



On Nonexpansive and Expansive Semigroup of Order-Preserving Total Mappings in Waist Metric Spaces

I. F. Usamot^a, O. T. Wahab^{b,*}, S. M. Alata^c, K. R. Tijani^d

^a Department of Mathematics, University of Ilorin, Ilorin, Nigeria

^b Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria

^c Department of Computer Science, College of Arabic and Islamic Legal Studies, Ilorin, Nigeria

^d Department of Mathematical Sciences, Osun State University, Osogbo, Nigeria

Abstract

In this paper, we introduce nonexpansive and expansive semigroup of order-preserving total mappings (ONT_n) and (OET_n), respectively, to prove some fixed point theorems in waist metric spaces. We examine the existence of mappings that satisfy the conditions ONT_n and OET_n . We also prove that every semigroup of order-preserving total mappings OT_n has fixed point properties and that the set of fixed points is closed and convex. The present study generalised many previous results on semigroup of order-preserving total mappings OT_n . Efficacy of the results was justified with some practical examples.

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1. Introduction

In the last four decades, semigroup of mappings is one of the areas of application in fixed point theory. In algebra, a semigroup is simply a set S with an associative binary operation. A subset $P \subset S$ is called a subsemigroup of S if it is closed under the binary operation on S .

Let X_n be an ordered finite set in a standard way and let $\alpha : Dom(\alpha) \subseteq X_n \rightarrow X_n$ be a self-map. The map α is called a full or total transformation of X_n if $Dom(\alpha) = X_n$. It is said to be partial if $Dom(\alpha) \subsetneq X_n$. Otherwise, it is called partial one-to-one or strictly partial. The set of full transformations on

X_n , denoted by \mathcal{T}_n , forms a semigroup under the composition of mappings called the full transformation semigroup. The set of partial and partial one-to-one transformations on X_n , denoted by \mathcal{P}_n and \mathcal{I}_n , respectively, also form a semigroup under the composition of mappings.

The semigroup of order-preserving full transformation of X_n is defined by

$$OT_n = \{ \alpha \in \mathcal{T}_n : x \leq y \Rightarrow \alpha x \leq \alpha y, \text{ for all } x, y \in X_n \}.$$

Let α be a transformation in OT_n . A point $x^* \in X_n$ is said to be fixed if it coincides with the image αx^* . For $\alpha \in OT_n$, the fix of α is given by $Fix(\alpha) = \{ x \in X_n : \alpha x = x \}$.

Let OCT_n and OC^*T_n be subsemigroups of OT_n , then a map-

*Corresponding author tel. no: +2348035224754.

Email address: taofeek.wahab@kwasu.edu.ng (O. T. Wahab)

ping $\alpha \in OCT_n$ is called a 'contraction' if

$$|x\alpha - y\alpha| \leq |x - y| \quad \text{for all } x, y \in X_n \quad (1)$$

and a mapping $\alpha \in OC^*\mathcal{T}_n$ is called 'contractive' if

$$|x\alpha - y\alpha| \geq |x - y| \quad \text{for all } x, y \in X_n \quad (2)$$

It is worthy to note there is a difference between 'contraction' and 'contractive' transformations in semigroup and deterministic fixed points. We refer to the following references for standard concepts and terminologies in the semigroup, (see [1, 2, 3, 4, 5]).

If α preserves only distance (with no order), then it satisfies both (1) and (2). This is called an isometry semigroup,

$$|x\alpha - y\alpha| = |x - y| \quad \text{for all } x, y \in X_n. \quad (3)$$

Several fixed point results have been proved on semigroup for the family of isometries under the asymptotic nonexpansive operators [6, 7, 8, 9], Lipschitzian semigroup of mappings [10, 11], commutative semigroup [12] etc. Worthy to mention a few recent studies of fixed points when the parameter set of semigroups is equal to $\{0, 1, 2, 3, \dots\}$ and $T_n = T^n$ is the n -th iterate of asymptotic pointwise contractions and asymptotic nonexpansive mappings in metric spaces (see [9, 13]). Also, a procedure for constructing and finding the cardinality of order-preserving total transformations with finite fixed points have been considered in [14, 15]. However, we observed through a survey that few or no record of results concerning the existence of the fixed points of a semigroup of order-preserving total mappings. In this respect, existence of semigroup of order-preserving full transformation (which double as a nonlinear operator on X_n) is studied in the present paper. The intuitive notion of semigroup is integrated into a more robust geometric structure to unify some results in the semigroup theory. In concrete, the paper introduces some fixed point theorems for the non-expansive (and expansive) semigroup of order-preserving mappings to prove some existence of fixed points of the elements of subsemigroups OCT_n (and $OC^*\mathcal{T}_n$).

We recall Banach's contraction mapping principle [16] which has been used in many areas of applied sciences to study the existence properties of nonlinear operators.

Definition 1. Let (E, d) be a metric space. A map $T : E \rightarrow E$ is called contraction on E if there exists a constant $\lambda \in [0, 1)$ such that for all $x, y \in E$,

$$d(Tx, Ty) \leq \lambda d(x, y) \quad (4)$$

If the condition (4) is weakened, that is $\lambda = 1$, then it reduces to a nonexpansive mapping

$$d(Tx, Ty) \leq d(x, y) \quad (5)$$

Otherwise, it is an expansive mapping. We note that every mapping $T \in OCT_n$ is a nonexpansive mapping and every mapping $T \in OC^*\mathcal{T}_n$ is an expansive mapping. The inclusion in both cases are strict. For few older results on the family of nonexpansive mappings, see [6, 17, 18, 19, 20, 21].

2. Waist Metric Space

Let $\alpha : Dom(\alpha) \rightarrow Im(\alpha)$ be a map in $O\mathcal{T}_n$, where $Dom(\alpha), Im(\alpha) \subset X$. The right waist and left waist of $Dom(\alpha)$ are given, respectively, by

$$w^+(Dom(\alpha)) = \max \{|x| : x \in Dom(\alpha)\}$$

and

$$w^-(Dom(\alpha)) = \min \{|x| : x \in Dom(\alpha)\}.$$

Similarly, the right and left waist of $Im(\alpha)$ are, respectively, given by

$$w^+(\alpha) = \max \{|y| : y \in Im(\alpha)\} \quad \text{and} \quad w^-(\alpha) = \min \{|y| : y \in Im(\alpha)\}.$$

In view of the above, we introduce a notion of distance function with the left waist $\omega^-(\cdot, \cdot)$ and right waist $\omega^+(\cdot, \cdot)$ terms as follow:

Definition 2. Let M be a non-empty ordered set and X be a finite subset of M . A function $\omega : X \times X \rightarrow X \cup \{0\}$ is called a right and left waist metric if for given transformation α and for each $x, y \in Dom(\alpha) \subseteq X$, the following conditions hold:

W1: $\omega^+(x, y)$ and $\omega^-(x, y)$ are finite and nonnegative integer;

W2: $\omega^+(x, x) = 0$ and $\omega^-(x, x) = 0$;

W3: $\omega^+(x, y) = \omega^+(y, x)$ and $\omega^-(x, y) = \omega^-(y, x)$;

W4: $\omega^+(x, y) + \omega^+(y, z) \geq \omega^+(x, z)$ and $\omega^-(x, y) + \omega^-(y, z) \geq \omega^-(x, z)$ for $x, y, z \in X$.

The pair $(M, \omega)_\alpha$ is called a waist metric space (WMS). WMS is a weakening form of the canonical metric space and it is classified as pseudometric space.

Example 1. Let $X = \{1, 2\} \subset M$ be endowed with the waist distance

$\omega_X^+(x, y) = \max \{|x - y| : x, y \in X\}$ and $\alpha = (1)(2)$, then $\omega_X^+(x, y)$ is a waist metric on X . Similarly for $\omega_X^-(x, y)$.

Example 2. Let α be a total map on set $X = \{1, 2, 3, 4, 5\} \subset M$ such that $\alpha = (1)(4)(2\ 1)(3\ 4)(5\ 4) \in O\mathcal{T}_n$, observe that $Im(\alpha) = \{1, 4\}$ and $Dom(\alpha) = \{1, 2, 3, 4, 5\}$. The following are verifiable:

i $w^+(Dom(\alpha)) = 5$ and $w^-(Dom(\alpha)) = 1$.

ii Both $\omega_X^+(x, y)$ and $\omega_X^-(x, y)$ are waist metric on X .

Remark 1. If $\alpha \in O\mathcal{T}_n$ for any given set X , then $\omega^-(x, y) = \omega^+(x, y)$. On the other hand, this is not so if α is a partial map $P\mathcal{T}_n$. Since the main focus of this present study is on the mappings in $O\mathcal{T}_n$, we denote $\omega(x, y)$ by a waist metric with no emphasis on left or right waist metric.

2.1. Completeness of $(M, \omega)_\alpha$

Let $\{x_k\}$ be a sequence in $X \subset (M, \omega)_\alpha$. Since X is a finite set, the convergent of $\{x_k\}$ is vacuously satisfied.

We present the following useful lemmas.

Lemma 1. A finite set $X \subset M$ is a closed set.

Proof: Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and let $X \cup X' = M$, where $X \cap X' = \emptyset$. For each $x_i \in M$, there is an ε -net such that $x_j \in B(x_i, \varepsilon) \subseteq X'$ for $i \neq j$. Observe that each $B(x_i, \varepsilon)$ is an open ball in M . Let $X' = \cup_{i \in \Delta} \{B(x_i, \varepsilon)\}$, then X' is the union of open balls which itself is an open set in M . Now,

$$M \setminus X' = \cap_{i \in \Delta} \{M \setminus B(x_i, \varepsilon)\} = \{x_1, x_2, \dots, x_n\}.$$

That is, $X = M \setminus X'$ is a complement of an open set. Thus, X is closed.

Remark 2. In Lemma 1, observe that for each $y \in X$, $B(x, \varepsilon) \cap X = \{x\}$. This means that no point in X is an accumulation point but every point in X is an isolated point. More so, any metric on a finite space induces a discrete topology (see [22]).

Definition 3. Let $(M, \omega)_\alpha$ be a WMS and $X \subset M$. A sequence $x_k \in X$ is said to be a Cauchy sequence in X if for given ε -net, there exist l and k with $l \geq k$ such that $x_l \in B(x_k, \varepsilon)$.

Lemma 2. Any convergent sequence in any metric space is a Cauchy sequence.

Definition 4. A waist metric space $(M, \omega)_\alpha$ is said to be complete if every Cauchy sequence in M converges to an element in M .

Theorem 1. Let $(M, \omega)_\alpha$ be a complete waist metric space and $X \subset M$. The subspace $(X, \omega)_\alpha$ is complete if and only if X is a closed subset of M .

The proof follows from Lemma 1 and 2. The following concepts are versions of some results in [23, 24].

Definition 5. Let $(M, \omega)_\alpha$ be a waist metric space. A mapping $u : M \times M \times [0, 1] \rightarrow M$ is called a convex structure on M if for all $x, y \in M$ and $\lambda \in [0, 1]$

$$\omega(z, u(x, y, \lambda)) \leq \lambda \omega(z, x) + (1 - \lambda) \omega(z, y)$$

holds for all $z \in M$. The waist metric space $(M, \omega)_\alpha$ together with a convex structure $u_\lambda = u(x, y, \lambda)$ is called a convex waist metric space.

In Definition 5, a convex waist metric space $(M, \omega, u)_\alpha$ satisfies the following:

$$\omega(u(x, p, \lambda), u(y, p, \lambda)) \leq \lambda \omega(x, y), \quad x, y, p \in M, \quad \lambda \in [0, 1]$$

$$\omega(x, y) = \omega(x, u(x, p, \lambda)) + \omega(u(y, p, \lambda), y), \quad x, y \in M, \quad \lambda \in [0, 1]$$

Definition 6. A nonempty subset X of a convex waist metric space $(M, \omega, u)_\alpha$ is said to be convex if $u(x, y, \lambda) \in X$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Definition 7. A nonempty subset X is said to be p -starshaped, where $p \in X$, provided $u(x, p, \lambda) \in X$ for all $x \in X$ and $\lambda \in [0, 1]$, that is, the segment $[p, x] = \{u(x, p, \lambda) : 0 \leq \lambda \leq 1\}$ joining p to x is contained in X for all $x \in X$.

The set X is said to be starshaped if it is p -starshaped for some $p \in X$.

Clearly, each convex waist metric space is starshaped but not conversely.

Lemma 3. Let $(M, \omega)_\alpha$ be a waist metric space. Then

$$\omega^2(z, u_{\frac{1}{2}}) \leq \frac{1}{2} \omega^2(z, x) + \frac{1}{2} \omega^2(z, y) - \frac{1}{4} \omega^2(x, y) \tag{6}$$

for all $x, y, z \in M$.

The proof follows from the parallelogram law.

Inequality (6) is similar to the (CN) inequality of Bruhat and Tits [26].

2.2. Nonexpansive Semigroup of Order-preserving Maps

Let S be a semitopological semigroup and X be a nonempty closed subset of a waist metric space $(M, \omega)_\alpha$. A family $\varphi = \{\alpha_s : s \in S\}$ of mappings of X into itself is called a semigroup if it satisfies the following:

S1: $x\alpha_s = x\alpha_t\alpha_r$ for all $s, t \in S$ and $x \in X$;

S2: for every $x \in X$ the mapping $s \rightarrow \alpha_s x$ from S into X is continuous.

The set of all fixed points of semigroup mappings is denoted by $F(\alpha_s) = \{x \in X : x\alpha_s = x \text{ for } s \in S\}$.

More so, any map $\alpha_s \in OCT_n$ or $\alpha_s \in OC^*T_n$ possesses the properties as stated in section one. Note that (i) the set of fixed points of order-preserving maps, denoted by $F_o(\alpha_s)$, is a subset of $F(\alpha_s)$. (ii) $F_o(\alpha_s) \subset F_o^n(\alpha_s)$ for all $n > 1$. (iii) If $F_o^n(\alpha_s)$ is singleton for some n , so does $F_o(\alpha_s)$.

Without loss of generality, the notion 'nonexpansive' connotes 'contraction' while 'expansive' connotes 'contractive'. In view of the above, we present some definitions of nonexpansive semigroup of order-preserving mappings in WMS under the property that both OCT_n and OC^*T_n have no common maps.

Definition 8. Let X be a closed subset of $(M, \omega)_\alpha$ and let S be a semitopological semigroup. The map $\alpha_s : X \rightarrow X$ is called a nonexpansive semigroup of order-preserving total map ONT_n if for $x, y \in X$ and $s \in S$,

$$\omega(x\alpha_s, y\alpha_s) \leq \omega(x, y). \tag{7}$$

Definition 9. Let X be a closed subset of $(M, \omega)_\alpha$ and let S be a semitopological semigroup. The map $\alpha_s : X \rightarrow X$ is called an expansive semigroup of order-preserving total map OET_n if for $x, y \in X$ and $s \in S$,

$$\omega(x\alpha_s, y\alpha_s) > \omega(x, y). \tag{8}$$

3. Main Results

The following lemma is useful in the proof of the main results.

Lemma 4. *If $\varphi = \{\alpha_s : s \in S\}$ is a semigroup of continuous mappings of X into itself and $\omega(\alpha_s x, y) \rightarrow 0$ as $s \rightarrow \infty_{\mathbb{R}}$ for $x, y \in X$, then $y \in F_o(\alpha_s) \subset X$.*

Proof: Let $\varepsilon > 0$ be given. By the continuity of α_t for $t \in S$, there exists $\tau > 0$ such that $\omega(\alpha_t x, \alpha_t y) < \frac{\varepsilon}{2}$ whenever $\omega(x, y) < \tau$ for $x, y \in X$.

Also, since $\omega(\alpha_s x, y) \rightarrow 0$ as $s \rightarrow \infty_{\mathbb{R}}$, then there exists $u \in S$ such that

$\omega(\alpha_{au} x, y) < \min\{\frac{\varepsilon}{2}, \tau\}$ for each $a \in S$. Thus, $\omega(\alpha_{tau} x, \alpha_t y) < \frac{\varepsilon}{2}$. Now,

$$\omega(y, \alpha_t y) \leq \omega(y, \alpha_{tau} x) + \omega(\alpha_{tau} x, \alpha_t y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $y \in F_o(\alpha_s)$.

Remark 3. *If $S = \mathbb{N}$, then the hypothesis on α_s would include asymptotic regularity condition. For example, see Theorem 6.7. in [25].*

In the next theorem, we let $A(X, \bar{c}o\{\alpha_s x\})$ denote the asymptotic center and $r(x_0, \bar{c}o\{\alpha_s x\})$ is the asymptotic radius.

Theorem 2. *Let X be a nonempty closed subset of a convex waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \varphi$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ONT_n$ for $s \in S$. If the set $\{\alpha_s x, s \in S\}$ is bounded for some $x \in X$ and $y \in A(X, \bar{c}o\{\alpha_s x\})$, then $y \in F_o(\alpha_s)$.*

Proof: Let $\{\alpha_s x, s \in S\}$ be a bounded net. Define $R := r(y, \bar{c}o\{\alpha_s x\})$ for $y \in A(X, \bar{c}o\{\alpha_s x\})$ with the property that $\omega(x, y) < R$.

If $R = 0$, then $\limsup \omega(\alpha_s x, y) = 0$ and by Lemma 4, the proof is complete. On the other hand, suppose $R > 0$ and $y \notin F_o(\alpha_s)$, then for given $\varepsilon > 0$ and a subnet $\{s_\beta\}$ in S , we have

$$\omega(\alpha_{s_\beta} y, y) > \varepsilon, \text{ for } s_\beta \in S.$$

Also, since $\omega(\alpha_s x, y) \rightarrow 0$ as $s \rightarrow \infty_{\mathbb{R}}$, then there exists $\gamma \in S$ such that, for choosing $\nu \geq 0$,

$$\omega(\alpha_\gamma x, y) < R + \nu. \tag{9}$$

Moreover, we have by hypothesis that

$$\begin{aligned} \omega(\alpha_s x, \alpha_s y) &\leq \limsup_{\beta} \omega(\alpha_\beta x, \alpha_\beta y) + \nu \\ &\leq \limsup \omega(x, y) + \nu \\ &= R + \nu \end{aligned} \tag{10}$$

By Lemma 3, (9) and (10), we have

$$\begin{aligned} \omega^2(u, \alpha_s x) &\leq \frac{1}{2} \omega^2(u, \alpha_s x) + \frac{1}{2} \omega^2(\alpha_s x, \alpha_s y) - \frac{1}{4} \omega^2(y, \alpha_s y) \\ &\leq \frac{1}{2} (R + \nu)^2 + \frac{1}{2} (R + \nu)^2 - \frac{1}{4} \varepsilon^2 \leq (R - \nu)^2 \end{aligned}$$

Thus, $\omega(u, \alpha_s x) < R - \nu$ which implies that

$$r(u, \alpha_s x) < r(y, \bar{c}o\{\alpha_s x\}).$$

is a contradiction. Hence, $y \in F_o(\alpha_s)$.

Necessary and sufficient result for Theorem 2 is presented as follow:

Theorem 3. *Let X be a nonempty closed subset of a convex waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \varphi$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ONT_n$ for $s \in S$. The set $\{\alpha_s x, s \in S\}$ is bounded for some $x \in X$ if and only if $F_o(\alpha_s)$ is nonempty.*

Proof: Assume that $\{\alpha_s x, s \in S\}$ is bounded for some $x \in X$, there is a unique element $y \in X$ for which $y \in A(X, \bar{c}o\{\alpha_s x\})$. By Theorem 2, $F_o(\alpha_s)$ is nonempty. The converse is obvious.

Remark 4. *If in Theorem 2, the boundedness assumption on $\{\alpha_s x, s \in S\}$ is dropped, then another suitable concept is stated in the next theorem.*

Theorem 4. *Let X be a closed subset of a complete waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \varphi$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ONT_n$ for $s \in S$. Then, α_s has at least one fixed point.*

Proof: For $\delta \in (0, 1]$, set $T_\delta = (1 - \delta)\alpha_s$. It follows that T_δ is a δ -contraction on X and by the Banach fixed point theorem, there exists x_δ for $\delta \in (0, 1]$ such that $T_\delta x_\delta = x_\delta$. Now, we show that for $\delta_k \rightarrow 0$, the net x_{δ_k} converges to p , where p is a fixed point of α_s . Indeed, for any arbitrary $u \in X$, we have

$$\begin{aligned} \omega^2(x_{\delta_k}, u) &= \omega^2(x_{\delta_k} - p, u - p) \\ &= \omega^2(x_{\delta_k}, p) + \omega^2(p, u) + 2\omega(x_{\delta_k} - p, u - p) \\ &\leq \omega^2(x_{\delta_k}, p) + \omega^2(p, u) \end{aligned}$$

By setting $u = \alpha_s p$, we have

$$\limsup (\omega^2(x_{\delta_k}, \alpha_s p) - \omega^2(x_{\delta_k}, p)) \leq \omega^2(p, \alpha_s p) \tag{11}$$

Also, since $T_{\delta_k} x_{\delta_k} = x_{\delta_k}$ and $\delta_k \rightarrow 0$ as $k \rightarrow \infty$, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) &= \lim_{k \rightarrow \infty} [\omega(x_{\delta_k}, T_{\delta_k} x_{\delta_k}) + \delta_k \omega(0, \alpha_s x_{\delta_k})] \\ &= \lim_{k \rightarrow \infty} \omega(x_{\delta_k}, T_{\delta_k} x_{\delta_k}) \rightarrow 0 \end{aligned} \tag{12}$$

On the other hand, since $\alpha_s \in ONT_n$, then

$$\omega(\alpha_s p, \alpha_s x_{\delta_k}) \leq \omega(p, x_{\delta_k}) \tag{13}$$

We have from (12) and (13) that

$$\omega(x_{\delta_k}, \alpha_s p) \leq \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) + \omega(x_{\delta_k}, p)$$

which further implies

$$\limsup [\omega(x_{\delta_k}, \alpha_s p) - \omega(x_{\delta_k}, p)] \leq \lim_{k \rightarrow \infty} \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) = 0 \tag{14}$$

Also, from (11) and (14), we obtain

$$\limsup \left(\omega^2(x_{\delta_k}, \alpha_s p) - \omega^2(x_{\delta_k}, p) \right) = \limsup \left(\omega(x_{\delta_k}, \alpha_s p) - \omega(x_{\delta_k}, p) \right) \times \left(\omega(x_{\delta_k}, \alpha_s p) + \omega(x_{\delta_k}, p) \right)$$

Thus,

$$\limsup \left(\omega^2(x_{\delta_k}, \alpha_s p) - \omega^2(x_{\delta_k}, p) \right) = \omega^2(p, \alpha_s p) = 0$$

and hence, p is the fixed point of α_s

Theorem 5. Let X be a closed subset of a complete waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \wp$ is a mapping satisfying (7), that is, $\alpha_s \in ON\mathcal{T}_n$ for $s \in S$. Then, the set $F_o(\alpha_s) \subset X$ is a nonempty closed convex set.

Proof: Since α_s satisfies (7), by Theorem 2, α_s has fixed point in X . It is left to show that $F_o(\alpha_s)$ is closed and convex. Firstly, we show that $F_o(\alpha_s)$ is closed. Let $\{x_t\}$ be a net in $F_o(\alpha_s)$ such that $x_t \rightarrow x$, then by hypothesis:

$$\omega(\alpha_t x, x) \leq \omega(\alpha_t x, x_t) + \omega(x_t, x) \rightarrow 0$$

This implies $\alpha_t x \rightarrow x \in F_o(\alpha_s)$. Hence, $F_o(\alpha_s)$ is closed. Also, let $x, y \in F_o(\alpha_s)$ and $\lambda \in [0, 1]$, we have $u_\lambda = u(x, y, \lambda) \in F_o(\alpha_s)$.

Indeed,

$$\omega(\alpha_s u_\lambda, x) = \omega(\alpha_s u_\lambda, \alpha_s x) \leq \omega(u_\lambda, x)$$

Similarly, $\omega(\alpha_s u_\lambda, y) \leq \omega(u_\lambda, y)$. Thus,

$$\omega(x, y) \leq \omega(x, \alpha_s u_\lambda) + \omega(\alpha_s u_\lambda, y) \leq \omega(x, y).$$

This shows that for some a, b with $0 \leq a, b \leq 1$, we have

$$\omega(x, \alpha_s u_\lambda) = a\omega(x, u_\lambda) \text{ and } \omega(y, \alpha_s u_\lambda) = b\omega(y, u_\lambda)$$

from which it follows that $\alpha_s u_\lambda \in F_o(\alpha_s)$.

By letting $S = \mathbb{N}$ in Theorems 4 and 5, we obtain the existence property of a nonexpansive semigroup of order-preserving total mapping in waist metric spaces as follow:

Corollary 1. Let X be a closed subset of a convex waist metric space (M, ω) . Suppose $\alpha \in \wp = \{\alpha^n : n \in \mathbb{N}\}$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself. Then, $F_o(\alpha_s) \neq \emptyset$ if and only if $\{\alpha^n x : n \in \mathbb{N}\}$ is bounded for some $x \in X$. Furthermore, $F_o(\alpha_s)$ is closed and convex.

Theorem 6. Let X be a closed subset of a complete waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \wp = \{\alpha_s : s \in S\}$ is a mapping satisfying (8). Then,

i. the map α_s exists;

ii. $F_o(\alpha_s)$ is nonempty.

Proof:

i. Let $x_1, x_2 \in X$ with $x_1 < x_2$. Suppose, on contrary, that $\alpha_s \notin OET_n$, then α_s is an order-reversing map, that is, $x_1 \alpha_s \geq x_2 \alpha_s$. By induction,

$$x_k \alpha_s \geq x_{k+1} \alpha_s, \text{ for } k = 1, 2, 3, \dots, n - 1$$

Define $x_{k+1} = x_k \alpha_s$. Using condition (8), there gives

$$\omega(x_{k+1}, x_{k+2}) \geq \omega^+(x_{k+1} \alpha_s, x_{k+2} \alpha_s) > \omega^+(x_{k+1}, x_{k+2})$$

This is a clear contradiction. Hence, $\alpha_s \in OET_n$.

ii. Let $\alpha_s \in OET_n$. Suppose that $F_o(\alpha_s)$ is empty, that is, there is no fixed β such that $\beta \in F_o(\alpha_s)$, this implies that $\omega^+(\beta, \beta \alpha_s) > \varepsilon$, for $\varepsilon > 0$. Let $\omega^+(x_k \alpha_s, \beta) = 0$, by condition (8), there results

$$\begin{aligned} \omega(x_k, \beta) &< \omega(x_k \alpha_s, \beta \alpha_s) \\ &\leq \omega(x_k \alpha_s, \beta) + \omega(\beta, \beta \alpha_s) \\ &= \omega(\beta, \beta \alpha_s) \end{aligned}$$

On the other hand, since α_t is continuous on X for each $t \in S$. Then, for $x_k \in X$ and $\tau > 0$, $\omega(x_k, \beta) > \tau$ implies that $\omega(x_k \alpha_t, \beta \alpha_t) > \frac{\varepsilon}{2}$.

Since $\omega^+(x_k \alpha_s, \beta) = 0$, then there exists $b \in S$ such that $\omega^+(x_k \alpha_{ab}, \beta) > \max\{\frac{\varepsilon}{2}, \delta\}$ for each $a \in S$. Intuitively, $\omega^+(x_k \alpha_t \alpha_{ab}, \beta \alpha_t) > \frac{\varepsilon}{2}$.

But then,

$$\omega^+(\beta \alpha_t, \beta) \leq \omega^+(\beta \alpha_t, x_k \alpha_{tab}) + \omega^+(x_k \alpha_{tab}, \beta)$$

and

$$\omega^+(\beta \alpha_t, x_k \alpha_{tab}) + \omega^+(x_k \alpha_{tab}, \beta) > \varepsilon$$

From the last two inequalities, we obtain $\omega^+(\beta \alpha_t, \beta) = \varepsilon$. This is a contradiction. Therefore, $\omega^+(\beta \alpha_s, \beta) = 0$ for each $s \in S$ and $\beta \in F_o(\alpha_s)$.

Remark 5. The convergence theorems (strong convergence and Δ -convergence) of the map $\alpha_s \in \wp$ satisfying the nonexpansive semigroup of order-preserving total mapping (7) in waist metric spaces can be routinely proved using the next lemma and their left.

Lemma 5. Let X be a closed subset of a convex waist metric space (M, ω) . Suppose α_s is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself with $F_o(\alpha_s) \neq \emptyset$. Then $\lim_s \omega(\alpha_s x, y)$ exists for each $y \in F_o(\alpha_s)$ and $s \in S$.

4. Practical Examples

The following three examples are considered to justify the theorems in the main results. The first two examples are from the same family for $n = 2$ and $n = 3$, respectively, while the third example is given for $n = 3$.

Example 3. Let S be a semigroup and $X = \{1, 2\}$. Let $\alpha_s : \{1, 2\} \rightarrow \{1, 2\}$ be the mapping $\alpha_s = (1)(2)$ in OT_2 , where $s \in S$, associated to $x \alpha_s = x^2 - 2x + 2$ in $X \subset (M, \omega)_\alpha$.

The map α_s is a nonexpansive order-preserving total mapping. Hence, it satisfies all hypotheses of Theorem 2 and 4 and the set $F_o(\alpha_s) = \{1, 2\}$.

Example 4. Let α_s be a mapping in OT_3 define by $\alpha : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ as $\alpha = (1)[3 \ 2 \ 1]$ which is equivalent to the map $x\alpha_s = \frac{x^2-3x}{2} + 2$.

Here, the map α_s is also a nonexpansive order-preserving total mapping, that is, $\alpha_s \in ONT_3$. Thus, it satisfies all hypotheses of Theorem 2 and 4. The set $F_o(\alpha_s) = \{1\}$.

Example 5. Let $\alpha_s : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ be given by the mapping $\alpha = (1)[2 \ 1](3)$ in OT_3 corresponding to the map $x\alpha_s = x^2 - 3x + 3$.

In Example 5, all hypotheses of Theorem 6 are satisfied since $\alpha_s \in OET_n$ with fixed point $F_o(\alpha_s) = \{1, 3\}$.

5. Concluding Remark

This study introduced the nonexpansive and expansive semigroup of order-preserving total mappings to prove some fixed point theorems in complete waist metric spaces. The study also considered some examples (see Examples 3, 4 and 5) on semigroup of order-preserving total mappings to validate the hypotheses of Theorems 4, 5 & 6. Results show that the nonexpansive (and expansive) semigroup of order-preserving mappings compares favorably with the semigroup of mappings OCT_n (and OC^*T_n). In fact, every mapping $\alpha_s \in \wp$ under the action of subsemigroups OCT_n and OC^*T_n is also in ONT_n and OET_n , respectively. However, this study only features some elements of subsemigroups of order-preserving full mappings in [27]. Therefore, future studies would be to establish some notions to study other elements of subsemigroups in \wp such as order-preserving partial mappings [1, 28, 15], order decreasing full mappings [29], symmetric inverse semigroups [5], orientation-preserving mappings [30], fence-preserving mappings [31] among others.

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