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Masses and thermal properties of a Charmonium and Bottomonium Mesons

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Abstract

In this research, we model Hulthén plus generalized inverse quadratic Yukawa potential to interact in a quark-antiquark system. The solutions of the Schrödinger equation are obtained using the Nikiforov-Uvarov method. The energy spectrum and normalized wave function were obtained. The masses of the heavy mesons for different quantum states such as 1S, 2S , 1P, 2P 3S, 4S, 1D, and 2D were predicted as 3.096 GeV, 3.686 GeV, 3.327 GeV, 3.774GeV, 4.040 GeV, 4.364GeV, 3.761 GeV, and 4.058 GeV respectively for charmonium ($c\bar{c}$). Also, for bottomonium ($b\bar{b}$) we obtained 9.460 GeV, 10.023 GeV, 9.841 GeV, 10.160 GeV, 10.345 GeV, 10.522 GeV, and 10.142GeV for different states of 1S , 2S , 1P , 2P , 3S , 4S , 1D respectively. The partition function was calculated from the energy spectrum, thereafter other thermal properties were obtained. The results obtained showed an improvement when compared with the work of other researchers and excellently agreed with experimental data with a percentage error of 1.60 % and 0.46 % for ($c\bar{c}$) and ($b\bar{b}$), respectively.

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Keywords: Schrödinger equation; Nikiforov-Uvarov method; heavy mesons; Thermal properties; Hulthén plus generalized inverse quadratic Yukawa potential.

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1. Introduction

In non-relativistic quantum mechanics, researchers work mostly on the solutions of the Schrödinger equation (SE) to obtain the eigenvalues and eigenfunction. These eigenvalues and eigenfunction can be used to study the mass spectra (MS) of the heavy mesons, and theoretic measures, among other things [1, 2, 3, 4]. In describing the interaction of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), a Cornell potential (CP) is used because of the Coulomb interaction and a confining term [5]. More so, in solving the SE with any potential of choice, analytical methods are employed. Most of the frequently used analytical methods are as follows, the Nikiforov-Uvarov (NU) method [6-12], the Nikiforov-Uvarov Functional Analysis (NUFA) method [13-16], the series expansion method (SEM) [17, 18, 19], Laplace transformation method (LTM) [20], Asymptoptic Iteration Method [21], WKB approximation method [22-24], among others [25-31]. The investigation of the MS with CP has gained remarkable interest, and has

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attracted the attention of many scholars [32-35]. For instance, Mutuk [36] solved the SE with CP using a neural network approach. The bottomonium, charmonium, and bottom-charmed spin-averaged spectra were calculated. Using, the vibrational method (VM) and supersymmetric quantum mechanics Vega and Flores, [37] obtained the analytical solutions of the SE with CP. The eigenvalues were used to calculate the MS of the HMs. Also, Kumar et al., [38] used the NUFA method to solve the SE with generalized Cornell potential. The result was used to predict the MS of the HMs. Furthermore, Hassanabadi et al. [39], used the VM to solve the SE with CP. The eigenvalues were used to calculate the mesonic wave function. In recent times, the study of MS of the HMs with exponential-type potentials has aroused the interest of scholars [40, 41]. Potential models such as Yukawa potential [42], Varshni [43], screened Kratzer potential [44], and so on have been used in the prediction of the MS of the HMs. For instance, Purohit et al [45] combined linear plus modified Yukawa potential to obtain the masses of the HMs through the solutions of the Klein-Gordon equation. The SE for most of the potentials with spin addition cannot be solved analytically; hence, numerical solutions such as Runge-Kutte approximation [46], Numerov matrix method [47], Fourier grid Hamiltonian method [48], and so on [49] are employed. Also, adding spin enables one to determine other properties of the mesons like decay properties and root mean square radii. However, we have considered our mesons as spinless particles for easiness [1, 40, 50, 51, 52]. Recently, researchers have studied thermal properties (TPs) of the HMs [40, 53]. For instance, Abu-Shady et al., [54] studied the TPs of HMs with extended Cornell potential. Also Inyang et al., [55] obtained the TPs of the HMs with exponential potential. Abu-Shady and Ezz-Alarab [40], solved the N-radial SE with the Trigonometric Rosen-Morse potential, and the obtained energy equation was used in studying TPs and masses of heavy and heavy-light mesons.

Cao et al., [56], studied the TPs of light mesons, including the temperature dependence of their masses by solving the spatial correlation. In recent times scholars, have proposed new potentials through the combination of two potentials [57, 58]. It's observed that the combination of at least two potential models has a propensity to fit experimental data more than a single potential [55]. Okoi et al.,[59] proposed the Hulthén plus generalized inverse quadratic Yukawa potential (HGIQYP) to obtain the bound state energy and expectation values. Hulthén potential [60] and generalized inverse quadratic Yukawa potential [61], is applied to diverse areas of physics such as nuclear and particle and so on.

The aim of this study is in three folds. First, the HGIQYP will be modeled to fit in the CP, secondly, the SE will be solved using the well-known NU method, and finally to predict the masses of charmonium and bottomonium mesons and calculate its TPs.

The HGIQYP takes the form [60]:

$$V(q) = -\frac{R_0 e^{\theta q}}{1 - e^{\theta q}} - R_1\left(1 + \frac{e^{\theta q}}{q}\right)^2 \tag{1}$$

where R_0 and R_1 are the strength of the potential, θ is the screening parameter and q is inter-nuclear distance. We model equation (1) to interact in the quark-antiquark system by expanding the exponential terms in a series of powers up to order three and have,

$$V(q) = -\frac{G_0}{q} + G_1 q + G_2 q^2 + \frac{G_3}{q^2} + G_4 \tag{2}$$

where

$$G_0 = \frac{R_0}{\sigma} + 2R_1 - 2R_1\alpha,\ G_1 = -\frac{\alpha R_0}{6} - R_1\alpha^2 \tag{3}$$
$$G_2 = \frac{R_1\alpha^3}{3},\ G_3 = -R_1,\ G_4 = \frac{R_0}{2} - R_1 + 2\alpha R_1 - 2R_1\alpha^2$$

2. The solutions of the Schrödinger equation with HGIQYP

In this research, the NU method is employed. The details are found in [62]. The SE is given as [63]

$$\frac{d^2U(q)}{dq^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{n\ell} - V(q)\right) - \frac{\ell(\ell+1)}{q^2}\right]U(q) = 0 \tag{4}$$

where ℓ is the angular momentum quantum number, μ is the reduced mass for the quark-antiquark particle, q is the inter-particle distance, and $\hbar$ is reduced plank constant. Then, we substitute equation (2) into equation (4), and got

$$\frac{d^2U(q)}{dq^2} + \left[\frac{2\mu E_{n\ell}}{\hbar^2} + \frac{2\mu G_0}{\hbar^2 q} - \frac{2\mu G_1 q}{\hbar^2} - \frac{2\mu G_2 q^2}{\hbar^2} - \frac{2\mu G_3}{\hbar^2 q^2} - \frac{2\mu G_4}{\hbar^2} - \frac{\ell(\ell+1)}{q^2}\right]U(q) = 0 \tag{5}$$

Transformation of q (in equation (8) to w coordinates yields equation (6),

$$w = \frac{1}{q},\ q > 0 \tag{6}$$

The second derivative of equation (6) is given as

$$\frac{d^2U(q)}{dq^2} = 2w^3\frac{dU(w)}{dw} + w^4\frac{d^2U(w)}{dw^2} \tag{7}$$

Replacing equations (6) and (7) in equation (8) gives;

$$\frac{d^2U(w)}{dw^2} + \frac{2w}{w^2}\frac{dU}{dw} + \frac{1}{w^4}\left[\frac{2\mu E_{nl}}{\hbar^2} + \frac{2\mu G_0 w}{\hbar^2} - \frac{2\mu G_1}{\hbar^2 w} - \frac{2\mu G_2}{\hbar^2 w^2} - \frac{2\mu G_3 w^2}{\hbar^2} - \frac{2\mu G_4}{\hbar^2} - l(l+1)w^2\right]U(w) = 0 \tag{8}$$

The approximation scheme (AS) is introduced by assuming that there is a characteristic radius r_0 of the meson. The AS is achieved by the expansion of $\frac{G_1}{w}$ and $\frac{G_1 2}{w^2}$ in a power series around r_0 ; i.e. around $\phi \equiv \frac{1}{r_0}$, up to the second-order [64].

By setting $y = w - \phi$ and around $y = 0$ we expand it in powers of series as:

$$\frac{G_1}{w} = \frac{G_1}{y+\phi} = \frac{G_1}{\phi\left(1+\frac{y}{\phi}\right)} = \frac{G_1}{\phi}\left(1+\frac{y}{\phi}\right)^{-1} \tag{9}$$

Equation (9) yields

$$\frac{G_1}{w} = G_1\left(\frac{3}{\phi} - \frac{3z}{\phi^2} + \frac{z^2}{\phi^3}\right) \tag{10}$$

Similarly,

$$\frac{G_2}{w^2} = G_2\left(\frac{6}{\phi^2} - \frac{8w}{\phi^3} + \frac{3w^2}{\phi^4}\right) \tag{11}$$

Placing equations (10) and (11) into equation (13), we obtain:

$$\frac{d^2U(w)}{dw^2} + \frac{2w}{w^2}\frac{dU(w)}{dw} + \frac{1}{w^4}\left[-\varepsilon + X_0 w - X_1 w^2\right]U(w) = 0 \tag{12}$$

where

$$-\varepsilon = \left(\frac{2\mu E_{nl}}{\hbar^2} - \frac{6\mu G_1}{\hbar^2\phi} - \frac{12\mu G_2}{\hbar^2\phi^2} - \frac{2\mu G_4}{\hbar^2}\right), \tag{13}$$
$$X_0 = \left(\frac{2\mu G_0}{\hbar^2} + \frac{6\mu G_1}{\hbar^2\phi^2} + \frac{16\mu G_2}{\hbar^2\phi^3}\right)$$
$$X_1 = \left(\frac{2\mu G_1}{\hbar^2\phi^3} + \frac{6\mu G_2}{\hbar^2\phi^4} + \frac{2\mu G_3}{\hbar^2} + \gamma\right),\ \gamma = l(l+1)$$

Connecting equation (12) and equation (1) of [62], we obtain

$$\tilde{\tau}(w) = 2w,\ \sigma(w) = z^2 \tag{14}$$
$$\tilde{\sigma}(w) = -\varepsilon + \alpha w - \beta w^2$$
$$\sigma'(w) = 2w,\ \sigma''(w) = 2$$

Plugging equation (23) into equation (8) of [56], gives

$$\pi(w) = \pm\sqrt{\varepsilon - X_0 w + (X_1 + k)\, w^2} \tag{15}$$

To determine k, in equation (15), the discriminant of the function equation (16) and equation (17) are obtained as

$$k = \frac{{X_0}^2 - 4X_1\varepsilon}{4\varepsilon} \tag{16}$$

$$\pi(w) = \pm\left(\frac{X_0 w}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{17}$$

For a physically acceptable solution, the negative part of equation (17) is required, upon differentiating we get.

$$\pi'_{-}(w) = -\frac{X_0}{2\sqrt{\varepsilon}} \tag{18}$$

Placing equation (23) and equation (17) into equation (6) of Ref. [62] gives

$$\tau(w) = 2w - \frac{X_0 w}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{19}$$

Equation (19) gives

$$\tau'(w) = 2 - \frac{X_0}{\sqrt{\varepsilon}} \tag{20}$$

Using equations (19) and equation (21) of Ref. [62], we have the following,

$$\lambda = \frac{{X_0}^2 - 4X_1\varepsilon}{4\varepsilon} - \frac{X_0}{2\sqrt{\varepsilon}} \tag{21}$$

$$\lambda_n = \frac{nX_0}{\sqrt{\varepsilon}} - n^2 - n \tag{22}$$

Equating equations (21) and (21), followed by the substitution of equations (5) and (14) yielded the energy spectrum of the HGIQYP

$$E_{nl} = \frac{R_0}{2} - R_1 + 2\vartheta R_1 - 2R_1\vartheta^2 - \frac{\vartheta R_0}{2\phi} - \frac{3R_1\vartheta^2}{\phi} + \frac{2R_1\vartheta^3}{\phi^2} \tag{23}$$
$$-\frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\phi^2}\left(-\frac{\vartheta R_0}{6} - R_1\vartheta^2\right) + \frac{2\mu}{\hbar^2}\left(\frac{R_0}{\vartheta} + 2R_1 - 2R_1\vartheta\right) + \frac{16\mu R_1\vartheta^3}{3\hbar^2\phi^3}}{n + \frac{1}{2} + \sqrt{\left(\frac{1}{2}+l\right)^2 + \frac{2\mu}{\hbar^2\phi^3}\left(-\frac{\vartheta R_0}{6} - R_1\vartheta^2\right) + \frac{2\mu R_1\vartheta^3}{\hbar^2\phi^4} - \frac{2\mu R_1}{\hbar^2}}}\right]^2$$

The wave function, is obtained by putting equations (23) and (17) into equation (4) of Ref. [62]

$$\frac{d\phi}{\phi} = \left(\frac{\varepsilon}{w^2\sqrt{\varepsilon}} - \frac{X_0}{2w\sqrt{\varepsilon}}\right)dw \tag{24}$$

Integration of equation (24) gives

$$\phi(w) = w^{-\frac{X_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{w\sqrt{\varepsilon}}} \tag{25}$$

Putting equations (23) and (19) into equation (26) of Ref. [62] we have

$$\rho(w) = w^{-\frac{X_0}{\sqrt{\varepsilon}}} e^{-\frac{2\varepsilon}{w\sqrt{\varepsilon}}} \tag{26}$$

The substitution of equations (23) and (26) into equation (5) of Ref.[62] gives

$$y_n(w) = B_n e^{\frac{2\varepsilon}{w\sqrt{\varepsilon}}} z^{\frac{X_0}{\sqrt{\varepsilon}}} \frac{d^n}{dw^n}\left[e^{-\frac{2\varepsilon}{w\sqrt{\varepsilon}}} w^{n-\frac{X_0}{\sqrt{\varepsilon}}}\right] \tag{27}$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) = \frac{1}{n!} e^{\frac{2\varepsilon}{w\sqrt{\varepsilon}}} w^{\frac{X_0}{\sqrt{\varepsilon}}} \frac{d^n}{dw^n}\left(e^{-\frac{2\varepsilon}{w\sqrt{z}}} w^{n-\frac{X_0}{\sqrt{\varepsilon}}}\right) \tag{28}$$

where $B_n = \frac{1}{n!}$. Hence,

$$y_n(w) \equiv L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) \tag{29}$$

The substitution of equations (25) and (29) into equation (2) of Ref. [62], gives the wave function in terms of associated Laguerre polynomials as

$$\psi(w) = N_{nl} w^{-\frac{X_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{w\sqrt{\varepsilon}}} L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) \tag{30}$$

where N_{nl} is normalization constant, which can be obtained from

$$\int_0^\infty |\psi_{nl}(r)|^2 dr = 1 \tag{31}$$

Inserting equation (30) into (31) with $w = 1/r$ gives

$$N_{nl}^2 \int_0^\infty r^{X_0/\sqrt{\varepsilon}} e^{-2\sqrt{\varepsilon}r}\left[L_n^{X_0/\sqrt{\varepsilon}}\left(2\sqrt{\varepsilon}r\right)\right]^2 dr = 1 \tag{32}$$

By using the transformation $x = 2\sqrt{\varepsilon}r$ we obtained the well-known standard integral of the Laguerre polynomials

$$\frac{N_{nl}^2}{\left(2\sqrt{\varepsilon}\right)^{X_0/\sqrt{\varepsilon}+1}} \int_0^\infty x^{X_0/\sqrt{\varepsilon}} e^{-x}\left[L_n^{X_0/\sqrt{\varepsilon}}(x)\right]^2 dx = 1 \tag{33}$$

The solution of the standard integral [65] is given as

$$\int_0^\infty x^{X_0/\sqrt{\varepsilon}} e^{-x}\left[L_n^{X_0/\sqrt{\varepsilon}}(x)\right]^2 dx = \frac{\Gamma\left(n + X_0/\sqrt{\varepsilon} + 1\right)}{\Gamma(n+1)} \tag{34}$$

Comparing equations (33) and (34) we obtained the normalization factor such that the total wave function of the mesons can be written in closed form as

$$\psi_{nl}(r) = \sqrt{\frac{\left(2\sqrt{\varepsilon}\right)^{X_0/\sqrt{\varepsilon}+1}\Gamma(n+1)}{\Gamma\left(n + X_0/\sqrt{\varepsilon} + 1\right)}} \times r^{X_0/2\sqrt{\varepsilon}} e^{-\sqrt{\varepsilon}r} L_n^{X_0/\sqrt{\varepsilon}}\left(2\sqrt{\varepsilon}r\right) \tag{35}$$

3. Thermal Properties of the SE with HGIQYP

To obtain the TPs of the heavy mesons, we first calculate the partition function. Equation 26 is written in the form

$$E_{nl} = Q_1 - \frac{\hbar^2}{8\mu}\left[\frac{Q_2}{(n+\theta)}\right]^2 \tag{36}$$

where

$$\theta = \frac{1}{2} + \sqrt{\left(\frac{1}{2}+l\right)^2 + \frac{2\mu}{\hbar^2\phi^3}\left(-\frac{\vartheta R_0}{6} - R_1\vartheta^2\right) + \frac{2\mu R_1\vartheta^3}{\hbar^2\phi^4} - \frac{2\mu R_1}{\hbar^2}} \tag{37}$$

$$Q_1 = \frac{R_0}{2} - R_1 + 2\vartheta R_1 - 2R_1\vartheta^2 - \frac{\vartheta R_0}{2\phi} - \frac{3R_1\vartheta^2}{\phi} + \frac{2R_1\vartheta^3}{\phi^2} \tag{38}$$

$$Q_2 = \frac{6\mu}{\hbar^2\phi^2}\left(-\frac{\vartheta R_0}{6} - R_1\vartheta^2\right) + \frac{2\mu}{\hbar^2}\left(\frac{R_0}{\vartheta} + 2R_1 - 2R_1\vartheta\right) + \frac{16\mu R_1\vartheta^3}{3\hbar^2\phi^3} \tag{39}$$

3.1. Partition function $Z(\beta)$

The partition function (PF) takes the form [54]:

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_{nl}} \tag{40}$$

where, $\beta = \frac{1}{K_0 T}$, K_0 is the Boltzmann constant, T is the absolute temperature, and λ is the maximum quantum number. Swapping equation (36) into equation (40) we obtain

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta\left(Q_1 - \frac{\hbar^2}{8\mu}\left[\frac{Q_2}{(n+\theta)}\right]^2\right)} \tag{41}$$

In the classical limit, the summation is replaced by an integral,

$$Z(\beta) = \int_\theta^{\lambda+\theta} e^{M_1\beta + \frac{N_1\beta}{\rho^2}} d\rho \tag{42}$$

where

$$n + \theta = \rho \quad M_1 = -Q_1 \quad N_1 = \frac{\hbar^2 Q_2^2}{8\mu} \tag{43}$$

From equation (42), the PF is obtain as,

$$Z(\beta) = e^{M_1\beta}\left(\frac{\rho e^{\frac{N_1\beta}{\rho^2}} - N_1\beta\sqrt{\pi}erfi\left(\frac{\sqrt{N_1\beta}}{\rho}\right)}{\sqrt{N_1\beta}}\right), \tag{44}$$

$$\theta \le \rho \le \lambda + \theta$$

Other TPs can be obtained as follows:

3.2. Mean energy $U(\beta)$

$$U(\beta) = -\frac{\partial}{\partial\beta} InZ(\beta), \tag{45}$$

3.3. Free energy $F(\beta)$

$$F(\beta) = -K_0 T \ln Z(\beta) \tag{46}$$

3.4. Entropy $S(\beta)$

$$S(\beta) = K_0 \ln Z(\beta) - K_0\beta\frac{\partial}{\partial\beta}\ln Z(\beta) \tag{47}$$

3.5. Specific heat capacity $C(\beta)$

$$C(\beta) = \frac{\partial U}{\partial T} = -K_0\beta^2\frac{\partial U}{\partial\beta} \tag{48}$$

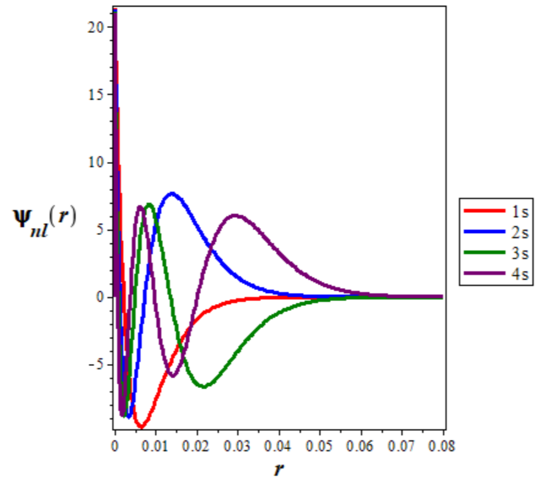

Figure 1: Wave function of charmonium against the radius at different s state

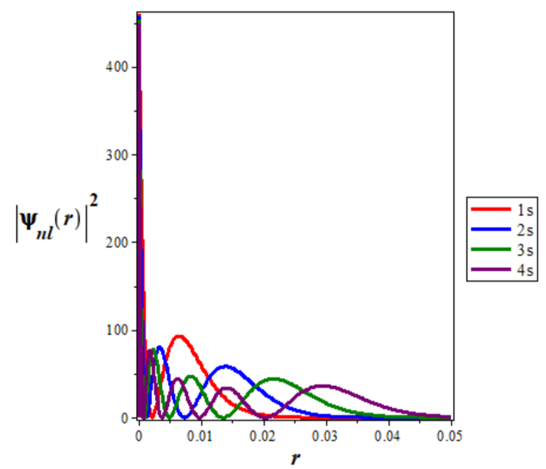

Figure 2: Probability density function of charmonium against the radius at different s state

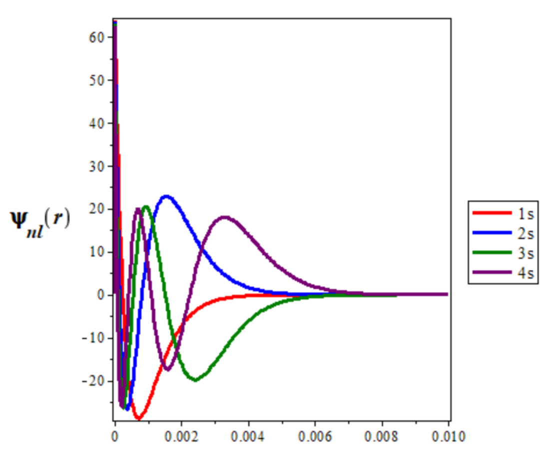

Figure 3: Wave function of bottomonium against the radius at different s state

4. Results and discussion

The prediction of the MS of the HMs is carried out using the relation [66, 67]

$$M = 2m + E_{nl} \tag{49}$$

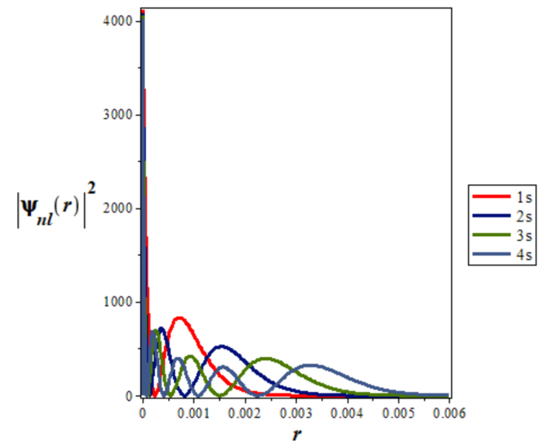

Figure 4: Probability density function of bottomonium against the radius at different s state

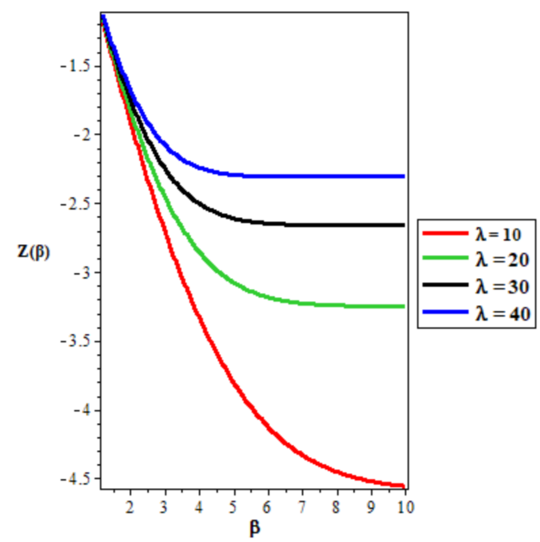

Figure 5: Partition function against β for different values of λ

where m is quark-antiquark mass and E_{nl} is energy eigenvalues. Plugging equation (26) into equation (49) gives:

$$M = 2m + \frac{R_0}{2} - R_1 + 2\vartheta R_1 - 2R_1\vartheta^2 - \frac{\vartheta R_0}{2\phi} - \frac{3R_1\vartheta^2}{\phi} + \frac{2R_1\vartheta^3}{\phi^2} \tag{50}$$

$$-\frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\phi^2}\left(-\frac{\vartheta R_0}{6}-R_1\vartheta^2\right)+\frac{2\mu}{\hbar^2}\left(\frac{R_0}{\vartheta}+2R_1-2R_1\vartheta\right)+\frac{16\mu R_1\vartheta^3}{3\hbar^2\phi^3}}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{2}+l\right)^2+\frac{2\mu}{\hbar^2\phi^3}\left(-\frac{\vartheta R_0}{6}-R_1\vartheta^2\right)+\frac{2\mu R_1\vartheta^3}{\hbar^2\phi^4}-\frac{2\mu R_1}{\hbar^2}}}\right]^2$$

The numerical values of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), are $m_b = 4.823\ GeV$ and $m_c = 1.209\ GeV$, and the corresponding reduced mass are $\mu_b = 2.4115\ GeV$ and $\mu_c = 0.6045\ GeV$, respectively [68]. The potential parameters were calculated by fitting with experimental data (ED) [69].

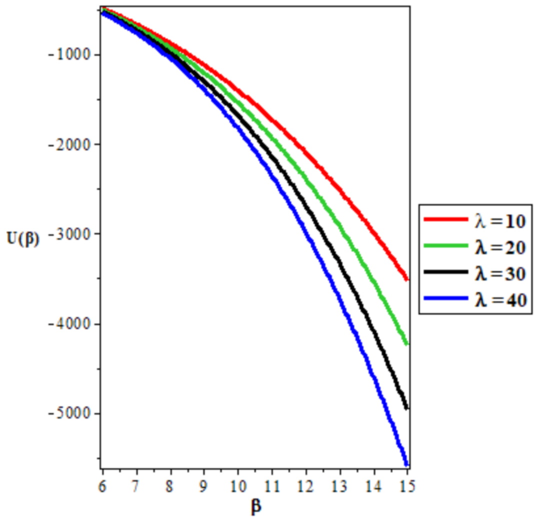

Figure 6: Mean energy function against β for different values of λ

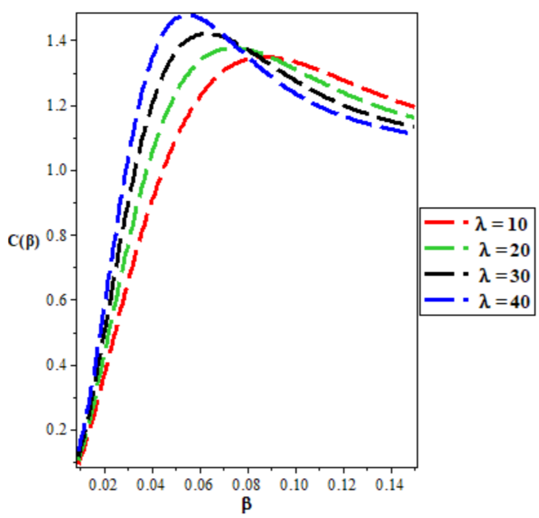

Figure 7: Specific heat against β for different values of λ

We observed that the results obtained from the prediction of MS of $(c\bar{c})$ and $(b\bar{b})$ mesons for different quantum states are in agreement with ED and are seen to be improved when compared with other theoretical predictions with different analytical methods from literature as shown in Tables 1 and 2.

The absolute deviation in the predicted results and ED is calculated using the following relation

$$\sigma = \frac{100}{N}\sum\left|\frac{T_P - T_{EXP.}}{T_{EXP.}}\right|, \tag{51}$$

where $T_{EXP.}$ is the ED, T_P is the predicted results and N is the number of ED and, also the percentage error (PE) is computed with the given relation [70]

$$error = \frac{\sum ARD}{\sum T_{EXP.}} \times 100\%. \tag{52}$$

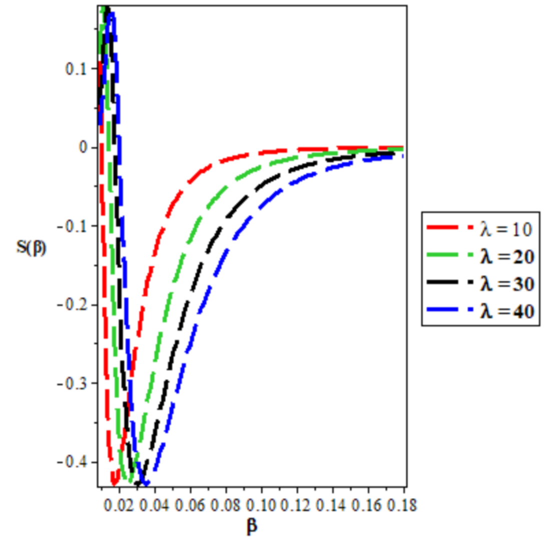

Figure 8: Entropy against β for different values of λ

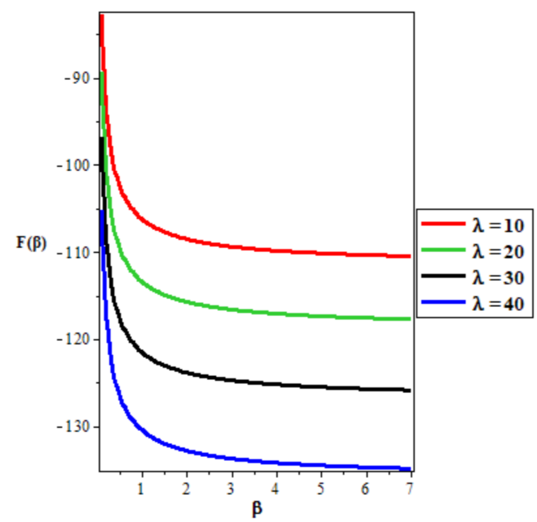

Figure 9: Free energy against β for different values of λ

The absolute deviation was calculated using equation (51) and the values for $(c\bar{c})$ and $(b\bar{b})$ are 1.601 GeV and 0.457 GeV, respectively. Also, the PE of the predicted results to the ED, was calculated using equation (52) and the results show that for $(c\bar{c})$ we have 1.60 % while that of the $(b\bar{b})$ is 0.46 %.

In Figures 1−4 we plotted the radial wave functions and probability densities for $(c\bar{c})$ and $(b\bar{b})$ mesons. Generally, the probability densities decay with the radial distance but the decay happens faster at a short distance for the low-lying quantum states compare to the high-lying states. Also, the peaks of the $(c\bar{c})$ mesons are higher than the corresponding $(b\bar{b})$ mesons and they tend towards shorter distances. The observed high peaks may be ascribed to their heavy masses.

The variation of the PF with respect to temperature (β) for different values of maximum quantum number(λ) is as shown

Table 1: Mass spectra of charmonium (in GeV) ($m_c = 1.209\ GeV, \mu = 0.6045\ GeV, \theta = 0.06, R_1 = -2.31507\ GeV, R_0 = -0.15571\ GeV, \phi = 0.67$)

State	Our work	[1]	[64]	[44]	Experiment [69]	Absolute Relative deviation (ARD)
1S	3.096	3.096	3.096	3.095922	3.096	0.000
2S	3.686	3.686	3.686	3.685893	3.686	0.000
1P	3.327	3.214	3.433	-	3.525	0.198
2P	3.774	3.773	3.910	3.756506	3.773	0.001
3S	4.040	4.275	3.984	4.322881	4.04	0.000
4S	4.364	4.865	4.150	4.989406	4.263	0.101
1D	3.761	3.412	3.767	-	3.770	0.009
2D	4.058	-	-	-	4.159	0.101

Table 2: Mass spectra of bottomonium (in GeV) ($m_b = 4.823\ GeV, \mu = 2.4115\ GeV, R_1 = -0.2542\ GeV, R_0 = -0.37717\ GeV, \phi = 0.67, \theta = 0.06, \hbar = 1$)

State	Our work	[1]	[64]	[44]	Experiment [69]	Absolute Relative deviation (ARD)
1S	9.460	9.46	9.460	9.515194	9.460	0.000
2S	10.023	10.023	10.023	10.01801	10.023	0.000
1P	9.841	9.492	9.840	-	9.899	0.058
2P	10.160	10.038	10.16	10.09446	10.260	0.100
3S	10.345	10.585	10.28	10.44142	10.355	0.010
4S	10.522	11.148	10.42	10.85777	10.580	0.058
1D	10.142	9.551	10.14	-	10.164	0.022

in Figure 5. In Figure 5, the PF decreases as λ is increased. The mean energy of the mesons is depicted in Figure 6, we found that as β increases, the mean energy decreases monotonically. The variation of heat capacity of the mesons is presented in Figure 7. We observed an increase in the heat capacity asβincreases for different valves of λ, and later begin to decrease and converges at a point. The plots of the behavior of the entropy as a function ofβ is presented in Figure 8. The entropy increases to a peak value before decreasing and converges. The free energy of the mesons is plotted against β in Figure 9. It was noticed that the free energy decreases at different values of λ.

5. Conclusion

In this study, the HGIQYP is model to interact in quark-antiquark system and the solutions were obtained using the NU method. The energy spectrum and normalized wave function were obtained. The energy spectrum was used to predict the MS of the HMs. Also, the PF was calculated from the energy spectrum, thereafter other TPs were obtained. The results obtained showed an improvement when compared with the work of other researchers and excellently agreed with ED with a PE of 1.60 % and 0.46 % for ($c\bar{c}$) and ($b\bar{b}$), respectively. The PE shows that the HGIQYP is fitted in the prediction of the MS of the HMs since the PE is less.

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