On Nonexpansive and Expansive Semigroup of Order-Preserving Total Mappings in Waist Metric Spaces

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Abstract

In this paper, we introduce nonexpansive and expansive semigroup of order-preserving total mappings (\(\text{ON}_{n}\)) and (\(\text{OE}_{n}\)), respectively, to prove some fixed point theorems in waist metric spaces. We examine the existence of mappings that satisfy the conditions \(\text{ON}_{n}\) and \(\text{OE}_{n}\). We also prove that every semigroup of order-preserving total mappings \(\text{OT}_{n}\) has fixed point properties and that the set of fixed points is closed and convex. The present study generalised many previous results on semigroup of order-preserving total mappings \(\text{OT}_{n}\). Efficacy of the results was justified with some practical examples.

DOI:10.46481/jnsps.2022.878

Keywords: Fixed point, semitopological semigroup, order-preserving total mappings, waist metric space, nonexpansive map.

Article History:
Received: 20 June 2022
Received in revised form: 08 November 2022
Accepted for publication: 09 November 2022
Published: 22 December 2022

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Communicated by: Joel Ndam

1. Introduction

In the last four decades, semigroup of mappings is one of the areas of application in fixed point theory. In algebra, a semigroup is simply a set \(S\) with an associative binary operation. A subset \(P \subset S\) is called a subsemigroup of \(S\) if it is closed under the binary operation on \(S\).

Let \(X_n\) be an ordered finite set in a standard way and let \(\alpha : \text{Dom}(\alpha) \subseteq X_n \rightarrow X_n\) be a self-map. The map \(\alpha\) is called a full or total transformation of \(X_n\) if \(\text{Dom}(\alpha) = X_n\). It is said to be partial if \(\text{Dom}(\alpha) \subset X_n\). Otherwise, it is called partial one-to-one or strictly partial. The set of full transformations on \(X_n\), denoted by \(\text{T}_n\), forms a semigroup under the composition of mappings called the full transformation semigroup. The set of partial and partial one-to-one transformations on \(X_n\), denoted by \(\text{P}_n\) and \(\text{I}_n\), respectively, also form a semigroup under the composition of mappings.

The semigroup of order-preserving full transformation of \(X_n\) is defined by

\[
\text{OT}_n = \{\alpha \in \text{T}_n : x \leq y \Rightarrow x\alpha \leq y\alpha, \text{ for all } x, y \in X_n\}.
\]

Let \(\alpha\) be a transformation in \(\text{OT}_n\). A point \(x^* \in X_n\) is said to be fixed if it coincides with the image \(\alpha\). For \(\alpha \in \text{OT}_n\), the fix of \(\alpha\) is given by \(\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\}\).

Let \(\text{OCT}_n\) and \(\text{OC}^*\text{T}_n\) be subsemigroups of \(\text{OT}_n\), then a map-
ping $\alpha \in OCT_n$ is called a 'contraction' if
\begin{equation}
|x\alpha - y\alpha| \leq |x - y| \quad \text{for all } x, y \in X_n
\end{equation}
and a mapping $\alpha \in OC^*T_n$ is called 'contractive' if
\begin{equation}
|x\alpha - y\alpha| \geq |x - y| \quad \text{for all } x, y \in X_n
\end{equation}
It is worthy to note there is a difference between 'contraction' and 'contractive' transformations in semigroup and deterministic fixed points. We refer to the following references for standard concepts and terminologies in the semigroup, (see [1, 2, 3, 4, 5]).

Several fixed point results have been proved on semigroup for the family of isometries under the asymptotic nonexpansive operators [6, 7, 8, 9], Lipschitzian semigroup of mappings [10, 11], commutative semigroup [12] etc. Worthy to mention a few recent studies of fixed points when the parameter set of semigroups is equal to $\{0, 1, 2, 3, \ldots\}$ and $T_n = T^n$ is the $n$-th iterate of asymptotic pointwise contractions and asymptotic nonexpansive mappings in metric spaces (see [9, 13]). Also, a procedure for constructing and finding the cardinality of order-preserving total transformations with finite fixed points have been considered in [14, 15]. However, we observed through a survey that few or no record of results concerning the existence of the fixed points of a semigroup of order-preserving total mappings. In this respect, existence of semigroup of order-preserving full transformation (which double as a nonlinear operator on $X_n$) is studied in the present paper. The intuitive notion of semigroup is integrated into a more robust geometric structure to unify some results in the semigroup theory. In concrete, the paper introduces some fixed point theorems for the nonexpansive (and expansive) semigroup of order-preserving mappings to prove some existence of fixed points of the elements of subsemigroups $OCT_n$ (and $OC^*T_n$).

We recall Banach’s contraction mapping principle [16] which has been used in many areas of applied sciences to study the existence properties of nonlinear operators.

**Definition 1.** Let $(E, d)$ be a metric space. A map $T : E \to E$ is called contraction on $E$ if there exists a constant $\lambda \in [0, 1)$ such that for all $x, y \in E$,
\begin{equation}
d(Tx, Ty) \leq \lambda d(x, y)
\end{equation}
If the condition (4) is weakened, that is $\lambda = 1$, then it reduces to a nonexpansive mapping
\begin{equation}
d(Tx, Ty) \leq d(x, y)
\end{equation}
Otherwise, it is an expansive mapping. We note that every mapping $T \in OCT_n$ is a nonexpansive mapping and every mapping $T \in OC^*T_n$ is an expansive mapping. The inclusion in both cases are strict. For few older results on the family of nonexpansive mappings, see [6, 17, 18, 19, 20, 21].

### 2. Waist Metric Space

Let $\alpha : Dom(\alpha) \to Im(\alpha)$ be a map in $OCT_n$, where $Dom(\alpha), Im(\alpha) \subset X$. The right waist and left waist of $Dom(\alpha)$ are given, respectively, by
\begin{equation}
w^+(Dom(\alpha)) = \max \{ |x| : x \in Dom(\alpha) \}
\end{equation}
and
\begin{equation}
w^-(Dom(\alpha)) = \min \{ |x| : x \in Dom(\alpha) \}.
\end{equation}
Similarly, the right and left waist of $Im(\alpha)$ are, respectively, given by
\begin{equation}
w^+(\alpha) = \max \{ |y| : y \in Im(\alpha) \} \quad \text{and} \quad w^- (\alpha) = \min \{ |y| : y \in Im(\alpha) \}.
\end{equation}

In view of the above, we introduce a notion of distance function with the left waist $\omega^-(\cdot, \cdot)$ and right waist $\omega^+(\cdot, \cdot)$ terms as follow:

**Definition 2.** Let $M$ be a non-empty ordered set and $X$ be a finite subset of $M$. A function $\omega : X \times X \to X \cup \{0\}$ is called a right and left waist metric if for given transformation $\alpha$ and for each $x, y \in Dom(\alpha) \subseteq X$, the following conditions hold:

- **W1:** $\omega^+(x, y)$ and $\omega^-(x, y)$ are finite and nonnegative integer;
- **W2:** $\omega^+(x, x) = 0$ and $\omega^-(x, x) = 0$;
- **W3:** $\omega^+(x, y) = \omega^+(y, x)$ and $\omega^-(x, y) = \omega^-(y, x)$;
- **W4:** $\omega^+(x, y) + \omega^+(y, z) \geq \omega^+(x, z)$ and $\omega^-(x, y) + \omega^-(y, z) \geq \omega^-(x, z)$ for $x, y, z \in X$.

The pair $(M, \omega)_\alpha$ is called a waist metric space (WMS). WMS is a weakening form of the canonical metric space and it is classified as pseudometric space.

**Example 1.** Let $X = \{1, 2\} \subset M$ be endowed with the waist distance
\begin{equation}
\omega^+_X(x, y) = \max \{ |x - y| : x, y \in X \} \quad \text{and} \quad \omega^-_X(x, y) \quad \text{is a waist metric on } X.
\end{equation}
Similarly for $\omega^-_X(x, y)$.

**Example 2.** Let $\alpha$ be a total map on set $X = \{1, 2, 3, 4, 5\} \subset M$ such that $\alpha = (1)(4)(2) \in OCT_n$, observe that $Im(\alpha) = \{1, 4\}$ and Dom($\alpha$) = $\{1, 2, 3, 4, 5\}$. The following are verifiable:

- i $w^+(Dom(\alpha)) = 5$ and $w^- (Dom(\alpha)) = 1$.
- ii Both $\omega^+_X(x, y)$ and $\omega^-_X(x, y)$ are waist metric on $X$.

**Remark 1.** If $\alpha \in OCT_n$ for any given set $X$, then $\omega^-(x, y) = \omega^+(x, y)$. On the other hand, this is not so if $\alpha$ is a partial map $PT_n$. Since the main focus of this present study is on the mappings in $OCT_n$, we denote $\omega(x, y)$ by a waist metric with no emphasis on left or right waist metric.
2.1. Completeness of \((M, \omega)_a\)

Let \(\{x_i\}\) be a sequence in \(X \subset (M, \omega)_a\). Since \(X\) is a finite set, the convergent of \(\{x_i\}\) is vacuously satisfied. We present the following useful lemmas.

**Lemma 1.** A finite set \(X \subset M\) is a closed set.

**Proof:** Let \(X = \{x_1, x_2, \ldots, x_n\}\) be a finite set and let \(X \cup X' = M\), where \(X \cap X' = \emptyset\). For each \(x_i \in M\), there is an \(\varepsilon\)-net such that \(x_i \in B(x_i, \varepsilon) \subseteq X'\) for \(i \neq j\). Observe that each \(B(x_i, \varepsilon)\) is an open ball in \(M\). Let \(X' = \bigcup_{i\in\Delta} \{B(x_i, \varepsilon)\}\), then \(X'\) is the union of open balls which itself is an open set in \(M\). Now, \(M \setminus X' = \cap_{i\in\Delta} \{M \setminus B(x_i, \varepsilon)\} = \{x_1, x_2, \ldots, x_n\}\).

That is, \(X = M \setminus X'\) is a complement of an open set. Thus, \(X\) is closed.

**Remark 2.** In Lemma 1, observe that for each \(y \in X\), \(B(x, \varepsilon) \cap X = \{x\}\). This means that no point in \(X\) is an accumulation point but every point in \(X\) is an isolated point. More so, any metric on a finite space induces a discrete topology (see [22]).

**Definition 3.** Let \((M, \omega)_a\) be a WMS and \(X \subset M\). A sequence \(x_k \in X\) is said to be a Cauchy sequence in \(X\) if for given \(\varepsilon\)-net, there exist \(l\) and \(k\) with \(l \geq k\) such that \(x_l \in B(x_k, \varepsilon)\).

**Lemma 2.** Any convergent sequence in any metric space is a Cauchy sequence.

**Definition 4.** A waist metric space \((M, \omega)_a\) is said to be complete if every Cauchy sequence in \(M\) converges to an element in \(M\).

**Theorem 1.** Let \((M, \omega)_a\) be a complete waist metric space and \(X \subset M\). The subspace \((X, \omega)_a\) is complete if and only if \(X\) is a closed subset of \(M\).

The proof follows from Lemma 1 and 2. The following concepts are versions of some results in [23, 24].

**Definition 5.** Let \((M, \omega)_a\) be a waist metric space. A mapping \(u : M \times M \times [0, 1] \to M\) is called a convex structure on \(M\) if for all \(x, y \in M\) and \(\lambda \in [0, 1]\)

\[
\omega(z, u(x, y, \lambda)) \leq \lambda \omega(z, x) + (1 - \lambda) \omega(z, y)
\]

holds for all \(z \in M\). The waist metric space \((M, \omega)_a\) together with a convex structure \(u_\lambda = u(x, y, \lambda)\) is called a convex waist metric space.

In Definition 5, a convex waist metric space \((M, \omega, u)_a\) satisfies the following:

\[
\omega(u(x, p, \lambda), u(y, p, \lambda)) \leq \lambda \omega(x, y), \quad x, y, p \in M, \quad \lambda \in [0, 1] \\
\omega(x, y) = \omega(x, u(x, p, \lambda)) + \omega(u(y, p, \lambda), y), \quad x, y \in M, \quad \lambda \in [0, 1]
\]

**Definition 6.** A nonempty subset \(X\) of a convex waist metric space \((M, \omega, u)_a\) is said to be convex if \(u(x, y, \lambda) \in X\) for all \(x, y \in X\) and \(\lambda \in [0, 1]\).

**Definition 7.** A nonempty subset \(X\) is said to be \(p\)-starshaped, where \(p \in X\), provided \(u(x, p, \lambda) \in X\) for all \(x \in X\) and \(\lambda \in [0, 1]\), that is, the segment \([p, x] = \{u(x, p, \lambda) : 0 \leq \lambda \leq 1\}\) joining \(p\) to \(x\) is contained in \(X\) for all \(x \in X\).

The set \(X\) is said to be starshaped if it is \(p\)-starshaped for some \(p \in X\).

Clearly, each convex waist metric space is starshaped but not conversely.

**Lemma 3.** Let \((M, \omega)_a\) be a waist metric space. Then

\[
\omega^2(z, u_1) \leq \frac{1}{2} \omega^2(z, x) + \frac{1}{2} \omega^2(z, y) - \frac{1}{4} \omega^2(x, y)
\]

for all \(x, y, z \in M\).

The proof follows from the parallelogram law. Inequality (6) is similar to the (CN) inequality of Bruhat and Tits [26].

2.2. Nonexpansive Semigroup of Order-preserving Maps

Let \(S\) be a semitopological semigroup and \(X\) be a nonempty closed subset of a waist metric space \((M, \omega)_a\). A family \(\varphi = \{\alpha_s : s \in S\}\) of mappings of \(X\) into itself is called a semigroup if it satisfies the following:

\(\text{S1: } x\alpha_s = x\alpha_t \alpha_s\) for all \(s, t \in S\) and \(x \in X\);

\(\text{S2: } \) for every \(x \in X\) the mapping \(s \to \alpha_s x\) from \(S\) into \(X\) is continuous.

The set of all fixed points of semigroup mappings is denoted by \(F(\alpha_s) = \{x \in X : x\alpha_s = x\ for\ s \in S\}\).

More so, any map \(\alpha_s \in OCT_n\) or \(\alpha_s \in OC^\ast T_n\) possesses the properties as stated in section one. Note that (i) the set of fixed points of order-preserving maps, denoted by \(F(\alpha_s)\), is a subset of \(F(\alpha_s)\). (ii) \(F_{\omega}(\alpha_s) \subset F^n_{\omega}(\alpha_s)\ for\ all\ n > 1\). (iii) If \(F^n_{\omega}(\alpha_s)\) is singleton for some \(n\), so does \(F_{\omega}(\alpha_s)\).

Without loss of generality, the notion ‘nonexpansive’ connotes ‘contraction’ while ‘expansive’ connotes ‘contractive’. In view of the above, we present some definitions of nonexpansive semigroup of order-preserving mappings in WMS under the property that both \(\overline{OCT}_n\) and \(\overline{OC}^\ast T_n\) have no common maps.

**Definition 8.** Let \(X\) be a closed subset of \((M, \omega)_a\) and let \(S\) be a semitopological semigroup. The map \(\alpha_s : X \to X\ is called a nonexpansive semigroup of order-preserving total map \(\text{ONT}_n\) if for \(x, y \in X\) and \(s \in S\),

\[
\omega(x\alpha_s, y\alpha_s) \leq \omega(x, y).
\]

**Definition 9.** Let \(X\) be a closed subset of \((M, \omega)_a\) and let \(S\) be a semitopological semigroup. The map \(\alpha_s : X \to X\ is called an expansive semigroup of order-preserving total map \(\text{OET}_n\) if for \(x, y \in X\) and \(s \in S\),

\[
\omega(x\alpha_s, y\alpha_s) > \omega(x, y).
\]
3. Main Results

The following lemma is useful in the proof of the main results.

**Lemma 4.** If \( \varphi = \{ \alpha_s : s \in S \} \) is a semigroup of continuous mappings of \( X \) into itself and \( \omega(\alpha_s x, y) \to 0 \) as \( s \to \infty \) for \( x, y \in X \), then \( y \in F_0(\alpha_s) \subset X \).

**Proof:** Let \( \varepsilon > 0 \) be given. By the continuity of \( \alpha_t \) for \( t \in S \), there exists \( \tau > 0 \) such that \( \omega(\alpha_t x, \alpha_t y) < \frac{\varepsilon}{2} \) whenever \( \omega(x, y) < \tau \) for \( x, y \in X \). Also, since \( \omega(\alpha_t x, y) \to 0 \) as \( s \to \infty \), then there exists \( u \in S \) such that
\[
\omega(\alpha_t u, y) < \min\{\frac{\varepsilon}{2}, \tau\}
\]
for each \( a \in S \). Thus, \( \omega(\alpha_t u, \alpha_t y) < \frac{\varepsilon}{2} \).

Now,
\[
\omega(y, \alpha_t y) \leq \omega(y, \alpha_t u) + \omega(\alpha_t u, \alpha_t y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that \( y \in F_0(\alpha_s) \).

**Remark 3.** If \( S = \mathbb{N} \), then the hypothesis on \( \alpha_s \) would include asymptotic regularity condition. For example, see Theorem 6.7. in [25].

In the next theorem, we let \( A(X, \co(\alpha_s x)) \) denote the asymptotic center and \( r(x_0, \co(\alpha_s x)) \) denote the asymptotic radius.

**Theorem 2.** Let \( X \) be a nonempty closed subset of a convex waist metric space \((M, \omega)_a\) and let \( S \) be a semitopological semigroup. Suppose \( \alpha_s \in \varphi \) is a nonexpansive semigroup of order-preserving total mapping \((7)\) of \( X \) into itself, that is, \( \alpha_s \in ONT_n \) for \( s \in S \). If the set \( \{\alpha_s x, s \in S\} \) is bounded for some \( x \in X \) and \( y \in A(X, \co(\alpha_s x)) \), then \( y \in F_0(\alpha_s) \).

**Proof:** Let \( \{\alpha_s x, s \in S\} \) be a bounded net. Define \( R := r(y, \co(\alpha_s x)) \) for \( y \in A(X, \co(\alpha_s x)) \) with the property that \( \omega(x, y) < R \).

If \( R = 0 \), then \( \lim \sup \omega(\alpha_s x, y) = 0 \) and by Lemma 4, the proof is complete. On the other hand, suppose \( R > 0 \) and \( y \notin F_0(\alpha_s) \), then for given \( \varepsilon > 0 \) and a subnet \( \{s_\beta\} \) in \( S \), we have
\[
\omega(\alpha_{s_\beta} y, y) > \varepsilon, \text{ for } s_\beta \in S.
\]
Also, since \( \omega(\alpha_s x, y) \to 0 \) as \( s \to \infty \), then there exists \( y \in S \) such that, for choosing \( \nu \geq 0 \),
\[
\omega(\alpha_{s_\beta} x, y) < R + \nu.
\]
Moreover, we have by hypothesis that
\[
\omega(\alpha_s x, \alpha_s y) \leq \lim \sup \omega(\alpha_{s_\beta} x, \alpha_{s_\beta} y) + \nu
\]
\[
= \lim \sup \omega(x, y) + \nu
\]
\[
= R + \nu
\]
(10)

By Lemma 3, (9) and (10), we have
\[
\omega^2(u, \alpha_s x) \leq \frac{1}{2} \omega^2(u, \alpha_s x) + \frac{1}{2} \omega^2(\alpha_s x, \alpha_s y) + \frac{1}{4} \omega^2(y, \alpha_s y)
\]
\[
\leq \frac{1}{2} (R + \nu)^2 + \frac{1}{2} (R + \nu)^2 + \frac{1}{4} \varepsilon^2 \leq (R + \nu)^2
\]
Thus, \( \omega(u, \alpha_s x) < R - \nu \) which implies that
\[
r(u, \alpha_s x) < r(y, \co(\alpha_s x)).
\]
is a contradiction. Hence, \( y \in F_0(\alpha_s) \).

**Remark 4.** If in Theorem 2, the boundedness assumption on \( \{\alpha_s x, s \in S\} \) is dropped, then another suitable concept is stated in the next theorem.

**Theorem 4.** Let \( X \) be a closed subset of a complete waist metric space \((M, \omega)_a\) and let \( S \) be a semitopological semigroup. Suppose \( \alpha_s \in \varphi \) is a nonexpansive semigroup of order-preserving total mapping \((7)\) of \( X \) into itself, that is, \( \alpha_s \in ONT_n \) for \( s \in S \). Then, \( \alpha_s \) has at least one fixed point.

**Proof:** For \( \delta \in (0, 1] \), let \( T_\delta \equiv (1 - \delta)\alpha_s \). It follows that \( T_\delta \) is a \( \delta \)-contraction on \( X \) and by the Banach fixed point theorem, there exists \( x_\delta \) for \( \delta \in (0, 1] \) such that \( T_\delta x_\delta = x_\delta \). Now, we show that for \( \delta_k \to 0 \), the net \( x_\delta \) converges to \( p \), where \( p \) is a fixed point of \( \alpha_s \). Indeed, for any arbitrary \( u \in X \), we have
\[
\omega^2(x_\delta, u) = \omega^2(x_\delta - p, u - p)
\]
\[
= \omega^2(x_\delta, p) + \omega^2(p, u) + 2 \omega(x_\delta - p, u - p)
\]
\[
\leq \omega^2(x_\delta, p) + \omega^2(p, u)
\]
By setting \( u = \alpha_s p \), we have
\[
\lim k \rightarrow \infty \left( \omega^2(x_\delta, \alpha_s p) - \omega^2(x_\delta, p) \right) \leq \omega^2(p, \alpha_s p)
\]
(11)
Also, since \( T_\delta x_\delta = x_\delta \) and \( \delta_k \to 0 \) as \( k \to \infty \), then
\[
\lim k \rightarrow \infty \omega(x_\delta, \alpha_s x_\delta) = \lim k \rightarrow \infty \left[ \omega(x_\delta, T_\delta x_\delta) + \delta_k \omega(0, \alpha_s x_\delta) \right] = \lim k \rightarrow \infty \omega(x_\delta, T_\delta x_\delta) \to 0
\]
(12)
On the other hand, since \( \alpha_s \in ONT_n \), then
\[
\omega(\alpha_s p, \alpha_s x_\delta) \leq \omega(p, x_\delta)
\]
(13)
We have from (12) and (13) that
\[
\omega(x_\delta, \alpha_s p) \leq \omega(x_\delta, \alpha_s x_\delta) + \omega(x_\delta, p)
\]
which further implies
\[
\lim k \rightarrow \infty \omega(x_\delta, \alpha_s p) \leq \lim k \rightarrow \infty \omega(x_\delta, \alpha_s x_\delta) = 0
\]
(14)
Also, from (11) and (14), we obtain
\[
\limsup \left( \omega^2(x_{i_j}, \alpha_s p) - \omega^2(x_{i_k}, p) \right) = \limsup \left( \omega(x_{i_j}, \alpha_s p) - \omega(x_{i_k}, p) \right)
\times \left( \omega(x_{i_j}, \alpha_s p) + \omega(x_{i_k}, p) \right)
\]
Thus,
\[
\limsup \left( \omega^2(x_{i_j}, \alpha_s p) - \omega^2(x_{i_k}, p) \right) = \omega^2(p, \alpha_s p) = 0
\]
and hence, \( p \) is the fixed point of \( \alpha_s \).

**Theorem 5.** Let \( X \) be a closed subset of a complete waist metric space \((M, \omega)\), and let \( S \) be a semitopological semigroup. Suppose \( \alpha_s \in \Phi \) is a mapping satisfying (7), that is, \( \alpha_s \in OET_n \) for \( s \in S \). Then, the set \( F_o(\alpha_s) \subset X \) is a nonempty closed convex set.

*Proof:* Since \( \alpha_s \) satisfies (7), by Theorem 2, \( \alpha_s \) has fixed point in \( X \). It is left to show that \( F_o(\alpha_s) \) is closed and convex. Firstly, we show that \( F_o(\alpha_s) \) is closed. Let \( \{x_t\} \) be a net in \( F_o(\alpha_s) \) such that \( x_t \to x \), then by hypothesis:
\[
\omega(\alpha_s, x_t, x_t) + \omega(x_t, x) \to 0
\]
This implies \( \alpha_s x \to x \in F_o(\alpha_s) \). Hence, \( F_o(\alpha_s) \) is closed. Also, let \( x, y \in F_o(\alpha_s) \) and \( \lambda \in [0, 1] \), we have \( u_\lambda = u(x, y, \lambda) \in F_o(\alpha_s) \).

Indeed,
\[
\omega(\alpha_s u_\lambda, x) = \omega(\alpha_s u_\lambda, \alpha_s x) \leq \omega(u_\lambda, x)
\]
Similarly, \( \omega(\alpha_s u_\lambda, y) \leq \omega(u_\lambda, y) \). Thus,
\[
\omega(x, y) \leq \omega(x, \alpha_s u_\lambda) + \omega(\alpha_s u_\lambda, y) \leq \omega(x, y).
\]
This shows that for some \( a, b \) with \( 0 \leq a, b \leq 1 \), we have
\[
\omega(x, \alpha_s u_\lambda) = a \omega(x, u_\lambda) \quad \text{and} \quad \omega(y, \alpha_s u_\lambda) = b \omega(y, u_\lambda)
\]
from which it follows that \( \alpha_s u_\lambda \in F_o(\alpha_s) \).

By letting \( S = \mathbb{N} \) in Theorems 4 and 5, we obtain the existence property of a nonexpansive semigroup of order-preserving total mapping in waist metric spaces as follow:

**Corollary 1.** Let \( X \) be a closed subset of a convex waist metric space \((M, \omega)\). Suppose \( \alpha \in \Phi = \{ \alpha^n : n \in \mathbb{N} \} \) is a nonexpansive semigroup of order-preserving total mapping (7) of \( X \) into itself. Then, \( F_o(\alpha_s) \neq \emptyset \) if and only if \( \{\alpha^n x : n \in \mathbb{N}\} \) is bounded for some \( x \in X \). Furthermore, \( F_o(\alpha_s) \) is closed and convex.

**Theorem 6.** Let \( X \) be a closed subset of a complete waist metric space \((M, \omega)\), and let \( S \) be a semitopological semigroup. Suppose \( \alpha_s \in \Phi = \{ \alpha_s : s \in S \} \) is a mapping satisfying (8). Then,

i. the map \( \alpha_s \) exists;
ii. \( F_o(\alpha_s) \) is nonempty.

*Proof:*

i. Let \( x_i, x_s \in X \) with \( x_i < x_s \). Suppose, on contrary, that \( \alpha_s \notin OET_n \), then \( \alpha_s \) is an order-reversing map, that is, \( x_i \alpha_s \geq x_s \alpha_s \).

By induction, \( x_k \alpha_s \geq x_{k+1} \alpha_s \), for \( k = 1, 2, 3, \ldots, n - 1 \).

Define \( x_{i+1} = x_i \alpha_s \). Using condition (8), there given
\[
\omega(x_{i+1}, x_{i+2}) \geq \omega^+(x_{i+1} \alpha_s x_{i+2} \alpha_s) > \omega^+(x_{i+1}, x_{i+2})
\]
This is a clear contradiction. Hence, \( \alpha_s \in OET_n \).

ii. Let \( \alpha_s \in OET_n \). Suppose that \( F_o(\alpha_s) \) is empty, that is, there is no fixed \( \beta \) such that \( \beta \in F_o(\alpha_s) \), this implies that \( \omega^+(\beta, \beta \alpha_s) > \epsilon \), for \( \epsilon > 0 \). Let \( \omega^+(x_i \alpha_s, \beta) = 0 \), by condition (8), there results
\[
\omega(x_i, \beta) < \omega(x_i \alpha_s, \beta \alpha_s) \leq \omega(x_i, \beta) + \omega(\beta, \beta \alpha_s) = \omega(\beta, \beta \alpha_s)
\]
On the other hand, since \( \alpha_s \) is continuous on \( X \) for each \( t \in S \), then, for \( x_t \in X \) and \( \tau > 0 \), \( \omega(x_t, \beta) > \tau \) implies that \( \omega(x_t, \beta \alpha_s) > \frac{\tau}{2} \).

Since \( \omega^+(x_i, \alpha_s, \beta) = 0 \), then there exists \( \beta \in S \) such that \( \omega^+(x_i, \alpha_{\beta \alpha_s}) > \frac{\tau}{2} \).

But then,
\[
\omega^+(\beta \alpha_s, \beta) \leq \omega^+(\beta \alpha_s, x_i \alpha_{\beta \alpha_s}) + \omega^+(x_i \alpha_{\beta \alpha_s}, \beta)
\]
and
\[
\omega^+(\beta \alpha_s, x_i \alpha_{\beta \alpha_s}) + \omega^+(x_i \alpha_{\beta \alpha_s}, \beta) > \epsilon
\]
From the last two inequalities, we obtain \( \omega^+(\beta \alpha_s, \beta) = \epsilon \). This is a contradiction. Therefore, \( \omega^+(\beta \alpha_s, \beta) = 0 \) for each \( s \in S \) and \( \beta \in F_o(\alpha_s) \).

**Remark 5.** The convergence theorems (strong convergence and \( \Delta \)-convergence) of the map \( \alpha_s \in \Phi \) satisfying the nonexpansive semigroup of order-preserving total mapping (7) in waist metric spaces can be routinely proved using the next lemma and their left.

**Lemma 5.** Let \( X \) be a closed subset of a convex waist metric space \((M, \omega)\). Suppose \( \alpha_s \) is a nonexpansive semigroup of order-preserving total mapping (7) of \( X \) into itself with \( F_o(\alpha_s) \neq \emptyset \). Then \( \lim \omega(x_i \alpha_s, y) \) exists for each \( y \in F_o(\alpha_s) \) and \( s \in S \).

4. Practical Examples

The following three examples are considered to justify the theorems in the main results. The first two examples are from the same family for \( n = 2 \) and \( n = 3 \), respectively, while the third example is given for \( n = 3 \).

**Example 3.** Let \( S \) be a semigroup and \( X = \{1, 2\} \). Let \( \alpha_s : \{1, 2\} \to \{1, 2\} \) be the mapping \( \alpha_s = (1)(2) \) in \( OT_2 \), where \( s \in S \), associated to \( xx_s = x^2 - 2x + 2 \) in \( X \in (M, \omega)_s \).

The map \( \alpha_s \) is a nonexpansive order-preserving total mapping. Hence, it satisfies all hypotheses of Theorem 2 and 4 and the set \( F_o(\alpha_s) = \{1, 2\} \).
Example 4. Let $\alpha_s$ be a mapping in $OT_3$ define by $\alpha : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ as $\alpha = (1)(3\ 2\ 1)$ which is equivalent to the map $x\alpha_s = x^2 - 3x + 3$.

Here, the map $\alpha_s$ is also a nonexpansive order-preserving total mapping, that is, $\alpha_s \in ONT_3$. Thus, it satisfies all hypotheses of Theorem 2 and 4. The set $F_{\alpha}(\alpha_s) = \{1\}$.

Example 5. Let $\alpha_s : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ be given by the mapping $\alpha = (1)(2\ 1)(3)$ in $OT_3$ corresponding to the map $x\alpha_s = x^2 - 3x + 3$.

In Example 5, all hypotheses of Theorem 6 are satisfied since $\alpha_s \in OET_n$ with fixed point $F_{\alpha}(\alpha_s) = \{1, 3\}$. 

5. Concluding Remark

This study introduced the nonexpansive and expansive semigroup of order-preserving total mappings to probe some fixed point theorems in complete waist metric spaces. The study also considered some examples (see Examples 3, 4 and 5) on semigroup of order-preserving total mappings to validate the hypotheses of Theorems 4, 5 & 6. Results show that the nonexpansive (and expansive) semigroup of order-preserving mappings compares favorably with the semigroup of mappings $OCT_n$ (and $OC^+T_n$). In fact, every mapping $\alpha_s \in \varphi$ under the action of subsemigroups $OCT_n$ and $OC^+T_n$ is also in $ONT_n$ and $OET_n$, respectively. However, this study only features some elements of subsemigroups of order-preserving full mappings in [27]. Therefore, future studies would be to establish some notions to study other elements of subsemigroups in $\varphi$ such as order-preserving partial mappings [1, 28, 15], order decreasing full mappings [29], symmetric inverse semigroups [5], orientation-preserving mappings [30], fence-preserving mappings [31] among others.

Acknowledgements

The authors wish to thank Prof. K. Rauf for his mentor-ship.

References