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Review

Strong Convergence Theorems for Split Common Fixed Point Problem of Bregman Generalized Asymptotically Nonexpansive Mappings in Banach Spaces

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Abstract

In this paper, a new iterative scheme is introduced and also strong convergence theorems for solving split common fixed point problem for uniformly continuous Bregman generalized asymptotically nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces are presented. The results are proved without the assumption of semicompactness property and or Opial condition.

Keywords: Split common fixed point problem, Bregman distance, Generalized asymptotically nonexpansive mapping, Strong convergence, uniformly convex and uniformly smooth Banach space, Algorithm

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1. Introduction

Throughout this paper, E_1 and E_2 are uniformly convex and uniformly smooth Banach spaces. Let us make some conventions: we always use $p, q \in (1, \infty)$ as conjugate exponents so that $\frac{1}{p} + \frac{1}{q} = 1$, where by $q = \frac{p}{p-1}$, $pq = p+q$, $(p-1)(q-1) = 1$. Furthermore, for real value a, b, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ which is to be understood componentwise in case of sequences and pointwise in case of functions.

Throughout this paper, let $p, r \geq 1$, given sequences of nonempty closed convex subsets $\{C_i\}_{i=1}^{p}$ and $\{Q_i\}_{i=1}^{r}$ of E_1 and E_2, respectively,

Definition 1.1. *[1] Multiple set split feasibility problem (MSSFP) is formulated as finding a point $x^* \in E_1$ with the property*

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{r} Q_j. \tag{1}$$

Definition 1.2. *[2] If in a MSSEP (1) $p = r = 1$, we get what is called the split feasibility problem (SFP), which is to find a point $x^* \in E$, with the property*

$$x^* \in C \text{ and } Ax^* \in Q. \tag{2}$$

Definition 1.3. *[3] Split common fixed point problem (SCFPP) is formulated as finding a point $x^* \in E_1$, with the property*

$$x^* \in \bigcap_{i=1}^{p} Fix(U_i) \text{ and } Ax^* \in \bigcap_{j=1}^{r} Fix(T_j),$$

where each $U_i : E_1 \longrightarrow E_1$ $(i = 1, 2, \cdots p)$ and $T_j : E_2 \longrightarrow E_2$

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$(j = 1, 2, \cdots r)$ *are some (nonlinear) mappings.*

Definition 1.4. *[3] two-set SCFPP is formulated as finding a point $x^* \in E_1$, with the property*

$$x^* \in Fix(U) \text{ and } Ax^* \in Fix(T) \tag{3}$$

where $U : E_1 \longrightarrow E_1$ and $T : E_2 \longrightarrow E_2$ are (nonlinear) mappings.

As stated in [4], let E be a Banach space with norm $\|\cdot\|$. Let C be a nonempty closed convex subset of E and E^* denote the dual space of E. Let $B : C \longrightarrow E^*$ be a nonlinear mapping and $F : C \times C \longrightarrow \Re$ be a bifunction. The generalized equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \langle Bx, y - x\rangle \geq 0 \forall y \in C. \tag{4}$$

Now, let

$$F(x, y) = f(x, y) + g(x, y) \tag{5}$$

where $f, g : C \times C \longrightarrow \Re$ are two bifunctions satisfying the following special properties $(A_1) - (A_4), (B_1) - (B_3) and (C)$;

$$\begin{cases} (A_1) - f(x, y) = 0, \forall x \in C; \\ (A_2) - f \text{is maximal monotone;} \\ (A_3) - \forall x, y, z \in C \text{we have} \limsup_{n\to 0^+}(f(tz + (1 - t)x, y) \\ \qquad \leq f(x, y)); \\ (A_4) - \forall x \in C, \text{the function} y \mapsto f(x, y) \text{is convex and} \\ \text{weakly lower semi-continuous;} \\ (B_1) - g(x, x) = 0 \forall x \in C; \\ (B_2) - g \text{is maximal monotone and monotone,} \\ \qquad \text{and weakly upper semi-continuous} \\ \qquad \text{in the first variable;} \\ (B_3) - g \text{is convex in the second variable;} \\ (C) - \text{for fixed} \lambda > 0 and x \in C, \text{there exists a bounded set} \\ K \subset C \text{and} a \in K \text{such that} f(a, z) + g(z, a) + \frac{1}{\lambda}(a - z, z - x) \\ \qquad < 0 \forall x \in C \backslash K. \end{cases} \tag{6}$$

The well known generalized mixed equilibrium problem is to find an $x \in C$ such that

$$f(x, y) + g(x, y) + \langle Bx, y - x\rangle \geq 0 \forall y \in C. \tag{7}$$

If $B \equiv 0$, the problem (7) reduces into mixed equilibrium problem for f and g, denoted by $MEP(f, g)$, which is to find $x \in C$ such that

$$f(x, y) + g(x, y) \geq 0, \forall y \in C. \tag{8}$$

If $g \equiv 0$ and $B \equiv 0$, (7) reduces into equilibrium problem for f, denoted by EP(f), which is to find $x \in C$ such that

$$f(x, y) \geq 0 \ \forall \ y \in C. \tag{9}$$

In 1994, the SFP (2) was introduced by Censor and Elfving [1]. Using the CQ-algorithm for solving the SFP (2), in 2002 Byrne [2] proposed that which generates the new iterate as follows

$$\text{choosing arbitrarily } x_1 \in H_1,$$
$$x_{n+1} = P_C[x_n - \gamma A^T(I - P_Q)Ax_n]$$

where $\gamma \in (0, \frac{2}{L})$, L denotes the largest eigenvalue of the matrix A^TA.

In 2004, Yang [5] presented a relaxed CQ-algorithm for solving the SFP, where at n-th iteration, the projections onto C and Q were replaced with the halfspaces C_n and Q_n, respectively.

In 2005, Qu and Xiu [6] proposed a modified relaxed CQ-algorithm

$$\text{choosing arbitrarily } x_1 \in H_1,$$
$$x_{n+1} = P_{C_n}[x_n - \alpha_n A^T(I - P_{Q_n})Ax_n].$$

In 2007, Frank Schopfer[7] developed iterative methods for the solution of the SFP (2) in Banach spaces and also analyse stability and regularizing properties. these iterative methods are as follows.

$$\text{choosing arbitrarily } x_1 \in E_1,$$
$$x_{n+1} = J^*_{E_1}(J_{E_1}(x_n) - \mu_n A^* J_{E_2}(Ax_n - P_Q(Ax_n)).$$

The concept of SCFPP in finite dimensional Hilbert spaces, say H_1 and H_2, was first introduced by Censor and Segal in 2009[1], who invented an algorithm of the two-set SCFPP which generate a sequence $\{x_n\}$ according to the following iterative procedure:

$$\text{choosing arbitrarily } x_1 \in H_1,$$
$$x_{n+1} = U(x_n + \gamma A^*(I - T)Ax_n), n \geq 1$$

where the initial guess $x_0 \in H_1$ is chosen arbitrarily and $0 < \gamma < \frac{1}{\|A\|^2}$ and $A : H_1 \longrightarrow H_2$ as a bounded linear operator having A^* as the adjoint operator of A. And $U : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ are (nonlinear) mappings.

In 2010, Moudafi [8] proposed the following iteration method to approximate a SCFPP of demicontractive mappings in Hilbert spaces;

$$\begin{cases} x_1 \in H_1 \text{ is arbitrarily chosen,} \\ u_n = x_n + \gamma A^*(I - T)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Uu_n, \end{cases} \tag{10}$$

where he proved that $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$, where $U : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ are two demicontractive mappings, $A : H_1 \longrightarrow H_2$ is a bounded linear operator and H_1 and H_2 are two Hilbert spaces.

Using the iterative scheme (10), in 2011, Moudafi[8] also obtained a weak convergence theorem for the split common

fixed poin problem of quasi-nonexpansive mapping in Hilbert spaces.

In 2012, Chang *et al.* [9] again, using (10) proved the weakly convergence of the sequence $\{x_n\}$ to the split common fixed point $x^* \in \Gamma$ of asymptotically quasi-nonexpansive mapping in Hilbert spaces. But these authors could only obtain strong convergence theorem if those mappings and spaces are semi-compacts and Hilbert spaces, respectively.

In 2015, Zhang *et al* [10] introduced the iterative scheme which guarantees the strong converges for SCFP of the asymptotically nonexpansive mapping in Hilbert spaces, without assumption of semi-compactness. The sequence is defined as follows;

$$\begin{cases} \text{choosing arbitrarily } x_1 \in H_1, C_1 = H_1, \\ z_n = x_n + \lambda A^*(T_2^n - I)Ax_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)T_1^n(z_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le k_n\|z_n - v\|, \|z_n - v\| \le k_n\|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), n \ge 1 \end{cases}$$

where A^* denote the adjoint of A, $\lambda \in (0, \frac{1}{\|A^*\|^2})$ and $\{\alpha_n\} \subset (0,\tau) \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $k_n = max\{k_n^{(1)}, k_n^{(2)}\}$, $n \ge 1$.

Takahashi [11], in 2015, also obtained a similar result for split common null point problem by using the following hybrid and shrinking projection methods, respectively.

$$\begin{cases} \text{choosing arbitrarily } x_1 \in H_1, \\ z_n = P_C(x_n - rA^*J_F(Ax_n - P_QAx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in H_2 : \langle x_n - z, x_1 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \forall n \in N, \end{cases}$$

and

$$\begin{cases} \text{choosing arbitrarily } x_1 \in H_1, \\ z_n = P_C(x_n - rA^*J_F(Ax_n - P_QAx_n)), \\ C_n = \{z \in H_1 : \|z_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \forall n \in N, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ for some $a \in R$ and $0 < r\|A\|^2 < 2$.

In 2015, Tang *et al.*[12] introduced the split common fixed point problem (SCFP) for an asymptotical nonexpansive mapping S and a $\tau-$quasi-pseudocontractive mapping T in the setting of two Banach spaces, E_1 and E_2, by using the sequence $\{x_n\}$ defined as follows;

$$\begin{cases} \text{choosing arbitrarily } x_1 \in E_1, \\ z_n = x_n + \gamma J_1^{-1}A^*J_2(T-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n S^n(z_n), n \ge 1 \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\gamma \subset (0, min\{\frac{1-2k^2}{\|A\|^2}, \frac{1-\tau}{\|A\|^2}\})$, $L = \sup_{n\ge1}\{l_n\}$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$.

In this paper, the modified algorithm of Zhang *et al.* for the solution of SCFP in Hilbert spaces is studied. Hence, the Bregman generalized asymptotically nonexpansive mapping is used to obtain the strong convergence for SCFPP (6) in uniformly convex and uniformly smooth Banach spaces, without the assumption of semi-compactness property and or without the assumption of Opial condition.

2. Preliminaries

Throughout this paper, the classes of Banach spaces we will deal with are [13];

Definition 2.1. *A Banach space E is said to be uniformly convex, if* $\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\|; \|x\| = \|y\| = 1, \|x-y\| \ge \epsilon$, *where* $0 \le \epsilon \le 2$ *and* $0 < \delta_E(\epsilon) \le 1\}$.

Definition 2.2. *A Banach space E is said to be uniformly smooth, if* $\lim_{r\to0}(\frac{\rho_E(r)}{r}) = 0$ *where* $\rho_E(r) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1, \|y\| \le r; 0 \le r < \infty$ *and* $0 \le \rho_E(r) < \infty\}$. *Moreover,*

1. ρ_E *is continuous, convex and nondecreasing with* $\rho_E(0) = 0$ *and* $\rho_E(r) \le r$
2. *The function* $r \mapsto \frac{\rho_E(r)}{r}$ *is nondecreasing and fulfills* $\frac{\rho_E(r)}{r} > 0$ *for all* $r > 0$
3. $\lim_{r\to0}\frac{\rho_E(r)}{r} = 0$.

Throughout this paper, some important mappings we will be using are;

Definition 2.3. *[14] For each* $p > 1$, *let* $g(t) = t^{p-1}$ *be a gauge function* $g : \mathfrak{R}^+ \longrightarrow \mathfrak{R}^+$ *such that* $g(0) = 0$ *and* $\lim_{n\to\infty} g(t) = \infty$. *we defined the ganeralized daulity map by* $J_p : E \longrightarrow 2^{E^*}$ *by*

$$J_{g(t)} = J_p(x) = \{x^* \in E^*; \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = g(\|x\|) = \|x\|^{p-1}\}.$$

The ganeralized daulity mapping has the following basic properties [7].

Lemma 2.1. *For every* $x \in E$ *the set* $J_E^p(x)$ *is not empty and convex.*

Lemma 2.2. $J_E^p(x)$ *is homogeneous of degree* $p-1$, *i.e.*

$$J_E^p(\lambda x) = |\lambda|^{p-1} sgn(\lambda) J_E^p(x) \ \forall x \in E, \lambda \in \mathfrak{R}.$$

Lemma 2.3. *If* $J_{E^*}^p(x)$ *is the duality mapping of* E^* *with gauge function* $t \mapsto t^{q-1}$ *then* $x^* \in J_E^p(x)$ *iff* $x \in J_{E^*}^q(x^*)$.

Lemma 2.4. *[7] In smooth Banach space, the Bregman distance of x to y with respect to the function* $f(x) = \frac{1}{p}\|x\|^p$ *is defined by*

$$\triangle_p(x,y) = \frac{1}{q}\|x\|^p - \langle J^p(x), y\rangle + \frac{1}{p}\|y\|^p.$$

Definition 2.4. *Let E be a smooth Banach space. Let* $\triangle_p$ *be a Bregman distance. A mapping* $T : E \longrightarrow E$ *is said to be a Bregman generalized asymptotically nonexpansive with* $\{k_n\}$ *and* $\{\mu_n\}$ *if there exists nonnegative real sequences* $\{k_n\}$ *and* $\{\mu_n\}$ *with* $\lim_{n\to\infty} k_n = 0$ *and* $\lim_{n\to\infty}\mu_n = 0$ *such that* $\triangle_p(T^n(x), T^n(y)) \le k_n \triangle_p(x,y) + \triangle_p(x,y) + \mu_n \ \forall (x,y) \in E \times E$.

Definition 2.5. *[15] Let E be a reflexive, strictly convex and smooth Banach space. Then for every closed convex subset $C \subset E$ and $x \in E$ there exists a unique element $P_C(x) \in C$ such that $\|x - P_C(x)\| = \min_{y\in C} \|x - y\|$ is called the metric projection of x onto C. Moreover, if J^p is a duality mapping of E, then $x_0 \in C$ is the metric projection of x onto C iff*

$$\langle J^p(x_0 - x), y - x_0\rangle \geq 0 \ \forall y \in C.$$

Definition 2.6. *[7] Let E be a reflexive, strictly convex and smooth Banach space and J^p be a duality mapping of E. Then for every closed convex subset $C \subset E$ and $x \in E$ there exists a unique element $\Pi_C^p(x) \in C$ such that $\triangle_p(x, \Pi_C^p(x)) = \min_{y\in C} \triangle_p(x, y)$. $\Pi_C^p(x)$ is called the Bregman projection of x onto C, with respect to the function $f(x) = \frac{1}{p}\|x\|^p$. Moreover $x_0 \in C$ is the Bregman projection of x onto C iff*

$$\langle J^p(x_0) - J^p(x), y - x_0\rangle \geq 0$$

or equivalently

$$\triangle_p(x_0, y) \leq \triangle_p(x, y) - \triangle_p(x, x_0) \ for \ every \ y \in C.$$

Lemma 2.5. *[7] The Bregman projection and the metric projection are related via $P_C(x) - x = \Pi_{C-x}^p(0)$, $\forall x \in E$. Especially we have $P_C(0) = \Pi_C^p(0)$ and thus $\|\Pi_C^p(0)\| = \min_{y\in C} \|y\|$.*

The Bregman distance has the following properties [7].

Lemma 2.6. *$\forall x, y \in E$ and $\{x_n\} \in E$, $\triangle_p(x, y) \geq 0$ and $\triangle_p(x, y) = 0 \Leftrightarrow x = y$.*

Lemma 2.7. *$\forall x, y \in E$ and $\lambda \geq 0$ $\triangle_p(-x, -y) = \triangle_p(x, y)$ and $\triangle_p$ is positively homogeneous of degree p, i.e. $\triangle_p(\lambda x, \lambda y) = \lambda^p \triangle_p (x, y)$.*

Lemma 2.8. *$\triangle_p$ is continuous in both arguments and it is strictly convex, weakly lower semicontinuous and Gateaux differentiable with respect to the second variable with derivative $\frac{\partial}{\partial y} \triangle_p(x, y) = J^p(y) - J^p(x)$.*

Lemma 2.9. *$\triangle_p$ satisfies the Three-point property that generalizes the "Law of cosines":*

$$\triangle_p(x, y) = \triangle_p(x, z) + \triangle_p(z, y) - \langle x - z, J^p y - J^p z\rangle$$

Lemma 2.10. *Let X be reflexive, smooth and strictly convex. Then for all $x, y \in X$ the following holds:*

$$\triangle_p(x, y) + \triangle_p(y, x) = \langle J^p(x) - J^p(y), x - y\rangle$$

Lemma 2.11. *Consider the following assertions;*

1. $\lim_{n\to\infty} \|x_n - x\| = 0$
2. $\lim_{n\to\infty} \|x_n\| = \|x\|$ *and* $\lim_{n\to\infty}\langle J^p(x_n), x\rangle = \langle J^p(x), x\rangle$
3. $\lim_{n\to\infty} \triangle_p(x_n, x) = 0$.

The implication (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are valid. If E is uniformly convex then the assertions are equivalent.

Lemma 2.12. *Let us write $\triangle_q^*(x^*, y^*) = \frac{1}{p}\|x^*\|_{E^*}^q - \langle J_{E^*}^q, y^*\rangle + \frac{1}{q}\|y^*\|_{E^*}^q$ for the Bregman distance on the dual space E^* with respect to the function $f^*(x^*) = \frac{1}{q}\|x^*\|_{E^*}^q$. Then we have*

$$\triangle_p(x, y) = \triangle_q^*(x^*, y^*). for \ x^* = J_E^p(x)$$

$$\Leftrightarrow J_{E^*}^q(x^*) = x \ and \ y^* = J_E^p(y) = y$$

.

$$\Leftrightarrow J_{E^*}^q(y^*) = y.$$

Lemma 2.13. *$\prod_C^p(x)$ maps bounded sets onto bounded sets; more precisely we have*

$$\|\Pi_C^p(x)\| \leq (2^{q-1}\|x\|) \vee (3\|\Pi_C^p(0)\|) \forall x \in E.$$

Throughout this paper, some important characteristic inequalities we will be using are;

Lemma 2.14. *[16] In the case of uniformly convex space E,with the duality map J^p of E, $\forall x, y \in E$ we have*

$$\begin{aligned}\|x - y\|^p &= \langle J^p(x - y), x - y\rangle \\ &\geq \|x\|^p - p\langle J^p(x), y\rangle + \sigma_p(x, y).\end{aligned}$$

with

$$\sigma_p(x, y) = pK_p \int_0^1 \frac{(\|x - ty\| \vee \|x\|)^p}{t} \lambda_E\left(\frac{t\|y\|}{2(\|x - ty\| \vee \|x\|)}\right) dt$$

where by

$$\begin{aligned}K_p \quad = \quad & 4(2 + \sqrt{4}) \min\{\frac{1}{2}p(p - 1) \wedge 1, (\frac{1}{2}p \wedge 1)(p - 1), \\ & (p - 1)(1 - (\sqrt{3} - 1)^q), 1 - (1 + (2 - \sqrt{3})q)^{1-p}\}.\end{aligned}$$

Lemma 2.15. *[16] In the case of uniformly smooth space, E,with the duality map J^p of E, $\forall x, y \in E$ we have*

$$\begin{aligned}\|x - y\|^p &= \langle J^p(x - y), x - y\rangle \\ &\leq \|x\|^p - p\langle J^p(x), y\rangle + \bar{\sigma}_p(x, y).\end{aligned}$$

with

$$\bar{\sigma}_p(x, y) = pG_p \int_0^1 \frac{(\|x - ty\| \vee \|x\|)^p}{t} \rho_E\left(\frac{t\|y\|}{2(\|x - ty\| \vee \|x\|)}\right) dt$$

where by

$$G_p = 8 \vee 64cK_p^{-1}$$

with

$$\begin{aligned}K_p \quad = \quad & 4(2 + \sqrt{4}) \min\{\frac{1}{2}p(p - 1) \wedge 1, (\frac{1}{2}p \wedge 1)(p - 1), \\ & (p - 1)(1 - (\sqrt{3} - 1)^q), 1 - (1 + (2 - \sqrt{3})q)^{1-p}\}\end{aligned}$$

and

$$c = 4\frac{\lambda_0}{\sqrt{1 + \lambda_0^2} - 1} \prod_{j=1}^{\infty}\left(1 + \frac{15}{2^{j+2}}\lambda_0\right)$$

with

$$\lambda_0 = \frac{\sqrt{339} - 18}{30}.$$

Lemma 2.16. *[17] Let E be a real uniformly convex Banach space. For arbitrary $r > 1$, let $B_r(0) = \{x \in E : \|x\| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$$

such that for every $x, y \in B_r(0), f_x \in J_p(x), f_y \in J_p(y)$, the following inequality hold:

$$\|\lambda x+(1-\lambda)y\|^p \leq \lambda\|x\|^p+(1-\lambda)\|y\|^p-(\lambda^p(1-\lambda)+(1-\lambda)^p\lambda)g(\|x-y\|)$$

and,

$$\langle x - y, f_x - f_y\rangle \geq g(\|x - y\|).$$

Lemma 2.17. *[17] Let E be a real uniformly smooth Banach space. For arbitrary $r > 1$, let $B_r(0) = \{x \in E : \|x\| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$$

such that for every $x, y \in B_r(0), f_x \in J_q(x), f_y \in J_q(y)$, the following inequality hold:

$$\|\lambda x+(1-\lambda)y\|^q \geq \lambda\|x\|^q+(1-\lambda)\|y\|^q-(\lambda^q(1-\lambda)+(1-\lambda)^q\lambda)g(\|x-y\|)$$

and,

$$\langle x - y, f_x - f_y\rangle \leq g(\|x - y\|).$$

3. Main results

Theorem 3.1. *Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces, $A : E_1 \longrightarrow E_2$ is bounded and linear operator such that $A(C)$, for $C \subset E_1$, is closed and convex, $U : E_1 \longrightarrow E_1$ be a uniformly continuous Bregman generalized asymptotically nonexpansive operator with the sequences $\{k_n^{(1)}\} \subset [0, \infty)$ and $\{\mu_n^{(1)}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} k_n^{(1)} = 0$ and $\lim_{n\to\infty}\mu_n^{(1)} = 0$, respectively, and $T : E_2 \longrightarrow E_2$ be a uniformly continuous Bregman generalized asymptotically nonexpansive operator with the sequences $\{k_n^{(2)}\} \subset [0, \infty)$ and $\{\mu_n^{(2)}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} k_n^{(2)} = 0$ and $\lim_{n\to\infty}\mu_n^{(2)} = 0$, respectively, and, for $p, q \in (1, \infty)$, $\Pi^p_{AC} : E_2 \longrightarrow AC$ be a Bregman projection onto a subset AC, and $Fix(U) \neq \phi$ and $Fix(\Pi^p_{AC}T) \neq \phi$ respectively, and $(I - U)$ and $(I - \Pi^p_{AC}T)$ be demiclosed at zero. Let $x_1 \in E_1$ be chosen arbitrarily and let $C_1 = E_1$ and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} z_n = J^q_{E_1^*}(J^p_{E_1}x_n - \lambda_n A^* J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n), \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1 - \alpha_n)J^p_{E_1}(U^n(z_n))), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \leq [k_n + 1]\triangle_p(z_n, v) + \mu_n; \\ \triangle_p(z_n, v) \leq \triangle_p(x_n, v)\}, \\ x_{n+1} = \Pi^p_{C_{n+1}}(x_1), n \geq 1 \end{cases} \tag{11}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|}\frac{1}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}, \ x_n \neq 0 \\ \frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n-\Pi^p_{AC_n}T^nAx_n\rangle^{p-1}}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^p}, x_n = 0 \end{cases} \tag{12}$$

and $\gamma \in (0, 1)$ and $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q\|A\|} \times \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle}{\|x_n\|^p\|J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\|}, \tag{13}$$

and A^* denote the adjoint of A, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\liminf_{n\to\infty}\alpha_n(1 - \alpha_n) > 0$, $k_n = (k_n^{(1)} \vee k_n^{(2)})$, $n \geq 1$. If $\Gamma = \{v \in Fix(U) : Av \in Fix(\Pi^p_{AC}T)\} \neq \phi$, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Proof:

Assume that $\|\lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n)\| = 0 \Leftrightarrow x_n \in \Gamma = \{x_n \in Fix(U) : Ax_n \in Fix(\Pi^p_{AC}T)\} \neq \phi$ then from 11 we have that $z_n = x_n$ and $\triangle_p(x_n, z_n) = 0$ and the conclusion follows immediately.

Now assume that $\|\lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n)\| \neq 0$ then we will divide the proof into five steps.

Step one: We first show that C_n is closed and convex, for any $n \geq 1$.

By induction hypothesis;

since $C_1 = E_1$, so C_1 is closed and convex.

Assuming that C_k is closed and convex.

We show that C_{k+1} is closed and convex, for some $k \geq 0$.

We first show that C_{k+1} is closed.

Let $\{x_m\}$ be a sequence in C_{k+1} such that $x_m \to v \ as \ n \to \infty$. We need to show that $v \in C_{k+1}$. From 11 and continuity of Bregman distance and the boundedness of $\{x_m\}$, we have that

$$\begin{aligned} \triangle_p(z_n, v) &= \lim_{m\to\infty} \triangle_p(z_n, x_m) \\ &\leq \lim_{m\to\infty} \triangle_p(x_n, x_m) = \triangle_p(x_n, v). \end{aligned} \tag{14}$$

Similarly, we have that

$$\begin{aligned} \triangle_p(y_n, v) &= \lim_{m\to\infty} \triangle_p(y_n, x_m) \\ &\leq \lim_{m\to\infty}([k_n + 1]\triangle_p(z_n, x_m) + \mu_n) = [k_n + 1](\triangle_p(x_n, v) + \mu_n. \end{aligned} \tag{15}$$

From (14) and (15), we have that $v \in C_{k+1}$.

Next we show that C_{k+1} is convex.

Let $v_1, v_2 \in C_{k+1}$ and $t \in (0, 1)$; putting $v = tv_1 + (1 - t)v_2$, it suffices to show that $v \in C_{k+1}$. Let $x_k \neq 0$, then

$$\begin{aligned} \triangle_p(z_k, v) &= \triangle_p(z_k, tv_1 + (1 - t)v_2) \\ &= \frac{1}{q}\|J^p_{E_1}x_k - \lambda_k A^*(J^p_{E_2}(I - \Pi^p_{AC_k}T^k)Ax_k)\|^q + \frac{1}{p}\|tv_1 + (1 - t)v_2\|^p \\ &\quad - \langle J^p_{E_1}x_k - \lambda_k A^*(J^p_{E_2}(I - \Pi^p_{AC_k}T^k)Ax_k), tv_1 + (1 - t)v_2\rangle \end{aligned} \tag{16}$$

$$= \frac{1}{q}\|J^p_{E_1}x_k - \lambda_k A^*(J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q + \frac{1}{p}\|tv_1+(1-t)v_2\|^p - \langle J^p_{E_1}x_k, tv_1+(1-t)v_2\rangle + \langle \lambda_k(J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tv_1 + (1-t)v_2\rangle.$$

Where

$$\begin{aligned}
&\lambda_k\langle A^*(J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tv_1+(1-t)v_2\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2+\Pi^p_{AC_k}T^kAx_k-\Pi^p_{AC_k}T^kAx_k\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2-\Pi^p_{AC_k}T^kAx_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, \Pi^p_{AC_k}T^kAx_k\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2-\Pi^p_{AC_k}T^kAx_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, \Pi^p_{AC_k}T^kAx_k-Ax_k+Ax_k\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2-\Pi^p_{AC_k}T^kAx_k\rangle\\
&- \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, \Pi^p_{AC_k}T^kAx_k-Ax_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2-Ax_k+Ax_k-\Pi^p_{AC_k}T^kAx_k\rangle\\
&- \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k-\Pi^p_{AC_k}T^kAx_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle\\
&= \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, (tAv_1+(1-t)Av_2-Ax_k)\\
&- (\Pi^p_{AC_k}T^kAx_k-Ax_k)\rangle - \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k-\Pi^p_{AC_k}T^kAx_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle\\
&= -\lambda_k\langle (J^p_{E_2}(\Pi^p_{AC_k}T^k-I)Ax_k, (tAv_1+(1-t)Av_2-Ax_k)\\
&- (\Pi^p_{AC_k}T^kAx_k-Ax_k)\rangle - \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k-\Pi^p_{AC_k}T^kAx_k\rangle\\
&+ \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle.
\end{aligned}$$

By the assumption that $A(C_k)$ is closed and convex and Lemma 2.5 and by the variational inequality for the Bregman projection of zero onto $A(C_k) - Ax_k$, as in Definition 2.6, we arrive at

$$\langle (J^p_{E_2}(\Pi^p_{AC_k}T^k - I)Ax_k, (tAv_1+(1-t)Av_2-Ax_k) - (\Pi^p_{AC_k}T^kAx_k - Ax_k)\rangle \geq 0$$

and therefore,

$$\begin{aligned}
&\lambda_k\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, tAv_1+(1-t)Av_2\rangle\\
&\leq \lambda_k\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle\\
&- \lambda_k\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k-\Pi^p_{AC_k}T^kAx_k\rangle. \qquad (17)
\end{aligned}$$

In addition, from Lemma 2.15, we have that

$$\begin{aligned}
&\frac{1}{q}\|J^p_{E_1}x_k - \lambda_k A^*(J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q\\
&\leq \frac{1}{q}\|J^p_{E_1}x_k\|^q - \lambda_k\langle x_k, A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\rangle\\
&+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_k, \lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\\
&= \frac{1}{q}\|J^p_{E_1}x_k\|^q - \lambda_k\langle Ax_k, J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\rangle\\
&+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_k, \lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\\
&= \frac{1}{q}\|x_k\|^p - \lambda_k\langle Ax_k, J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\rangle\\
&+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_k, \lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k) \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_k, \lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\\
&= G_q\int_0^1 \frac{(\|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\| \vee \|J^p_{E_1}x_k\|)^q}{t}\times\\
&\rho_{E^*}\left(\frac{t\|\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|}{(\|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\| \vee \|J^p_{E_1}x_k\|)}\right)dt, \qquad (19)
\end{aligned}$$

$$for\ every\ t \in [0,1].$$

But

$$\begin{aligned}
&\|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|\\
&\leq \|x_k\|^{p-1} + \|\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|.
\end{aligned}$$

From (12), suppose that

$$\lambda_k = \frac{\tau_k}{\|A\|}\frac{\|x_k\|^{p-1}}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}$$

then we have that

$$\|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\| \leq 2\|x_k\|^{p-1}$$

and that

$$\|x_k\|^{p-1} \leq \|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\| \vee \|J^p_{E_1}x_k\| \leq 2\|x_k\|^{p-1}. \qquad (20)$$

From Definition 2.2 (2), (12) and (20), we have that

$$\begin{aligned}
&\rho_{E_1^*}\left(\frac{t\|\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|}{(\|J^p_{E_1}x_k - t\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\| \vee \|J^p_{E_1}x_k\|)}\right)\\
&\leq \rho_{E_1^*}\left(\frac{t\|\lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|}{\|x_k\|^{p-1}}\right)\\
&= \rho_{E_1^*}(t\tau_k). \qquad (21)
\end{aligned}$$

Substituting Eqs. (20) and (21) into (19), and since $t \leq \tau_k$ and $\rho_{E_1^*}$ is nondecreasing function then, we have that

$$\begin{aligned}
&\frac{1}{q}\bar{\sigma}_q(J^p_{E_2}x_k, \lambda_k A^*J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\\
&\leq 2^qG_q\|x_k\|^{(p-1)q}\int_0^{\tau_k}\frac{\rho_{E_1^*}(t)}{t}dt\\
&\leq 2^qG_q\|x_k\|^p\int_0^{\tau_k}\frac{\rho_{E_1^*}(\tau_k)}{\tau_k}dt\\
&= 2^qG_q\|x_k\|^p\rho_{E_1^*}(\tau_k). \qquad (22)
\end{aligned}$$

Substituting Eqs. (22) into (18), we have that

$$\begin{aligned}
&\frac{1}{q}\|J^p_{E_1}x_k - \lambda_k A^* J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q \\
&\le \frac{1}{q}\|x_k\|^p - \lambda_k\langle Ax_k, J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\rangle + 2^q G_q\|x_k\|^p\rho_{E_1^*}(\tau_k) \\
&= \frac{1}{q}\|J^p_{E_1}x_k\|^q - \lambda_k\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle \\
&+ 2^q G_q\|x_k\|^p\rho_{E_1^*}(\tau_k). \qquad (23)
\end{aligned}$$

Substituting (23) and (17) into (16), we have that

$$\begin{aligned}
\triangle_p(z_k, tv_1+(1-t)v_2) &\le \frac{1}{q}\|x_k\|^p + \frac{1}{p}\|tv_1+(1-t)v_2\|^p \\
&- \langle J^p_{E_1}x_k, tv_1+(1-t)v_2\rangle + 2^q G_q\|x_k\|^p\rho_{E_1^*}(\tau_k) \\
&- \lambda_n\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k), Ax_k - \Pi^p_{AC_k}T^k Ax_k\rangle \\
&= \triangle_p(x_k, tv_1+(1-t)v_2) + 2^q G_q\|x_k\|^p\rho_{E_1^*}(\tau_k) \\
&- \lambda_n\langle (J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k), Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle. \qquad (24)
\end{aligned}$$

Substituting (13) and (12) into (24), we have that

$$\begin{aligned}
\triangle_p(z_k, tv_1+(1-t)v_2) &\le \triangle_p(x_k, tv_1+(1-t)v_2) \\
&+ \frac{\gamma\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k Ax_k\rangle}{\|A\|\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\|} \qquad (25) \\
&- \frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle}{\|A^*\|\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|} \\
&= \triangle_p(x_k, tv_1+(1-t)v_2) - [1-\gamma] \\
&\times \frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle}{\|A\|\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k)\|}. \qquad (26)
\end{aligned}$$

Therefore,

$$\triangle_p(z_k, tv_1+(1-t)v_2) \le \triangle_p(x_k, tv_1+(1-t)v_2). \qquad (27)$$

Let $x_k = 0$, we have

$$\triangle_p(x_k, tv_1+(1-t)v_2) = \frac{1}{p}\|tv_1+(1-t)v_2\|^p \qquad (28)$$

and from (28), we have

$$\begin{aligned}
\triangle_p(z_k, tv_1+(1-t)v_2) &= \frac{1}{q}\|\lambda_k A^* J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q \\
&+ \triangle_p(x_k, tv_1+(1-t)v_2) \\
&+ \lambda_k\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k), tAv_1+(1-t)Av_2\rangle. \qquad (29)
\end{aligned}$$

Substituting (17) into (29), we have that

$$\begin{aligned}
\triangle_p(z_k, tv_1+(1-t)v_2) &\le \frac{1}{q}\|\lambda_k A^* J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q \\
&+ \triangle_p(x_k, tv_1+(1-t)v_2) + \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle \\
&- \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle. \qquad (30)
\end{aligned}$$

But, from (12), we have that

$$\begin{aligned}
&\frac{1}{q}\|\lambda_k A^* J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^q \\
&= \frac{1}{q}\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p}. \qquad (31)
\end{aligned}$$

Substituting (31) into (30), we have that

$$\begin{aligned}
&\triangle_p(z_k, tv_1+(1-t)v_2) \\
&\le \frac{1}{q}\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p} \\
&+ \triangle_p(x_k, tv_1+(1-t)v_2) + \lambda_k\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k\rangle \\
&- \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle \\
&\le (1-\frac{1}{p})\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p} \\
&+ \triangle_p(x_k, tv_1+(1-t)v_2) + \lambda_k\langle A^* J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, x_k\rangle \\
&- \frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p} \\
&\times \langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle \\
&\le (1-\frac{1}{p})\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p} \\
&+ \triangle_p(x_k, tv_1+(1-t)v_2) + \lambda_k\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|\|Ax_k\| \\
&- \frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p} \\
&\times \langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle \\
&= \triangle_p(x_k, tv_1+(1-t)v_2) \\
&- \frac{1}{p}\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k, Ax_k - \Pi^p_{AC_k}T^k)Ax_k\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_k}T^k)Ax_k\|^p}.
\end{aligned}$$

This implies that

$$\triangle_p(z_k, tv_1+(1-t)v_2) \le \triangle_p(x_k, tv_1+(1-t)v_2). \qquad (32)$$

In addition, it follows from Eq. (11), Definition 2.4 and Lemma 2.16 that

$$\begin{aligned}
&\triangle_p(y_k, tv_1+(1-t)v_2) \\
&= \triangle_p(J^q_{E_1^*}(\alpha_k J^p_{E_1}(z_k) + (1-\alpha_k)J^p_{E_1}(U^k(z_k)), tv_1+(1-t)v_2) \\
&= \frac{1}{q}\|\alpha_k J^p_{E_1}(z_k) + (1-\alpha_k)J^p_{E_1}(U^k(z_k)\|^q \\
&- \langle \alpha_k J^p_{E_1}(z_k) + (1-\alpha_k)J^p_{E_1}(U^k(z_k), tv_1+(1-t)v_2\rangle \\
&+ \frac{1}{p}\|tv_1+(1-t)v_2\|^p \\
&= \frac{1}{q}\|\alpha_k J^p_{E_1}(z_k) + (1-\alpha_k)J^p_{E_1}(U^k(z_k)\|^p - \alpha_k\langle J^p_{E_1}z_k, tv_1+(1-t)v_2\rangle
\end{aligned}$$

$$
\begin{aligned}
&- (1-\alpha_k)\langle J^p_{E_1}(U^k(z_k)), tv_1 + (1-t)v_2\rangle + \frac{1}{p}\|tv_1 + (1-t)v_2\|^p \\
&\leq \frac{1}{q}\alpha_k\|z_k\|^p + \frac{1}{q}(1-\alpha_k)\|U^k z_k\|^p - (\alpha_k^q(1-\alpha_k) + (1-\alpha_k)^q\alpha_k) \\
&\times g(\|z_k - U^k z_k\|) - \alpha_k\langle J^p_{E_1}(z_k), tv_1 + (1-t)v_2\rangle \\
&- (1-\alpha_k)\langle J^p_{E_1}(U^k(z_k)), tv_1 + (1-t)v_2\rangle + \frac{1}{p}\|tv_1 + (1-t)v_2\|^p \\
&= \alpha_k\Big(\frac{1}{p}\|tv_1 + (1-t)v_2\|^p - \langle J^p_{E_1}(z_k), tv_1 + (1-t)v_2\rangle + \frac{1}{q}\|z_k\|^p\Big) \\
&+ (1-\alpha_k)\Big(\frac{1}{p}\|tv_1 + (1-t)v_2\|^p - \langle J^p_{E_1}(U^k(z_k)), tv_1 + (1-t)v_2\rangle \\
&+ \frac{1}{q}\|U^k z_k\|^p\Big) - (\alpha_k^q(1-\alpha_k) + (1-\alpha_k)^q\alpha_k)g(\|z_k - U^k z_k\|) \\
&= \alpha_k \triangle_p (z_k, tv_1 + (1-t)v_2) + (1-\alpha_k) \triangle_p (U^k(z_k), tv_1 \\
&+ (1-t)v_2) - (\alpha_k^p(1-\alpha_k) + (1-\alpha_k)^p\alpha_k) \times g(\|z_k - U^k z_k\|) \\
&\leq \alpha_k \triangle_p (z_k, tv_1 + (1-t)v_2) + (1-\alpha_k)(k_k \triangle_p (z_k, tv_1 \\
&+ (1-t)v_2) + \triangle_p(z_k, tv_1 + (1-t)v_2) + \mu_k) - (\alpha_k^q(1-\alpha_k) \\
&+ (1-\alpha_k)^q\alpha_k)g(\|z_k - U^k z_k\|) \\
&\leq k_k \triangle_p (z_k, tv_1 + (1-t)v_2) + \triangle_p(z_k, tv_1 + (1-t)v_2) + \mu_k \\
&- (\alpha_k^q(1-\alpha_k) + (1-\alpha_k)^q\alpha_k)g(\|z_k - U^k z_k\|) \\
&\leq [k_k + 1] \triangle_p (z_k, tv_1 + (1-t)v_2) + \mu_k. \qquad (33)
\end{aligned}
$$

From (27), (32) and (33), we have that
$v \in C_{k+1}$. Hence, C_{k+1} is convex.
Therefore C_n is closed and convex for each $n \in \mathbb{N}$

Step two: We prove $\Gamma \subset C_n$ for any $n \geq 1$. Clearly $\Gamma \subset C_1$. Now assuming that $\Gamma \subset C_n$ for some $n \geq 1$. Let $v \in \Gamma$ then from (11) and Lemma 2.15, having considered $x_n \neq 0$, we get

$$
\begin{aligned}
&\triangle_p(z_n, v) \\
&= \frac{1}{q}\|J^p_{E_1}x_n - \lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^k)Ax_n)\|^q + \frac{1}{p}\|v\|^p \\
&- \langle J^p_{E_1}x_n - \lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n), v\rangle \qquad (34) \\
&= \frac{1}{q}\|J^p_{E_1}x_n - \lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\|^q + \frac{1}{p}\|v\|^p \\
&- \langle J^p_{E_1}x_n, v\rangle + \langle \lambda_n(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, v\rangle.
\end{aligned}
$$

Where

$$
\begin{aligned}
&\lambda_n\langle A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, v\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av + \Pi^p_{AC_n}T^nAx_n - \Pi^p_{AC_n}T^nAx_n\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av - \Pi^p_{AC_n}T^nAx_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, \Pi^p_{AC_n}T^nAx_n\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av - \Pi^p_{AC_n}T^nAx_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, \Pi^p_{AC_n}T^nAx_n - Ax_n + Ax_n\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av - \Pi^p_{AC_n}T^nAx_n\rangle \\
&- \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, \Pi^p_{AC_n}T^nAx_n - Ax_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av - Ax_n + Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle \\
&- \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle \\
&= \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, (Av - Ax_n) - (\Pi^p_{AC_n}T^nAx_n - Ax_n)\rangle \\
&- \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle \\
&= -\lambda_n\langle(J^p_{E_2}(\Pi^p_{AC_n}T^n - I)Ax_n, (Av - Ax_n) - (\Pi^p_{AC_n}T^nAx_n - Ax_n)\rangle \\
&- \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle \\
&+ \lambda_n\langle(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle.
\end{aligned}
$$

By the assumption that $A(C_n)$ is closed and convex and Lemma 2.5 and by the variational inequality for the Bregman projection of zero onto $A(C_n) - Ax_n$, as in Definition 2.6, we arive at

$$
\langle(J^p_{E_2}(\Pi^p_{AC_n}T^n - I)Ax_n, (Av - Ax_n) - (\Pi^p_{AC_n}T^nAx_n - Ax_n)\rangle \geq 0
$$

and therefore,

$$
\begin{aligned}
&\lambda_n\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Av\rangle \\
&\quad\leq \lambda_n\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle \\
&\quad- \lambda_n\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle. \qquad (35)
\end{aligned}
$$

In addition, from Lemma 2.15, we have that

$$
\begin{aligned}
&\frac{1}{q}\|J^p_{E_1}x_n - \lambda_n A^*(J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\|^q \\
&\quad\leq \frac{1}{q}\|J^p_{E_1}x_n\|^q - \lambda_n\langle x_n, A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\rangle \\
&\quad+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, \lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n) \\
&\quad= \frac{1}{q}\|J^p_{E_1}x_n\|^q - \lambda_n\langle Ax_n, J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\rangle \\
&\quad+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, \lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n) \\
&\quad= \frac{1}{q}\|x_n\|^p - \lambda_n\langle Ax_n, J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\rangle \\
&\quad+ \frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, \lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n) \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{q}\bar{\sigma}_q(J^p_{E_1}x_n, \lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n) \\
&\quad= G_q\int_0^1 \frac{(\|J^p_{E_1}x_n - t\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\| \vee \|J^p_{E_1}x_n\|)^q}{t}\times \\
&\rho_{E^*}\left(\frac{t\|\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\|}{(\|J^p_{E_1}x_n - t\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\| \vee \|J^p_{E_1}x_n\|)}\right)dt, \qquad (37)
\end{aligned}
$$

$$
for\ every\ t \in [0, 1].
$$

But

$$\|J^p_{E_1}x_n - t\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\| \le \|x_n\|^{p-1} + \|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|.$$

From (12), suppose that

$$\lambda_n = \frac{\tau_n}{\|A\|}\frac{\|x_n\|^{p-1}}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}$$

then we have

$$\|J^p_{E_1}x_n - t\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\| \le 2\|x_n\|^{p-1}$$

and that

$$\|x_n\|^{p-1} \le \|J^p_{E_1}x_n - t\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\| \vee \|J^p_{E_1}x_n\| \le 2\|x_n\|^{p-1}. \tag{38}$$

From Definition 2.2(2), we have that from Eqs. (38) and (13)

$$\begin{aligned}&\rho_{E_1^*}\left(\frac{t\|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}{(\|J^p_{E_1}x_n - t\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\| \vee \|J^p_{E_1}x_n\|)}\right)\\ &\le \rho_{E_1^*}\left(\frac{t\|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}{\|x_n\|^{p-1}}\right)\\ &= \rho_{E_1^*}(t\tau_n).\end{aligned} \tag{39}$$

Substituting Eqs. (38) and (39) into (37), and since $t \le \tau_n$ and $\rho_{E_1^*}$ is nondecreasing then, we have that

$$\begin{aligned}&\frac{1}{q}\bar{\sigma}_q(J^p_{E_2}x_n, \lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n)\\ &\le 2^q G_q\|x_n\|^{(p-1)q}\int_0^\tau \frac{\rho_{E_1^*}(t)}{t}dt\\ &\le 2^q G_q\|x_n\|^p\int_0^\tau \frac{\rho_{E_1^*}(\tau_n)}{\tau_n}dt\\ &\le 2^q G_q\|x_n\|^p\rho_{E_1^*}(\tau_n).\end{aligned} \tag{40}$$

Substituting Eq. (40) into (36), we have that

$$\begin{aligned}&\frac{1}{q}\|J^p_{E_1}x_n - \lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^q\\ &\le \frac{1}{q}\|x_n\|^p - \lambda_n\langle Ax_n, J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\rangle + 2^q G_q\|x_n\|^p\rho_{E_1^*}(\tau_n)\\ &= \frac{1}{q}\|J^p_{E_1}x_n\|^q - \lambda_n\langle (J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle\\ &+ 2^q G_q\|x_n\|^p\rho_{E_1^*}(\tau_n).\end{aligned} \tag{41}$$

Substituting Eqs. (35) and (41) into (34), we have that

$$\begin{aligned}\triangle_p(z_n, v) &\le \frac{1}{q}\|x_n\|^p + \frac{1}{p}\|v\|^p\\ &- \langle J^p_{E_1}x_n, v\rangle + 2^q G_q\|x_n\|^p\rho_{E_1^*}(\tau_n)\\ &- \lambda_n\langle (J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n), Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle\\ &= \triangle_p(x_n, v) + 2^q G_q\|x_n\|^p\rho_{E_1^*}(\tau_n)\\ &- \lambda_n\langle (J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n), Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle.\end{aligned} \tag{42}$$

Substituting (13) and (12) into (42), we have that

$$\begin{aligned}&\triangle_p(z_n, v)\\ &\le \triangle_p(x_n, v) + \frac{\gamma\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle}{\|A\|\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n)\|}\\ &- \frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle}{\|A^*\|\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}\\ &= \triangle_p(x_n, v) - [1-\gamma]\\ &\times \frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle}{\|A^*\|\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|}.\end{aligned} \tag{43}$$

Therefore,

$$\triangle_p(z_n, v) \le \triangle_p(x_n, v). \tag{44}$$

For $x_n = 0$ we have

$$\triangle_p(x_n, v) = \frac{1}{p}\|v\|^p \tag{45}$$

and from (45), we have

$$\begin{aligned}\triangle_p(z_n, v) &= \frac{1}{q}\|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^q + \triangle_p(x_n, v)\\ &+ \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n), Av\rangle.\end{aligned} \tag{46}$$

Substituting Eq. (35) into Eq. (46), we have that

$$\begin{aligned}\triangle_p(z_n, v) &\le \frac{1}{q}\|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^q\\ &+ \triangle_p(x_n, v) + \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle\\ &- \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle\end{aligned} \tag{47}$$

But, from (13), we have that

$$\begin{aligned}&\frac{1}{q}\|\lambda_n A^* J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^q\\ &= \frac{1}{q}\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n)Ax_n\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^nAx_n\|^p}.\end{aligned} \tag{48}$$

Substituting (48) into (47), we have that

$$\begin{aligned}&\triangle_p(z_n, v)\\ &\le \frac{1}{q}\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^p}\\ &+ \triangle_p(x_n, v) + \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n\rangle\\ &- \lambda_n\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle\\ &\le (1-\frac{1}{p})\frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle^p}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^p}\end{aligned} \tag{49}$$

$$
\begin{aligned}
&+ \triangle_p(x_n, v) + \lambda_n \langle A^* J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, x_n\rangle \\
&- \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_k} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\|^p} \\
&\times \langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle \\
&\leq (1 - \frac{1}{p}) \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\|^p} \\
&+ \triangle_p(x_n, v) + \lambda_n \|A^* J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\| \|x_n\| \\
&- \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\|^p} \\
&\times \langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle \\
&= \triangle_p(x_n, v) \\
&- \frac{1}{p}) \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n Ax_n\rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\|^p} \\
&= \triangle_p(x_n, v) - \frac{1}{p}) \frac{1}{\|A\|^p} \\
&\times \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n, Ax_n - \Pi^p_{AC_n} T^n) Ax_n\rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) Ax_n\|^p} \qquad (50)
\end{aligned}
$$

This implies that

$$\triangle_p(z_n, v) \leq \triangle_p(x_n, v). \qquad (51)$$

In addition, it follows from (11), Definition 2.4, Lemma 2.16 and $p \in (1, \infty)$ that

$$
\begin{aligned}
&\triangle_p(y_n, v) \\
&= \triangle_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1 - \alpha_n) J^p_{E_1}(U^n(z_n))), v) \\
&= \frac{1}{q} \|\alpha_n J^p_{E_1} z_n + (1 - \alpha_n) J^p_{E_1} U^z_n\|^q - \langle \alpha_n J^p_{E_1} z_n \\
&+ (1 - \alpha_n) J^p_{E_1} U^n z_n, v\rangle + \frac{1}{p}\|v\|^p \\
&= \frac{1}{q} \|\alpha_n J^p_{E_1} z_n + (1 - \alpha_n) J^p_{E_1} U^n z_n\|^p - \alpha_n \langle J^p_{E_1} z_n, v\rangle \\
&- (1 - \alpha_n)\langle J^p_{E_1}(U^n(z_n)), v\rangle + \frac{1}{p}\|v\|^p \\
&\leq \frac{1}{q}\alpha_n \|z_n\|^p + \frac{1}{q}(1 - \alpha_n)\|U^n z_n\|^p - (\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q \alpha_n) \\
&\times g(\|z_n - U^n z_n\|) - \alpha_n \langle J^p_{E_1}(z_n), v\rangle - (1 - \alpha_n)\langle J^p_{E_1}(U^n(z_n)), v\rangle \\
&+ \frac{1}{p}\|v\|^p \\
&= \alpha_n(\frac{1}{p}\|v\|^p - \langle J^p_{E_1}(z_n), v\rangle + \frac{1}{q}\|z_n\|^p) + (1 - \alpha_n)(\frac{1}{p}\|v\|^p \\
&- \langle J^p_{E_1}(U^n(z_n)), v\rangle + \frac{1}{q}\|U^n z_n\|^p) - (\alpha_n^q(1 - \alpha_n) \\
&+ (1 - \alpha_n)^q \alpha_n) g(\|z_n - U^n z_n\|) \\
&= \alpha_n \triangle_p(z_n, v) + (1 - \alpha_n) \triangle_p(U^n(z_n), v) \\
&- (\alpha_n^p(1 - \alpha_n) + (1 - \alpha_n)^p \alpha_n) \times g(\|z_n - U^n z_n\|) \\
&\leq \alpha_n \triangle_p(z_n, v) + (1 - \alpha_n)(k_n \triangle_p(z_n, v) + \triangle_p(z_n, v) + \mu_n) \\
&- (\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q \alpha_n) g(\|z_n - U^n z_n\|) \\
&\leq k_n \triangle_p(z_n, v) + \triangle_p(z_n, v) + \mu_n
\end{aligned}
$$

$$
\begin{aligned}
&- (\alpha_n^q(1 - \alpha_n) + (1 - \alpha_n)^q \alpha_n) g(\|z_n - U^n z_n\|) \qquad (52) \\
&\leq [k_n + 1] \triangle_p(z_n, v) + \mu_n. \qquad (53)
\end{aligned}
$$

From Eqs. (44), (51) and (53), we have that $v \in C_n$ and $\Gamma \subset C_n$ for any $n \geq 1$. *Step three:* we show that $\{x_n\}$ is Cauchy sequence. For $v \in C_n$, and $x_{n+1} = \Pi^p_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, then from Definition 2.6, we have that

$$
\begin{aligned}
0 &\leq \langle J^p_{E_1} \Pi^p_{C_{n+1}}(x_1) - J^p_{E_1}(x_1), v - \Pi^p_{C_{n+1}}(x_1)\rangle \\
&= \langle J^p_{E_1} \Pi^p_{C_{n+1}}(x_1), v\rangle - \langle J^p_{E_1}(x_1), v\rangle \\
&- \langle J^p_{E_1} \Pi^p_{C_{n+1}}(x_1), \Pi^p_{C_{n+1}}(x_1)\rangle + \langle J^p_{E_1}(x_1), \Pi^p_{C_{n+1}}(x_1)\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\langle J^p_{E_1} \Pi^p_{C_{n+1}}(x_1), \Pi^p_{C_{n+1}}(x_1)\rangle| &\leq |\langle J^p_{E_1} \Pi^p_{C_{n+1}}(x_1), v\rangle \\
&- \langle J^p_{E_1}(x_1), v\rangle + \langle J^p_{E_1}(x_1), \Pi^p_{C_{n+1}}(x_1)\rangle|.
\end{aligned}
$$

That is

$$\|\Pi^p_{C_{n+1}}(x_1)\|^p \leq \|\Pi^p_{C_{n+1}}(x_1)\|^{p-1}\|v\| + \|x_1\|^{p-1}\|v\| + \|x_1\|^{p-1}\|\Pi^p_{C_{n+1}}(x_1)\|. \qquad (54)$$

Observe that if

$$
\begin{aligned}
&\|\Pi^p_{C_{n+1}}(x_1)\|^{p-1} < 2\|x_1\|^{p-1} \\
&\textit{then we have } \|\Pi^p_{C_{n+1}}(x_1)\| < 2^{q-1}\|x_1\| \qquad (55)
\end{aligned}
$$

and so we are done. Otherwise, let

$$t_{x_1} = \left(\frac{\|\Pi^p_{C_{n+1}}(x_1)\|}{\|x_1\|}\right)^{p-1} \geq 2.$$

For $x_1 \neq 0$ and using (54) we have that

$$
\begin{aligned}
&\|\Pi^p_{C_{n+1}}(x_1)\|(\|\Pi^p_{C_{n+1}}(x_1)\|^{p-1} - \|x_1\|^{p-1}) \\
&\leq \|v\|(\|\Pi^p_{C_{n+1}}(x_1)\|^{p-1} + \|x_1\|^{p-1})
\end{aligned}
$$

and so

$$\|\Pi^p_{C_{n+1}}(x_1)\| \leq \|v\| \frac{\|\Pi^p_{C_{n+1}}(x_1)\|^{p-1} + \|x_1\|^{p-1}}{\|\Pi^p_{C_{n+1}}(x_1)\|^{p-1} - \|x_1\|^{p-1}}. \qquad (56)$$

From (56), since the function $h(t_{x_1}) = \frac{t_{x_1}+1}{t_{x_1}-1}$ is decreasing for $t_{x_1} > 1$ we arive at

$$\|\Pi^p_{C_{n+1}}(x_1)\| \leq \min_{v \in C_{n+1}} \|v\| h(t_{x_1}) \leq \min_{v \in C_{n+1}} \|v\| h(2) = 3 \min_{v \in C_{n+1}} \|v\|. \qquad (57)$$

If $x_1 = 0$ then from (54), we have that

$$\|\Pi^p_{C_{n+1}}(x_1)\| \leq \min_{v \in C_{n+1}} \|v\|. \qquad (58)$$

From (55), (57) and (58), we have that $x_{n+1} = \Pi^p_{C_{n+1}}(x_1)$ is

bounded.
For any $n \geq 1$, by Definition 2.6, we have that

$$\begin{aligned} 0 &\leq \triangle_p(J^p_{E_1}\Pi^p_{C_n}(x_1), \Pi^p_{C_{n+1}}(x_1)) \\ &\leq \triangle_p(x_1, \Pi^p_{C_{n+1}}(x_1)) - \triangle_p(x_1, \Pi^p_{C_n}(x_1)) \end{aligned} \tag{59}$$

From (59), we have that

$$\triangle_p(x_1, \Pi^p_{C_n}(x_1)) \leq \triangle_p(x_1, \Pi^p_{C_{n+1}}(x_1)).$$

Thus $\{\triangle_p(x_1, \Pi^p_{C_n}(x_1))\}$ is nondecreasing. Therefore by boundedness of $\Pi^p_{C_n}(x_1)$, we have that $\lim_{n\to\infty} \triangle_p(x_1, \Pi^p_{C_n}(x_1))$ exists. Let $m, n \in N$ $m < n$. From $x_n = \Pi^p_{C_n}(x_1) \subset C_n$ and (59), we have that

$$\triangle_p(\Pi^p_{C_n}(x_1), \Pi^p_{C_m}(x_1)) \leq \triangle_p(x_1, \Pi^p_{C_m}(x_1)) - \triangle_p(x_1, \Pi^p_{C_n}(x_1)). \tag{60}$$

Since $\lim_{n\to\infty} \triangle_p(x_1, \Pi^p_{C_n}(x_1))$ exists, it follows from (60) that

$$\triangle_p(\Pi^p_{C_m}(x_1), \Pi^p_{C_n}(x_1)) \to 0 \; as \; m, n \to \infty.$$

Therefore $\{\Pi^p_{C_n}(x_1)\}$ is a Cauchy sequence.

Step four: we will show that $\lim_{n\to\infty} \triangle_p(z_n, Uz_n) = 0$ and $\lim_{n\to\infty} \triangle_p(Ax_n, \Pi^p_{AC_n}T^nAx_n) = 0$. Let $x_{n+1} = \Pi^p_{C_{n+1}}(x_1) \subset C_{n+1} \subset C_n$.

From 11, we have that

$$\triangle_p(z_n, v) \leq \triangle_p(x_n, v) \Leftrightarrow \frac{1}{p}\|z_n - v\|^p \leq \frac{1}{p}\|x_n - v\|^p.$$

Therefore $\{z_n\}$ is bounded as $\{x_n\}$ is bounded.

Hence, from Lemma 2.9 and 2.10, we have

$$\begin{aligned} \triangle_p(z_n, x_n) &= \triangle_p(z_n, x_{n+1}) + \triangle_p(x_{n+1}, x_n) \\ &\quad - \langle z_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle \\ &\leq \triangle_p(x_n, x_{n+1}) + \triangle_p(x_{n+1}, x_n) \\ &\quad - \langle z_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle \\ &= \langle J^p x_{n+1} - J^p x_n, x_{n+1} - x_n, \rangle \\ &\quad - \langle z_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle \\ &\to 0 \; as \; n \to \infty. \end{aligned} \tag{61}$$

Now, for $x_n \neq 0$ for all $n \in N$, from (43), we have

$$\triangle_p(z_n, v) \leq \triangle_p(x_n, v) - [1-\gamma]\|Ax_n - \Pi^p_{AC_n}T^nAx_n\|^{p-1}.$$

This implies that

$$\begin{aligned} \|Ax_n - \Pi^p_{AC_n}T^nAx_n\| &\leq \left(\frac{\triangle_p(x_n, v) - \triangle_p(z_n, v)}{[1-\gamma]}\right)^{\frac{1}{p-1}} \\ &= \left(\frac{\frac{1}{q}(\|z_n\|^p - \|x_n\|^p) + \langle J^p_{E_1}z_n - J^p_{E_1}x_n, v\rangle}{[1-\gamma]}\right)^{\frac{1}{p-1}} \end{aligned}$$

and therefore we have that

$$\lim_{n\to\infty}\|Ax_n - \Pi^p_{AC_n}T^nAx_n\| = 0. \tag{62}$$

For $x_n = 0$ for all $n \in N$, from (49), we have

$$\begin{aligned} \triangle_p(z_n, v) &\leq \triangle_p(x_n, v) \\ &\quad - \frac{1}{p}\frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^nAx_n\rangle^p}{\|A\|^p\|J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n)\|^p}. \end{aligned}$$

This implies that

$$\begin{aligned} \|Ax_n - \Pi^p_{AC_n}T^nAx_n\| &\leq \frac{p^{\frac{1}{p^2-p}}}{\|A\|^p}\left(\triangle_p(x_n, v) - \triangle_p(z_n, v)\right)^{\frac{1}{p^2-p}} \\ &= \frac{p^{\frac{1}{p^2-p}}}{\|A\|^p}\left(\frac{1}{q}(\|z_n\|^p - \|x_n\|^p) + \langle J^p_{E_1}z_n - J^p_{E_1}x_n, v\rangle\right)^{\frac{1}{p^2-p}}. \end{aligned}$$

Therefore, we have that

$$\lim_{n\to\infty}\|Ax_n - \Pi^p_{AC_n}T^nAx_n\| = 0. \tag{63}$$

By Lemma 2.11, we have that

$$\lim_{n\to\infty} \triangle_p(Ax_n, \Pi^p_{AC_n}T^nAx_n) = 0. \tag{64}$$

On the other hand, since

$$\begin{aligned} &\triangle_p(y_n, x_n) = \triangle_p(y_n, x_{n+1}) + \triangle_p(x_{n+1}, x_n) - \langle y_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle \\ &\leq [1 + k_n]\triangle_p(z_n, x_{n+1}) + \langle J^p x_{n+1} - J^p x_n, x_{n+1} - x_n, \rangle \\ &- \langle z_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle + \mu_n \\ &\leq [1 + k_n]\triangle_p(x_n, x_{n+1}) + \langle J^p x_{n+1} - J^p x_n, x_{n+1} - x_n, \rangle \\ &- \langle z_n - x_{n+1}, J^p x_n - J^p x_{n+1}\rangle + \mu_n \\ &\to 0 \; as \; n \to \infty \end{aligned} \tag{65}$$

and

$$\begin{aligned} \triangle_p(y_n, z_n) &= \triangle_p(y_n, x_n) + \triangle_p(x_n, z_n) \\ &\quad - \langle y_n - x_n, J^p z_n - J^p x_n\rangle \to 0 \; as \; n \to \infty. \end{aligned} \tag{66}$$

Now, from (52), we have that

$$\begin{aligned} \triangle_p(y_n, v) &\leq [k_n + 1]\triangle_p(z_n, v) + \mu_n \\ &\quad (\alpha_n^p(1-\alpha_n) + (1-\alpha_n)^p\alpha_n)g(\|z_n - U^nz_n\|). \end{aligned}$$

This implies, From (44), we have that

$$\begin{aligned} g(\|z_n - U^nz_n\|) &\leq \frac{[k_n + 1]\triangle_p(z_n, v) + \mu_n - \triangle_p(y_n, v)}{(\alpha_n^p(1-\alpha_n) + (1-\alpha_n)^p\alpha_n)} \\ &= \frac{\frac{1}{q}([k_n + 1]\|z_n\|^p - \|y_n\|^p) + \langle J^p y_n - J^p z_n, v\rangle - k_n\langle J^p z_n, v\rangle}{(\alpha_n^p(1-\alpha_n) + (1-\alpha_n)^p\alpha_n)}. \\ &+ \frac{\frac{k_n}{p}\|v\|^p + \mu_n}{(\alpha_n^p(1-\alpha_n) + (1-\alpha_n)^p\alpha_n)}. \end{aligned}$$

Since $\alpha_n(1-\alpha_n) > 0$, and g is continuous, strictly increasing and convex function, then we have that

$$\lim_{n\to\infty} g\|z_n - U^n(z_n)\| = 0$$

and

$$\lim_{n\to\infty}\|z_n - U^n(z_n)\| = 0. \tag{67}$$

By Lemma 2.11, we have that

$$\lim_{n\to\infty} \triangle_p\,(z_n, U^n z_n) = 0. \tag{68}$$

Since T and U are continuous, then from (64) and (68), we have that

$$\lim_{n\to\infty} \triangle_p\,(Ax_n, \Pi^p_{AC}TAx_n) = 0 \tag{69}$$

and

$$\lim_{n\to\infty} \triangle_p\,(z_n, Uz_n) = 0. \tag{70}$$

Step five: we will show that $\{x_n\}$ converges strongly to an element of Γ. Since $\{x_n\}$ is Cauchy sequence, we may assume that $x_n \longrightarrow x^*$, from (61) we have $z_n \longrightarrow x^*$, which implies that $z_n \rightharpoonup x^*$. So it follows from (70) and the demicloseness of $(I-U)$ at zero that $x^* \in Fix(U)$. In addition, since A is bounded linear operator, we have that $\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$. Hence, it follows from (69), and demicloseness of $(I - \Pi^p_{AC}T)$ at zero that $Ax^* \in Fix(\Pi^p_{AC}T)$. This means that $x^* \in \Gamma$ and $\{x_n\}$ converges strongly to $x^* \in \Gamma$. The proof is completed.

In theorem 3.1, as $U = T$, $E_1 = E_2 = E$ and $A = I$, we have the following result.

Corollary 3.1.1. *Let E be a uniformly convex and uniformly smooth Banach spaces, $T : E \longrightarrow E$ be a uniformly continuous Bregman generalized asymptotically nonexpansive operator with the sequences $\{k_n\} \subset [0,\infty)$ and $\{\mu_n\} \subset [0,\infty)$ satisfying $\lim_{n\to\infty} k_n = 0$ and $\lim_{n\to\infty}\mu_n = 0$, respectively, and, for $p, q \in (1,\infty)$, $\Pi^p_C : E \longrightarrow C$ be a Bregman projection onto a subset $C \in E$, and $Fix(T) \neq \phi$ and $(I - T)$ be demiclosed at zero. Let $x_1 \in E$ be chosen arbitrarily and let $C_1 = E$. Define a sequence $\{x_n\}$ as follows;*

$$\begin{cases} z_n = J^q_{E_1^*}(J^p_{E_1}x_n - \lambda_n J^p_{E_2}(I - \Pi^p_{C_n}T^n)x_n), \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1-\alpha_n)J^p_{E_1}(T^n(z_n))), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \le [k_n + 1]\,\triangle_p\,(z_n, v) + \mu_n; \\ \triangle_p(z_n, v) \le \triangle_p(x_n, v)\}, \\ x_{n+1} = \Pi^p_{C_{n+1}}(x_1), n \ge 1 \end{cases} \tag{71}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|J^p_{E_2}(I-\Pi^p_{C_n}T^n)x_n\|},\ x_n \neq 0 \\ \frac{\langle J^p_{E_2}(I-\Pi^p_{C_n}T^n)x_n, x_n - \Pi^p_{C_n}T^n x_n\rangle^{p-1}}{\|J^p_{E_2}(I-\Pi^p_{C_n}T^n)x_n\|^p}, x_n = 0 \end{cases}$$

and $\gamma \in (0,1)$ and $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q} \times \frac{\langle J^p_{E_2}(I - \Pi^p_{C_n}T^n)x_n, x_n - \Pi^p_{C_n}T^n x_n\rangle}{\|x_n\|^p \|J^p_{E_2}(I - \Pi^p_{C_n}T^n)x_n\|},$$

and $\{\alpha_n\} \subset (0,1)$ satisfies $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. If $Fix(T) = \{v \in E : v = Tv\} \neq \phi$, then $\{x_n\}$ converges strongly to $x^* \in Fix(T)$.

In theorem 3.1, when U and T are two Bregman symptotically nonexpansive mappings, the following result holds.

Corollary 3.1.2. *Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces, $A : E_1 \longrightarrow E_2$ is bounded and linear operator, such that $A(C)$, for $C \subset E_1$, is closed and convex, $U : E_1 \longrightarrow E_1$ be a uniformly continuous Bregman asymptotically nonexpansive operator with the sequence $\{k_n^{(1)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(1)} = 1$, and $T : E_2 \longrightarrow E_2$ be a uniformly continuous Bregman asymptotically nonexpansive operator with the sequence $\{k_n^{(2)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(2)} = 1$, respectively and, for $p, q \in (1,\infty)$, $\Pi^p_{AC} : E_2 \longrightarrow AC$ be a Bregman projection onto a subset AC, and $Fix(U) \neq \phi$ and $Fix(\Pi^p_{AC}T) \neq \phi$ respectively, and $(I - U)$ and $(I - \Pi^p_{AC}T)$ be demiclosed at zero. Let $x_1 \in E_1$ be chosen arbitrarily and let $C_1 = E_1$. Define a sequence $\{x_n\}$ as follows;*

$$\begin{cases} z_n = J^q_{E_1^*}(J^p_{E_1}x_n - \lambda_n A^* J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n), \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1-\alpha_n)J^p_{E_1}(U^n(z_n))), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \le k_n\,\triangle_p\,(z_n, v) \le k_n\,\triangle_p\,(x_n, v)\}, \\ x_{n+1} = \Pi^p_{C_{n+1}}(x_1), n \ge 1 \end{cases} \tag{72}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|}\frac{1}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|},\ x_n \neq 0 \\ \frac{1}{\|A\|^p}\frac{\langle J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n Ax_n\rangle^{p-1}}{\|J^p_{E_2}(I-\Pi^p_{AC_n}T^n)Ax_n\|^p}, x_n = 0 \end{cases}$$

and $\gamma \in (0,1)$ and $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q\|A\|} \times \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n, Ax_n - \Pi^p_{AC_n}T^n Ax_n\rangle}{\|x_n\|^p \|J^p_{E_2}(I - \Pi^p_{AC_n}T^n)Ax_n\|},$$

and A^* denote the adjoint of A, and $\{\alpha_n\} \subset (0,1)$ satisfies $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $k_n = (k_n^{(1)} \vee k_n^{(2)})$, $n \ge 1$. If $\Gamma = \{v \in Fix(U) : Av \in Fix(\Pi^p_{AC}T)\} \neq \phi$, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

In theorem 3.1, when U and T are two ϕ-asymptotically nonexpansive mappings, the following result holds.

Corollary 3.1.3. *Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces, $A : E_1 \longrightarrow E_2$ is bounded and linear operator, such that $A(C)$, for $C \subset E_1$, is closed and convex, $U : E_1 \longrightarrow E_1$ be a uniformly continuous ϕ-asymptotically nonexpansive operator with the sequence $\{k_n^{(1)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(1)} = 1$, and $T : E_2 \longrightarrow E_2$ be a uniformly continuous ϕ-asymptotically nonexpansive operator with the sequence $\{k_n^{(2)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(2)} = 1$, respectively and, for $p, q \in (1,\infty)$, $\Pi^p_{AC} : E_2 \longrightarrow AC$ be a generalized projection onto a subset AC, and $Fix(U) \neq \phi$ and $Fix(\Pi^p_{AC}T) \neq \phi$ respectively, and $(I - U)$ and $(I - \Pi^p_{AC}T)$ be demiclosed at zero. Let $x_1 \in E_1$ be chosen arbitrarily and let*

$C_1 = E_1$. *Define a sequence* $\{x_n\}$ *as follows;*

$$\begin{cases} z_n = J_{E_1^*}^{-1}(J_{E_1}x_n - \lambda_n A^* J_{E_2}(I - \Pi_{AC_n}^p T^n)Ax_n), \\ y_n = J_{E_1^*}^{-1}(\alpha_n J_{E_1}(z_n) + (1-\alpha_n)J_{E_1}(U^n(z_n))), \\ C_{n+1} = \{v \in C_n : \phi(y_n, v) \le k_n\phi(z_n, v) \le k_n\phi(x_n, v)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), n \ge 1 \end{cases} \tag{73}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|}\frac{1}{\|J_{E_2}(I-\Pi_{AC_n}^p T^n)Ax_n\|}, \ x_n \ne 0 \\ \frac{1}{\|A\|^p}\frac{\langle J_{E_2}(I-\Pi_{AC_n}^p T^n)Ax_n, Ax_n - \Pi_{AC_n}T^n Ax_n\rangle}{\|J_{E_2}(I-\Pi_{AC_n}T^n)Ax_n\|}, x_n = 0 \end{cases}$$

and $\gamma \in (0,1)$ *and* $\tau_n = \frac{1}{\|x_n\|}$ *are chosen such that*

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2G\|A\|} \times \frac{\langle J_{E_2}(I - \Pi_{AC_n}T^n)Ax_n, Ax_n - \Pi_{AC_n}T^n Ax_n\rangle}{\|x_n\|\|J_{E_2}(I - \Pi_{AC_n}T^n)Ax_n\|},$$

and A^* *denote the adjoint of* A, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $k_n = (k_n^{(1)} \vee k_n^{(2)})$, $n \ge 1$. *If* $\Gamma = \{v \in Fix(U) : Av \in Fix(\Pi_{AC}T)\} \ne \phi$, *then* $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$.

In theorem 3.1, when E_1 and E_2 are real Hilbert spaces H_1 and H_2, respectively, we have the following result.

Corollary 3.1.4. *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \longrightarrow H_2$ *is bounded and linear operator,* $U : H_1 \longrightarrow H_1$ *be a uniformly continuous generalized asymptotically nonexpansive operator with the sequences* $\{k_n^{(1)}\} \subset [0,\infty)$ *and* $\{\mu_n^{(1)}\} \subset [0,\infty)$ *satisfying* $\lim_{n\to\infty}k_n^{(1)} = 0$ *and* $\lim_{n\to\infty}\mu_n^{(1)} = 0$, *respectively, and* $T : H_2 \longrightarrow H_2$ *be a uniformly continuous generalized asymptotically nonexpansive operator with the sequences* $\{k_n^{(2)}\} \subset [0,\infty)$ *and* $\{\mu_n^{(2)}\} \subset [0,\infty)$ *satisfying* $\lim_{n\to\infty}k_n^{(2)} = 0$ *and* $\lim_{n\to\infty}\mu_n^{(2)} = 0$, *respectively, and* $(I-U)$ *and* $(I-T)$ *be demiclosed at zero. Let* $x_1 \in H_1$ *be chosen arbitrarily and let* $C_1 = H_1$ *and the sequence* $\{x_n\}$ *be defined as follows;*

$$\begin{cases} z_n = x_n - \lambda_n A^*(I - T^n)Ax_n), \\ y_n = \alpha_n z_n + (1-\alpha_n)U^n(z_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le [k_n+1]\|z_n - v\| + \mu_n; \\ \|z_n - v\| \le [k_n+1]\|x_n - v\| + \mu_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), n \ge 1 \end{cases} \tag{74}$$

where $\lambda_n \in (0, \frac{1}{\|A^*\|^2})$ *and* A^* *denote the adjoint of* A, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, *and* $k_n = (k_n^{(1)} \vee k_n^{(2)})$, $n \ge 1$. *If* $\Gamma = \{v \in Fix(U) : Av \in Fix(T)\} \ne \phi$, *then* $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$.

In theorem 3.1, when U and T are two nonexpansive mappings, the following result holds.

Corollary 3.1.5. *Let* E_1 *and* E_2 *be two uniformly convex and uniformly smooth Banach spaces,* $A : E_1 \longrightarrow E_2$ *is bounded and linear operator, such that* $A(C)$, *for* $C \subset E_1$, *is closed and convex,* $U : E_1 \longrightarrow E_1$ *be a uniformly continuous Bregman nonexpansive operator, and* $T : E_2 \longrightarrow E_2$ *be a uniformly continuous Bregman nonexpansive operator, and, for* $p, q \in (1,\infty)$, $\Pi_{AC}^p : E_2 \longrightarrow AC$ *be a Bregman projection onto a subset AC, and* $Fix(U) \ne \phi$ *and* $Fix(\Pi_{AC}^p T) \ne \phi$ *respectively, and* $(I-U)$ *and* $(I - \Pi_{AC}^p T)$ *be demiclosed at zero. Let* $x_1 \in E_1$ *be chosen arbitrarily and let* $C_1 = E_1$ *and the sequence* $\{x_n\}$ *be defined as follows;*

$$\begin{cases} z_n = J_{E_1^*}^q(J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p(I - \Pi_{AC_n}^p T)Ax_n), \\ y_n = J_{E_1^*}^q(\alpha_n J_{E_1}^p(z_n) + (1-\alpha_n)J_{E_1}^p(U(z_n))), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \le \triangle_p(z_n, v) \le \triangle_p(x_n, v)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^p(x_1), n \ge 1 \end{cases} \tag{75}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|}\frac{1}{\|J_{E_2}^p(I-\Pi_{AC_n}^p T)Ax_n\|}, \ x_n \ne 0 \\ \frac{1}{\|A\|^p}\frac{\langle J_{E_2}^p(I-\Pi_{AC_n}^p T)Ax_n, Ax_n - \Pi_{AC_n}^p TAx_n\rangle^{p-1}}{\|J_{E_2}^p(I-\Pi_{AC_n}^p T)Ax_n\|^p}, x_n = 0 \end{cases}$$

and $\gamma \in (0,1)$ *and* $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ *are chosen such that*

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q\|A\|} \times \frac{\langle J_{E_2}^p(I - \Pi_{AC_n}^p T)Ax_n, Ax_n - \Pi_{AC_n}^p TAx_n\rangle}{\|x_n\|^p\|J_{E_2}^p(I - \Pi_{AC_n}^p T)Ax_n\|},$$

and A^* *denote the adjoint of* A, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. *If* $\Gamma = \{v \in Fix(U) : Av \in Fix(\Pi_{AC}^p T)\} \ne \phi$, *then* $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$.

4. Application to the mixed equilibrium problem

Lemma 4.1. *[4] Let E be a reflexive, strictly convex and smooth Banach space, and let C be a nonempty closed convex subset of E. Let* $f, g : C \times C \longrightarrow \Re$ *be two bifunctions which satisfy the conditions* $(A_1) - (A_4), (B_1) - (B_3)$*and*$(C)$, *in (6), then for every* $x \in E$ *and* $r > 0$, *there exists a unique point* $z \in C$ *such that*

$$f(z,y) + g(z,y) + \frac{1}{r}\langle y - z, jz - jx\rangle \ge 0 \forall y \in C\}$$

In Reich, S, Sabach, S (2010) [18], when $f(x) = \frac{1}{p}\|x\|^p$ then we have the following Lemma.

Lemma 4.2. *Let E be a reflexive, strictly convex and smooth Banach space, and let C be a nonempty closed convex subset of E. Let* $f, g : C \times C \longrightarrow \Re$ *be two bifunctions which satisfy the conditions* $(A_1) - (A_4), (B_1) - (B_3)$*and*$(C)$, *in (6), then for every* $x \in E$ *and* $r > 0$, *define a mapping* $S_r : E \longrightarrow C$ *as follows;*

$$S_r(x) = \{x \in C : f(z,y) + g(z,y) + \frac{1}{r}\langle y - z, J_E^p z - J_E^p x\rangle \ge 0 \forall y \in C\}$$

then the following hold;

1. S_r *is a single-valued;*
2. S_r *is a Bregman firmly nonexpansive-type mapping, i.e.*

   $\forall x, y \in E \langle S_r x - S_r y, J_E^p S_r x - J_E^p S_r y\rangle \le \langle S_r x - S_r y, J_E^p x - J_E^p y\rangle;$

   or equivalently
   $\triangle_p(S_r x, S_r y) + \triangle_p(S_r y, S_r x) + \triangle_p(S_r x, x) + \triangle_p(S_r y, y) \le \triangle_p(S_r x, y) + \triangle_p(S_r y, x)$

3. $F(S_r) = MEP(f, g)$;

4. $MEP(f, g)$ *is closed and convex;*

5. *for all* $x \in E$ *and for all* $v \in F(S_r)$, $\triangle_p(v, S_r x) + \triangle_p(S_r x, x) \leq \triangle_p(v, x)$.

Theorem 4.1. *Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces, $A : E_1 \longrightarrow E_2$ is bounded and linear operator, such that $A(C)$, for $C \subset E_1$, is closed and convex, $T : E_2 \longrightarrow E_2$ be a uniformly continuous Bregman nonexpansive operator, and, for $p, q \in (1, \infty)$, $\Pi^p_{AC} : E_2 \longrightarrow AC$ be a Bregman projection onto a subset AC, and $f, g : C \times C \longrightarrow \Re$ be two bifunctions which satisfy the conditions $(A_1) - (A_4), (B_1) - (B_3)$ and (C), in (6), for $C \times C \subset E_1 \times E_1$, assume that $C = MEP(f, g) \neq \phi$ and $Fix(T) \neq \phi$, taking $C_1 = E_1$, the sequence $\{x_n\}$ is defined as follows;*

$$\begin{cases} \text{choosing arbitrarily } x_1 \in E_1, \\ z_n = J^q_{E_1^*}(J^p_{E_1} x_n - \lambda_n A^* J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n), \\ f(S_r(x_n), b) + g(S_r(x_n), b) + \frac{1}{r_n}\langle b - S_r(x_n), J^p_{E_1}(S_r(x_n)) \\ -J^p_{E_1}(x_n)\rangle \geq 0 \ \forall b \in E_1 \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1 - \alpha_n) J^p_{E_1}(S_r(z_n))), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \leq \triangle_p(z_n, v) \leq \triangle_p(x_n, v)\} \\ x_{n+1} = \Pi^p_{C_{n+1}}(x_1), n \geq 1, \end{cases} \tag{76}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|} \frac{1}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n\|}, \ x_n \neq 0 \\ \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n, A x_n - \Pi^p_{AC_n} T^n A x_n\rangle^{p-1}}{\|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n\|^p}, x_n = 0 \end{cases}$$

and $\gamma \in (0, 1)$ and $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q \|A\|} \times \frac{\langle J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n, A x_n - \Pi^p_{AC_n} T^n A x_n\rangle}{\|x_n\|^p \|J^p_{E_2}(I - \Pi^p_{AC_n} T^n) A x_n\|},$$

and A^* denote the adjoint of A, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. If $\Gamma = \{v \in MEP(f, g) : Av \in Fix(\Pi^p_{AC} T)\} \neq \phi$, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Proof: It follows from the lemma 4.2 that $Fix(S_r) = MEP(f, g)$ is nonempty, closed and convex and S_r is a firmly nonexpansive mapping. Meanwhile S_r is assume to be continuous. Hence all conditions in corollary 3.1.5 are satisfied. The conclusion of theorem 4.1 can be directly obtained from corollary 3.1.5.

Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. $A : E_1 \longrightarrow E_2$ is bounded and linear operator. Assume that $f, g : C \times C \longrightarrow \Re$ be two bifunctions and $f', g' : Q \times Q \longrightarrow \Re$ be another two bifunctions. The split mixed equilibrium problem (SMEP) is to find an element $v \in C$ such that

$$f(v, y) + g(v, y) \geq 0 \forall y \in C \tag{77}$$

and such that $Av \in Q$ solve

$$f(Av, Ay) + g(Av, Ay) \geq 0 \forall Ay \in Q \tag{78}$$

Let $\omega = \{v \in MEP(f, g) : Av \in MEP(f', g')\}$ denote the solution set of the SMEP.

Corollary 4.1.1. *Let E_1 and E_2 be two uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. $A : E_1 \longrightarrow E_2$ is bounded and linear operator, such that $A(C)$ is closed and convex, and, such that $A(C)$, for $C \subset E_1$, is closed and convex, for $p, q \in (1, \infty)$, $\Pi^p_{AC} : E_2 \longrightarrow AC$ be a Bregman projection onto a subset AC. Assume that $f, g : C \times C \longrightarrow \Re$ be two bifunctions and $f', g' : Q \times Q \longrightarrow \Re$ be another two bifunctions which satisfy the conditions $(A_1) - (A_4), (B_1) - (B_3)$ and (C), in (6), for $C \times C \subset E_1 \times E_1$, assuming $MEP(f, g) \neq \phi$ and $MEP(\Pi^p_{AC}, f', g') \neq \phi$, respectively, taking $C_1 = E_1$, the sequence $\{x_n\}$ is defined as follows;*

$$\begin{cases} \text{choosing arbitrarily } x_1 \in E_1, \\ f(S_r^{f+g}(x_n), b) + g(S_r^{f+g}(x_n), b) + \frac{1}{r_n}\langle b - S_r^{f+g}(x_n), J^p_{E_1}(S_r^{f+g}(x_n)) \\ -J^p_{E_1}(x_n)\rangle \geq 0 \ \forall b \in E_1 \\ f(S_r^{f'+g'}(Ax_n), b) + g(S_r^{f'+g'}(Ax_n), b) + \frac{1}{r_n}\langle b - S_r^{f'+g'}(Ax_n), J^p_{E_2}(S_r^{f'+g'}(Ax_n)) \\ -J^p_{E_2}(Ax_n)\rangle \geq 0 \ \forall \ b \in E_2 \\ z_n = J^q_{E_1^*}(J^p_{E_1}(x_n) - \lambda A^*(J^p_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n)), \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(z_n) + (1 - \alpha_n) J^p_{E_1}(S_r^{f+g}(z_n)), \\ C_{n+1} = \{v \in C_n : \triangle_p(y_n, v) \leq \triangle_p(z_n, v) \leq \triangle_p(x_n, v)\} \\ x_{n+1} = \prod^p_{C_{n+1}}(x_1), n \geq 1, \end{cases} \tag{79}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|} \frac{1}{\|J^p_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n\|}, \ x_n \neq 0 \\ \frac{1}{\|A\|^p} \frac{\langle J^p_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n), A x_n - \Pi^p_{AC} S^{f'+g'}_{r_n} A x_n\rangle^{p-1}}{\|J^p_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n\|^p}, \ x_n = 0 \end{cases}$$

and $\gamma \in (0, 1)$ and $\tau_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\tau_n) = \frac{\gamma}{2^q G_q \|A\|} \frac{\langle J_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n, A x_n - \Pi^p_{AC} S^{f'+g'}_{r_n} A x_n\rangle}{\|x_n\|^p \|J^p_{E_2}(I - \Pi^p_{AC} S^{f'+g'}_{r_n}) A x_n\|}$$

and A^* denote the adjoint of A, and $\{r_n\} \subset (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $n \geq 1$. If $\omega = \{v \in MEP(f, g) : Av \in MEP(\Pi^p_{AC}, f', g')\} \neq \phi$, then $\{x_n\}$ converges strongly to $x^* \in \omega$.

5. An Example

In this section we discuss how to apply Theorem 3.1 on the following example.

Let $E_1 = E_2 = C_1 = (-\infty, \infty)$, $A = A^* = I$, $p = q = 2$, and $\alpha_n = \frac{1}{2^n}$. Define

$$U : E_1 \longrightarrow E_1 \text{ by } U(x) = \begin{cases} x, x \in (-\infty, 0] = B_1 \\ \frac{1}{2}x, x \in (0, \infty) = B_2, \end{cases}$$

$$A : E_1 \longrightarrow E_2 \text{ by } A(x) = x, x \in (-\infty, \infty),$$

$$T : E_2 \longrightarrow E_2 \; by \; T(x) = \begin{cases} x, x \in (-\infty, 0] = D_1 \\ \frac{1}{4}x, x \in (0, \infty) = D_2, \end{cases}$$

$$\Pi_C^2 : E_2 \longrightarrow C \; by \; \Pi_C^2(x) = \begin{cases} 0, x \in (-\infty, 0) \\ x, x \in [0, \infty), \end{cases}$$

$$\Pi_C^2 T : E_2 \longrightarrow C \; by \; \Pi_C^2 T(x) = \begin{cases} x, x \in \{0\} \\ 0, x \in (-\infty, 0) \\ \frac{1}{4}x, x \in (0, \infty). \end{cases}$$

Now we check whether or not the mappings U and T are uniformly continuous Bregman generalized asymptotically nonexpansive. To do this it suffices to check for each map the following four cases.

Case I: if $x, y \in B_1$ then

$$\triangle_2(U^n(x), U^n(y)) = \triangle_2(x, y) \leq [\frac{1}{2^n} + 1] \triangle_2 (x, y) + \frac{1}{2^n}; \quad (80)$$

Case II: if $x, y \in B_2$ then

$$\begin{aligned} \triangle_2(U^n(x), U^n(y)) &= \triangle_2 (\frac{1}{2^n}x, \frac{1}{2^n}y) \\ &= \frac{1}{2^{2n}} \triangle_2 (x, y) \leq [\frac{1}{2^n} + 1] \triangle_2 (x, y) + \frac{1}{2^n}; \end{aligned} \quad (81)$$

Case III: if $x \in B_2$ and $y \in B_1$, then

$$\begin{aligned} \triangle_2(U^n(x), U^n(y)) &= \triangle_2(\frac{1}{2^n}x, y) = \frac{1}{2^{2n}}\frac{1}{2}|x|^2 - \frac{1}{2^n}\langle x, y\rangle + \frac{1}{2}|y|^2 \\ &\leq \frac{1}{2}|x|^2 - \langle x, y\rangle + \frac{1}{2}|y|^2 = \triangle_2(x, y) \leq [\frac{1}{2^n} + 1] \triangle_2 (x, y) + \frac{1}{2^n}; \end{aligned} \quad (82)$$

Case IV: if $x \in B_1$ and $y \in B_2$, then

$$\begin{aligned} \triangle_p(U^n(x), U^n(y)) &= \triangle_2(x, \frac{1}{2^n}y) \\ &= \frac{1}{2}|x|^2 - \frac{1}{2^n}\langle x, y\rangle + \frac{1}{2}\frac{1}{2^{2n}}|y|^2 \\ &\leq \frac{1}{2}|x|^2 - \langle x, y\rangle + \frac{1}{2}|y|^2 = \triangle_2(x, y) \leq [\frac{1}{2^n} + 1] \triangle_2 (x, y) + \frac{1}{2^n}. \end{aligned} \quad (83)$$

Since it is clear that U is uniformly continuous, then from (80), (81), (82) and (83) U is a uniformly continuous Bregman generalized asymptotically nonexpansive mapping. Similarly, it is also clear that T is a uniformly continuous Bregman generalized asymptotically nonexpansive mapping. Next, we simplify the scheme as follows.

Clearly $Fix(U) = (-\infty, 0]$, $Fix(T) = (-\infty, 0]$ and $Fix(\Pi_C^2 T) = \{0\}$ such that $\Gamma = \{x \in Fix(U); Ax \in Fix(\Pi_C^2 T)\} = \{0\}$. Now we have that

$$\lambda_n = \begin{cases} \frac{1}{|x_n|}, x_n \in (-\infty, 0) \\ \frac{4}{3x_n}, x_n \in (0, \infty) \\ 1, x_n \in \{0\}. \end{cases}$$

$$z_n = \begin{cases} x_n + 1, x_n \in (-\infty, 0) \\ x_n - 1, x_n \in (0, \infty) \\ 0, x_n \in \{0\}, \end{cases}$$

$$y_n = \begin{cases} x_n + 1, x_n \in (-\infty, 0) \\ (\frac{2}{2^n} - \frac{1}{2^{2n}})(x_n - 1), x_n \in (0, \infty) \\ 0, x_n \in \{0\}, \end{cases}$$

$$C_{n+1} = \{v \in C_n; \begin{cases} \frac{1}{2}|x_n + 1 - v|^2 \leq [\frac{1}{2^n} + 1]\frac{1}{2}|x_n + 1 - v|^2 + \frac{1}{2^n}; \\ \frac{1}{2}|x_n + 1 - v|^2 \leq \frac{1}{2}|x_n - v|^2, x_n \in (-\infty, 0) \\ \\ \frac{1}{2}|(\frac{2}{2^n} - \frac{1}{2^{2n}})(x_n - 1) - v|^2 \leq [\frac{1}{2^n} + 1]\frac{1}{2}|x_n - 1 - v|^2 + \frac{1}{2^n}; \\ \frac{1}{2}|x_n - 1 - v|^2 \leq \frac{1}{2}|x_n - v|^2, x_n \in (0, \infty) \\ \\ \frac{1}{2}|0 - v|^2 \leq [\frac{1}{2^n} + 1]\frac{1}{2}|0 - v|^2 + \frac{1}{2^n}, x_n \in \{0\}\}, \end{cases}$$

$$x_{n+1} = \Pi_{C_{n+1}}^2(x_1), n \geq 1.$$

Now, take the initial point $x_1 = 0.5$ from positive real numbers, the numerical experiment result is given below;

$z_1 = -0.5$; $y_1 = -0.375$; $C_2 = (-\infty, 0]$; $x_2 = 0$; $z_2 = 0$; $y_2 = 0$; $x_3 = x_2 = 0$.

This implies that zero is in the solution set Γ. Similarly, from the negative real numbers, let $x_1 = -0.5$ then we arrive at the same fixed point zero as follows;

$z_1 = 0.5$; $y_1 = 0.5$; $C_2 = [0, \infty)$; $x_2 = 0$; $z_2 = 0$; $y_2 = 0$; $x_3 = x_2 = 0$.

6. Conclusion

In summary, the validity of the proposed scheme is provided by step one and two in the manuscript. Strong convergence of the proposed sequence is achieved in step three. Approximate fixed point sequence of the proposed mappings is given by step four. Application of the demicloseness principle for the proposed mappings is discussed in step five. Furthermore, in this paper, the strong convergence for SCFPP of Bregman generalized asymptotically nonexpansive mapping is obtained in uniformly convex and uniformly smooth Banach spaces, without the assumption of semi-compactness property and or without the assumption of Opial condition. This infer that, the main result presented here generalizes and extends that of Zhang *et al.* [10] and the references therein. This is true by the fact that the map presented here is more general than asymptotically nonexpansive map which was used by Zhang et. al. So also the space used here generalizes the one applied in the work of Zhang etal. Moreso, the notion of Bregman distance used here is the general case of the classical norm distance used in the result of Zhang *et al.*

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