2	0.7752	0.1457	0.0108	(0.4663,1.0110)	
	ϕ_1	0.0009	0.0007	0.0000	(0.0001,0.0026)	
	ϕ_2	0.0010	0.0008	0.0000	(0.0000,0.0031)	

Table 1 give the posterior summary statistics for the bivariate Nadarajah-Haghighi distribution considering the *FGM* copula function compared to the aforementioned bivariate distributions. The table also gives the MC error for each posterior estimate. The MC error of the posterior estimates of the fitted models were all less than $\frac{1}{20}$ of standard deviation of the estimates, this showed that the posterior estimates have reasonably good precision. The results also showed that, the *FGMBNH* distribution is more efficient than the *FGMBE*, *FGMBW*, *FGMBGE* and *FGMBMW* distributions, since it has the least information criteria value. Furthermore, the estimates of the dependent parameter (λ) are similar for all the distributions considered.

Table 2 give the posterior summary statistics for the bivariate Nadarajah-Haghighi distribution considering the product copula function compared to the aforementioned bivariate distributions. Similar to the results for the distributions based on the *FGM* copula, the *FGMBNH* distribution is more efficient than the aforementioned distributions since it has the least information criteria value. Comparing between the estimates of the bivariate distributions considering the *FGM* and product copula functions reveal that, the estimates for each *FGM* bivariate distribution are similar to its corresponding product bivariate distribution. Furthermore, comparing between the fits of the bivariate distributions considering the *FGM* copula and the bivariate distributions considering the product copula reveal that, except for the *FGMBMW* distribution, the fits of the bivariate distributions considering the *FGM* copula are more efficient than the bivariate distributions considering the product copula function since they have the least information criteria values.

Furthermore, the data is fitted to the *FGMBNH* distribu-

Table 3. posterior summaries considering *FGM* copula in the presence of covariate - Kidney data

model	parameter	median	sd	MC error	95% *CrI*	DIC
model I	λ	0.5223	0.3526	0.0145	(-0.3288,0.9705)	682.7
	ϕ_{10}	-0.3115	0.2658	0.0111	(-0.8238,0.2320)	
	ϕ_{11}	0.5135	0.2207	0.0081	(0.0716,0.9167)	
	ϕ_{12}	-0.0015	0.0070	0.0002	(-0.0149,0.0126)	
	ϕ_{20}	-0.1204	0.2739	0.0107	(-0.6340,0.4389)	
	ϕ_{21}	0.0129	0.2110	0.0074	(-0.4173,0.4083)	
	ϕ_{22}	-0.0035	0.0077	0.0002	(-0.0181,0.0124)	
	θ_1	0.0137	0.0111	0.0004	(0.0035,0.0450)	
	θ_2	0.0111	0.0076	0.0002	(0.0033,0.0323)	
model II	λ	0.7428	0.2967	0.0052	(-0.1071,0.9901)	708.9
	ϕ_1	0.2761	0.0417	0.0007	(0.2032,0.3684)	
	ϕ_2	0.3175	0.0541	0.0009	(0.2226,0.4331)	
	θ_{10}	-0.2323	0.3129	0.0046	(-0.8392,0.3709)	
	θ_{11}	0.2237	0.2903	0.0047	(-0.3455,0.7947)	
	θ_{12}	-0.0452	0.0111	0.0002	(-0.0662,-0.0226)	
	θ_{20}	-0.2378	0.3175	0.0053	(-0.8479,0.4009)	
	θ_{21}	0.0416	0.2903	0.0045	(-0.5374,0.6097)	
	θ_{22}	-0.0582	0.0105	0.0002	(-0.0780,-0.0371)	
model III	λ	0.7477	0.2645	0.0024	(0.0022,0.9893)	709.1
	ϕ_{10}	-1.1140	0.1865	0.0039	(-1.4900,-0.7674)	
	ϕ_{11}	0.3020	0.2131	0.0022	(-0.1292,0.7034)	
	ϕ_{12}	0.0010	0.0080	0.0002	(-0.0138,0.0178)	
	ϕ_{20}	-0.9954	0.1906	0.0034	(-1.3930,-0.6387)	
	ϕ_{21}	0.0777	0.2234	0.0027	(-0.3676,0.5077)	
	ϕ_{22}	0.0040	0.0087	0.0002	(-0.0125,0.0218)	
	θ_{10}	-0.4131	0.3111	0.0038	(-1.0280,0.2000)	
	θ_{11}	0.1566	0.2959	0.0032	(-0.4234,0.7434)	
	θ_{12}	-0.0537	0.0135	0.0003	(-0.0803,-0.0273)	
	θ_{20}	-0.3647	0.3090	0.0030	(-0.9615,0.2536)	
	θ_{21}	-0.0040	0.2932	0.0030	(-0.5818,0.5714)	
	θ_{22}	-0.0687	0.0127	0.0002	(-0.0935,-0.0434)	

Table 4. posterior summaries considering *FGM* copula - Tobacco-stained-fingers data

model	parameter	median	sd	MC error	95% CrI	DIC
FGM BNH	λ	0.9036	0.1220	0.0019	(0.5478,0.9967)	627.4
	ϕ_1	0.2200	0.0723	0.0016	(0.1334,0.4149)	
	ϕ_2	0.2115	0.0446	0.0007	(0.1497,0.3258)	
	θ_1	1.4660	0.8177	0.0151	(0.4833,3.6510)	
	θ_2	1.4890	0.7025	0.0123	(0.6029,3.2950)	
FGM BE	λ	0.9089	0.1155	0.0014	(0.5742,0.9966)	659.4
	θ_1	0.1417	0.0227	0.0003	(0.1017,0.1900)	
	θ_2	0.1037	0.0122	0.0001	(0.0817,0.1292)	
FGM BW	β_1	0.6686	0.0882	0.0009	(0.5051,0.8524)	631.1
	β_2	0.6192	0.0665	0.0009	(0.4964,0.7597)	
	λ	0.9018	0.1244	0.0016	(0.5383,0.9963)	
	θ_1	0.2040	0.0369	0.0004	(0.1419,0.2863)	
	θ_2	0.2029	0.0325	0.0004	(0.1452,0.2723)	
FGM BGE	β_1	0.6528	0.0994	0.0013	(0.486,0.8726)	634.2
	β_2	0.5807	0.0742	0.0009	(0.4479,0.7384)	
	λ	0.9032	0.1205	0.0015	(0.5487,0.9963)	
	θ_1	0.0733	0.0258	0.0003	(0.0333,0.1336)	
	θ_2	0.0504	0.0133	0.0002	(0.0282,0.0801)	
FGM BMW	α_1	0.1898	0.0362	0.0006	(0.1273,0.2684)	634.4
	α_2	0.1911	0.0325	0.0005	(0.1348,0.2607)	
	β_1	0.6302	0.0926	0.0014	(0.4597,0.8225)	
	β_2	0.5728	0.0745	0.0013	(0.4284,0.7207)	
	ϕ_1	0.0213	0.0255	0.0004	(0.0008,0.0962)	
	ϕ_2	0.0173	0.0177	0.0003	(0.0008,0.0662)	
	λ	0.9010	0.1299	0.0022	(0.5132,0.9961)	

tion by taking sex and age as covariates. The covariates were assumed to have effect on ϕ_k, θ_k and on both ϕ_k and θ_k for $k = 1, 2$. The posterior summary statistics of the fits of the data are given in table 3. It is observed that the covariates have effect on ϕ_k.

Table 5. posterior summaries considering product copula - Tobacco-stained-fingers data

model	parameter	median	sd	MC error	95% CrI	DIC
PB NH	ϕ_1	0.1917	0.0626	0.0010	(0.1173,0.3612)	643.4
	ϕ_2	0.2027	0.0443	0.0008	(0.1434,0.3138)	
	θ_1	1.6020	0.9106	0.0132	(0.5268,4.0580)	
	θ_2	1.5930	0.7913	0.0136	(0.6229,3.5410)	
PBE	θ_1	0.1277	0.0214	0.0002	(0.0907,0.1738)	678.7
	θ_2	0.1027	0.0123	0.0001	(0.0806,0.1288)	
PBW	β_1	0.6412	0.0873	0.0009	(0.4799,0.8255)	647.6
	β_2	0.6080	0.0675	0.0007	(0.486,0.7496)	
	θ_1	0.1896	0.0353	0.0004	(0.1287,0.2658)	
	θ_2	0.2030	0.0333	0.0004	(0.1452,0.2758)	
PB GE	β_1	0.6278	0.0967	0.0011	(0.4634,0.8398)	650
	β_2	0.5673	0.0747	0.0008	(0.438,0.7287)	
	θ_1	0.0587	0.0227	0.0002	(0.0244,0.1125)	
	θ_2	0.0476	0.0131	0.0001	(0.0263,0.0771)	
PB MW	α_1	0.1771	0.0351	0.0005	(0.1186,0.2570)	650.9
	α_2	0.1916	0.0329	0.0004	(0.1343,0.2625)	
	β_1	0.6025	0.0929	0.0013	(0.4298,0.7945)	
	β_2	0.5646	0.0752	0.0009	(0.4180,0.7148)	
	ϕ_1	0.0211	0.0249	0.0003	(0.0008,0.0941)	
	ϕ_2	0.0176	0.0180	0.0002	(0.0008,0.0673)	

3.2. *Tobacco-stained-fingers data set*

In this section, we consider the tobacco stained finger data set obtained from [45]. The data consist of a sample of 143 smokers screened between March 2006 and January 2010 in a 180-bed community hospital in La Chauxde-Fonds, Switzerland. Information on death and hospital admission of the patients were collected until June 2014. For more details on this data set see [45].

To demonstrate the applicability of the introduced methodology using this data set, it was considered as T_1, the time before the first hospital readmission in smokers with stains on their fingers. On the other hand, the survival time of the patient with tobacco-tar stain on their fingers was considered as T_2. This data is censored in case of death before the closure date. Similar procedure was followed in analyzing this data set as in section 3.1.

The posterior summary statistics of the fits of the bivariate distributions to the tobacco-stained-fingers data are given in tables 4 and 5. The MC error of the posterior estimates of the fitted models considering the tobacco-stained-fingers data were all less than $\frac{1}{20}$ of standard deviation of the estimates, this showed that the posterior estimates have reasonably good precision. Comparing the information criteria values in tables 4 and 5 showed that, the introduced BNH distribution fits the tobacco-stained-fingers data more efficiently than the compared bivariate distributions. Also, comparing between fits of the bivariate distributions considering the FGM copula and the bivariate distributions considering the product copula showed that, the bivariate distributions considering the FGM copula

fits the tobacco-stained-fingers data better than the bivariate distributions considering the product copula function. For the FGM bivariate models, the estimates of the dependent parameter are similar for all the considered bivariate distributions. Moreover, the estimates of the shape and scale parameters of the bivariate distributions considering the FGM copula are similar to their bivariate distribution counterparts considering the product copula.

4. Conclusion

In real life applications, lifetime data of two organs affected by a given type of treatment or therapy applied to the same individual that present some type of weak dependence are usually obtained. In such a situation, the use of copula functions would be a very good alternative in modelling this type of bivariate lifetime data in presence of censored data and covariates. In this article, the analytical structures of the statistical methodology associated with the modelling of lifetimes assuming weak linear dependence was introduced. *FGM* copula function was used, however, there are many other copula functions that could be used to build new bivariate lifetime models depending on the assumptions established about the form of the relationship between the bivariate lifetimes (T_1 and T_2). Also, marginal *NH* distributions were used since the distribution usually provide goodness of fit for lifetime data, as constant, decreasing or increasing hazard functions, features that are characteristic of the studied variables (T_1 and T_2). Hence, other univariate lifetime distributions such as inverted *NH*, unit *NH*, discrete *NH* distributions could also be used. Two real data sets: Kidney data and Tobacco-stained-fingers data sets were used in demonstrating the applicability of the introduced methodology. The obtained results were compared assuming weak dependence that can be model using copula function, with those obtained considering independence between lifetimes. Moreover, the introduced methodology is further compared with some competing bivariate distributions. The proposed approach could be useful in many areas of interest, including medical sciences, physical sciences and engineering studies.

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Appendices

The following OpenBugs computational code can be used to obtained the posterior estimates from Bivariate Nadarajah-Haghighi distribution with parameters θ_1, θ_2, ϕ_1, ϕ_2 and λ.

```
# The model assuming product copula function
model
{
for (i in 1:N) {
s1[i]< -exp(1-pow((1+theta1*t1[i]), phi1))
s2[i]< -exp(1-pow((1+theta2*t2[i]), phi2))
b1[i]<- pow((1+theta1*t1[i]), phi1-1)
b2[i]<- pow((1+theta2*t2[i]), phi2-1)
f1[i]< -theta1*phi1*b1[i]*s1[i]
f2[i]< -theta2*phi2*b2[i]*s2[i]
ft1t2[i]< - f1[i]*f2[i]
del1[i]<- -f1[i]*s2[i]
del2[i]<- -f2[i]*s1[i]
st1t2[i]<- s1[i]*s2[i]
L1[i]< - pow(ft1t2[i], d1[i]*d2[i])*pow(st1t2[i],
(1-d1[i])*(1-d2[i]))*pow(del1[i], d1[i]*(1-d2[i]))*
pow( del2[i], d2[i]*(1-d1[i]))
L[i]<- abs(L1[i])
logL[i] < - log(L[i])

zeros[i] < - 0
zeros[i] ~ dloglik(logL[i])

}

# prior distributions
theta1 ~ dgamma(1,1)
theta2 ~ dgamma(1,1)
phi1 ~ dgamma(1,1)
phi2 ~ dgamma(1,1)
}

# The model assuming FGM copula function
model
{
for (i in 1:N) {
s1[i]< -exp(1-pow((1+theta1*t1[i]), phi1))
s2[i]< -exp(1-pow((1+theta2*t2[i]), phi2))
b1[i]<- pow((1+theta1*t1[i]), phi1-1)
b2[i]<- pow((1+theta2*t2[i]), phi2-1)
f1[i]< -theta1*phi1*b1[i]*s1[i]
f2[i]< -theta2*phi2*b2[i]*s2[i]
f12[i]<- exp(2-pow((1+theta1*t1[i]), phi1)-
pow((1+theta2*t2[i]), phi2))
ft1t2[i]< - theta1*theta2*phi1*phi2*f12[i]*
b1[i]*b2[i]*(1+lambda*(1-2*s1[i])*(1-2*s2[i]))
del1[i]<- f1[i]*s2[i]* (1+lambda*(1-2*s1[i])
*(1-s2[i]))
del2[i]<- f2[i]*s1[i]* (1+lambda*(1-2*s2[i])
*(1-s1[i]))
st1t2[i]<- f12[i]* (1+lambda*(1-s2[i])*(1-s1[i]))
L1[i]< - pow(ft1t2[i], d1[i]*d2[i])*pow(st1t2[i],
(1-d1[i])*(1-d2[i]))*pow(del1[i], d1[i]*(1-d2[i]))
*pow( del2[i], d2[i]*(1-d1[i]))
L[i]<- abs(L1[i])
logL[i] < - log(L1[i])

zeros[i] < - 0
zeros[i] ~ dloglik(logL[i])

}

# prior distributions
theta1 ~ dgamma(1,1)
theta2 ~ dgamma(1,1)
phi1 ~ dgamma(1,1)
phi2 ~ dgamma(1,1)
lambda~ dunif(-1,1)
```

```
}

# The model assuming FGM copula function assuming
covariate effect on the shape parameters

model
{
for (i in 1:N) {
phi1[i]<-exp(phi10+phi11*x1[i]+phi12*x2[i])
phi2[i]<-exp(phi20+phi21*x1[i]+phi22*x2[i])
s1[i]< -exp(1-pow((1+theta1*t1[i]), phi1[i]))
s2[i]< -exp(1-pow((1+theta2*t2[i]), phi2[i]))
b1[i]<- pow((1+theta1*t1[i]), phi1[i]-1)
b2[i]<- pow((1+theta2*t2[i]), phi2[i]-1)
f1[i]< -theta1*phi1[i]*b1[i]*s1[i]
f2[i]< -theta2*phi2[i]*b2[i]*s2[i]
f12[i]<- exp(2-pow((1+theta1*t1[i]), phi1[i])-
pow((1+theta2*t2[i]), phi2[i]))
ft1t2[i]< - theta1*theta2*phi1[i]*phi2[i]*f12[i]*
b1[i]*b2[i]*(1+lambda*(1-2*s1[i])*(1-2*s2[i]))
del1[i]<- f1[i]*s2[i]* (1+lambda*(1-2*s1[i])
*(1-s2[i]))
del2[i]<- f2[i]*s1[i]* (1+lambda*(1-2*s2[i])
*(1-s1[i]))
st1t2[i]<- f12[i]* (1+lambda*(1-s2[i])*(1-s1[i]))
L1[i]< - pow(ft1t2[i], d1[i]*d2[i])*pow(st1t2[i],
(1-d1[i])*(1-d2[i]))*pow(del1[i], d1[i]*(1-d2[i]))
*pow( del2[i], d2[i]*(1-d1[i]))
L[i]<- abs(L1[i])
logL[i] < - log(L1[i])

zeros[i] < - 0
zeros[i] ~ dloglik(logL[i])
}

# prior distributions
theta1 ~ dgamma(1,1)
theta2 ~ dgamma(1,1)
phi10~dnorm(0, 10)
phi11~dnorm(0, 10)
phi12~dnorm(0, 10)
phi20~dnorm(0, 10)
phi21~dnorm(0, 10)
phi22~dnorm(0, 10)
lambda ~ dunif(-1, 1)
}
```

In this codes, N is the sample size, S1[i] is the survival function given in equation (9), S2[i] is the survival function given in equation (10), f1[i] is the probability density function given in equation (7), f2[i] is the probability density function given in equation (8), ft1t2 [i] is the joint probability density function (pdf) given in equation (12), del1[i] is an expression given in equation (14), del2[i] is an expression given in equation (15), St1t2[i] is the joint survival function given in equation (11) and L[i] is the likelihood function given in equation (19). For the product copula function, the dependence parameter (lambda) assumes the value zero, and hence, the joint density function reduces to the product of the pdfs in equations (7) and (8). The joint survival function also reduces to the product of the survival functions in equations (9) and (10). For the models that assume covariate effect on the parameter(s) of the model, appropriate substitutions are made in the codes for the FGM copula function as discussed in the work.

Table 5. Kidney data

Patient	T_1	T_2	δ_1	δ_1	Sex	Age	Disease types
1	8	16	1	1	1	28	3
2	23	13	1	0	0	48	0
3	22	28	1	1	1	32	3
4	447	318	1	1	0	31.5	3
5	30	12	1	1	1	10	3
6	24	245	1	1	0	16.5	3
7	7	9	1	1	1	51	0
8	511	30	1	1	0	55.5	0
9	53	196	1	1	0	69	1
10	15	154	1	1	1	51.5	0
11	7	333	1	1	0	44	1
12	141	8	1	0	0	34	3
13	96	38	1	1	0	35	1
14	149	70	0	0	0	42	1
15	536	25	1	0	0	17	3
16	17	4	1	0	1	60	1
17	185	177	1	1	0	60	3
18	292	114	1	1	0	43.5	3
19	22	159	0	0	0	53	0
20	15	108	1	0	0	44	3
21	152	562	1	1	1	46.5	2
22	402	24	1	0	0	30	3
23	13	66	1	1	0	62.5	1
24	39	46	1	0	0	42.5	1
25	12	40	1	1	1	43	1
26	113	201	0	1	0	57.5	1
27	132	156	1	1	0	10	0
28	34	30	1	1	0	52	1
29	2	25	1	1	1	53	0
30	130	26	1	1	0	54	0
31	27	58	1	1	0	56	1
32	5	43	0	1	0	50.5	1
33	152	30	1	1	0	57	2
34	190	5	1	0	0	44.5	0
35	119	8	1	1	0	22	3
36	54	16	0	0	0	42	3
37	6	78	0	1	0	52	2
38	63	8	1	0	1	60	2

Table 5. Tobacco data

n	sex	age	T_1	δ_1	T_2	δ_2
control31	M	83	0.063014	1	0.013699	0
control34	M	36	6.69863	0	6.079452	0
control69	M	49	5.021918	0	4.40274	0
control55	M	39	5.59726	0	4.978082	0
control4	M	49	7.879452	0	3.876712	0
control39	F	85	5.309589	1	4.060274	0
control54	F	50	5.6	0	4.980822	0
control14	M	58	6.093151	1	2.490411	1
control24	F	80	0.065753	1	0.249315	0
control11	M	33	7.739726	0	0.646575	0
control5	M	64	1.405479	1	1.389041	1
control53	M	47	5.791781	0	3.4	0
control62	F	57	5.227397	0	4.545206	0
control27	M	46	7.435616	0	4.945206	0
control44	F	83	6.019178	0	1.654795	1
control17	M	64	0.668493	1	0.454795	1
control25	F	76	3.038356	1	0.227397	0
control16	M	64	0.347945	1	0.347945	0
control32	F	83	6.460274	0	2.753425	0
control57	M	71	5.536986	0	0.10137	1
control66	M	65	0.383562	1	0.375342	1
control8	M	50	7.605479	0	6.986301	0
control65	M	60	5.213699	0	4.594521	0
control59	F	41	5.536986	0	0.052055	0
control9	M	55	7.80274	0	7.183562	0
control33	F	35	6.506849	0	1.19726	0
control3	M	57	4.693151	1	2.139726	0
control63	M	55	0.575342	1	0.065753	0
control56	M	51	4.652055	1	4.109589	0
control7	F	51	7.80548	0	7.186301	0
control47	M	59	6.079452	0	2.767123	1
control71	M	50	4.978082	0	4.358904	0
control26	M	56	0.252055	1	0.252055	1
control15	M	70	0.005479	1	0.005479	0
control60	M	41	5.161644	0	4.542466	0
control36	F	82	2.736986	1	1.865753	0
control1	F	46	7.928767	0	0.654795	0
control68	M	74	5.410959	0	0.334247	1
control51	F	71	3.263014	1	3.126027	1
control46	F	46	0.257534	1	0.109589	0
control21	F	76	6.627397	1	2.161644	0
control41	F	59	6.243835	0	1.043836	1
control37	M	59	6.112329	0	3.271233	0
control18	F	58	7.835617	0	7.216438	0
control19	F	75	1.734247	1	0.69589	1
control67	F	68	5.367123	0	4.747945	0
control64	F	60	5.19726	0	4.578082	0
control45	F	71	6.172603	0	5.553425	0
control10	F	50	7.739726	0	7.120548	0
control61	M	64	5.254795	0	1.189041	0
control35	M	62	4.257534	1	2.553425	0

Table 5. Tobacco data cont.

n	sex	age	T_1	δ_1	T_2	δ_2
control13	M	37	3.512329	1	3.512329	0
control30	M	66	7.041096	0	1.252055	0
control70	F	83	0.167123	1	0.167123	0
control28	M	53	6.923288	0	0.052055	0
control2	M	50	7.969863	0	6.70411	0
control12	F	77	1.561644	1	1.49863	1
control58	F	81	5.534246	0	4.915069	0
control40	F	78	0.29589	1	0.167123	1
control29	M	65	6.926027	0	0.432877	1
control48	M	37	5.646575	0	0.638356	0
control43	M	64	0.243836	1	0.032877	1
control52	F	50	5.657534	0	5.038356	0
control49	M	76	0.682192	1	0.569863	0
control23	F	66	7.210959	0	0.216438	1
control20	M	66	7.827397	0	1.131507	0
control6	M	51	7.816438	0	0.276712	0
control50	M	40	5.619178	0	5	0
control22	F	65	7.128767	0	0.169863	0
control38	F	60	5.904109	0	3.413699	0
Tache40	M	68	1.183562	1	0.687671	1
Tache7	M	45	8.819178	0	1.747945	0
Tache64	M	54	7.419178	0	0.550685	0
Tache60	M	55	7.419178	0	1.59726	0
Tache61	F	71	7.512329	0	0.339726	0
Tache16	F	57	1.394521	1	0.071233	0
Tache57	M	64	0.315068	1	0.150685	1
Tache26	F	49	8.372602	0	0.463014	0
Tache73	M	65	2.180822	1	0.027397	0
Tache50	F	75	1.90411	1	1.69589	1
Tache42	M	73	7.260274	1	1.863014	1
Tache9	M	63	5.99726	1	0.482192	0
Tache25	M	62	0.128767	1	0.120548	1
Tache37	F	69	5.624658	1	0.320548	0
Tache6	M	50	8.819178	0	5.969863	0
Tache31	M	79	2.021918	1	0.076712	0
Tache67	F	88	1.780822	1	0.90411	0
Tache72	F	63	2.161644	1	0.071233	0
Tache8	M	71	8.873973	0	0.567123	1
Tache33	M	27	8.284931	0	7.665753	0
Tache30	M	56	8.265754	0	0.030137	0
Tache56	F	70	6.838356	1	2.616438	1
Tache13	M	69	7.882192	0	1.139726	1
Tache70	M	78	7.339726	0	1.936986	1
Tache19	M	50	0.679452	1	0.679452	0
Tache14	F	72	3.230137	1	0.8	0
Tache59	M	74	0.30137	1	0.30137	0
Tache22	M	52	2.123288	1	0.956164	1
Tache36	M	85	0.550685	1	0.139726	1
Tache53	M	50	3.60274	1	1.038356	0
Tache2	M	47	5.323287	1	4.482192	1

Table 5. Tobacco data cont.

n	sex	age	T_1	δ_1	T_2	δ_2
Tache18	M	69	8.452055	0	0.136986	0
Tache3	M	61	2.89589	1	1.649315	0
Tache1	M	67	0.071233	1	0.071233	0
Tache11	M	53	3.923288	1	1.994521	0
Tache35	F	44	8.30137	0	4.073973	0
Tache46	M	63	7.808219	0	0.09589	0
Tache47	F	66	7.769863	1	3.106849	1
Tache58	F	51	7.904109	0	0.424658	0
Tache21	F	69	2.232877	1	0.060274	0
Tache55	M	50	0.520548	1	0.109589	1
Tache29	M	45	0.052055	1	0.052055	0
Tache27	M	54	8.378082	0	1.032877	0
Tache5	M	38	8.821918	0	1.830137	0
Tache24	M	37	8.394521	0	7.775342	0
Tache51	M	62	7.547945	0	0.906849	0
Tache39	M	67	0.238356	1	0.013699	1
Tache54	M	53	7.616438	0	0.035616	0
Tache45	M	36	8.128767	0	7.509589	0
Tache20	M	49	8.380822	0	5.019178	0
Tache48	M	80	0.252055	1	0.016438	0
Tache4	F	64	0.136986	1	0.136986	0
Tache38	M	79	3.468493	1	0.39726	0
Tache49	M	67	7.654795	0	7.035616	0
Tache34	M	75	0.665753	1	0.386301	0
Tache65	F	63	7.128767	0	0.323288	0
Tache69	M	77	0.947945	1	0.071233	1
Tache17	M	83	6.306849	1	1.243836	0
Tache71	M	52	0.915069	1	0.868493	0
Tache41	M	59	7.846575	0	1.084931	0
Tache63	M	72	7.383562	0	0.213699	0
Tache15	M	50	1.20274	1	1.20274	0
Tache62	M	66	7.383562	0	5.243835	1
Tache12	F	75	0.021918	1	0.021918	0
Tache23	F	73	2.336986	1	2.336986	0
Tache66	M	44	7.189041	0	1.221918	0
Tache52	M	87	0.405479	1	0.405479	0
Tache10	F	69	2.386301	1	2.386301	0
Tache68	M	56	2.616438	1	2.610959	0
Tache28	M	54	8.40274	0	1.271233	1
Tache44	M	46	4.950685	1	1.991781	0
Tache43	F	69	8	0	1.30411	0
Tache32	F	71	5.134246	1	0.136986	1