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The Derivation of the Riemann Analytic Continuation Formula from the Euler's Quadratic Equation

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Abstract

The analysis of the derivation of the Riemann Analytic Continuation Formula from Euler's Quadratic Equation is presented in this paper. The connections between the roots of Euler's quadratic equation and the Analytic Continuation Formula of the Riemann Zeta equation are also considered. The method of partial summation is applied twice on the resulting series, thus leading to the Riemann Analytic Continuation Formula. A polynomial approach is anticipated to prove the Riemann hypothesis; thus, a general equation for the zeros of the Analytic Continuation Formula of the Riemann Zeta equation based on a polynomial function is also obtained. An expression in Terms of Prime numbers and their products is considered and obtained. A quadratic function, $G(t_n)$, that is required for Euler's quadratic equation (EQE) to give the Analytic Continuation Formula of the Riemann Zeta equation (ACF) is presented. This function thus allows a new way of defining the Analytic Continuation Formula of the Riemann Zeta equation (ACF) via this equivalent equation. By and large, the Riemann Zeta function is shown to be a type of L function whose solutions are connected to some algebraic functions. These algebraic functions are shown and presented to be connected to some polynomials. These Polynomials are also shown to be some of the algebraic functions' solutions. Conclusively, $\varsigma(z)$ is redefined as the product of a new function which is called $H(t_n, z)$ and this new function is shown to be dependent on the polynomial function, $G(t_n)$.

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1. Introduction

Many authors have recently presented some polynomial approaches to prove the Riemann Hypothesis [1-9]. Some of the proofs were based on Jensen polynomials, Laguerre polynomials as Jensen polynomials of Laguerre–Pólyaentire functions, and some worked on a general theorem which models such polynomials by Hermite polynomials. The authors presented an Approximation to Zeros of the Riemann Zeta Function using Fractional Calculus [10]. This allowed the authors to construct a fractional iterative method to nd the zeros of functions in which it is possible to avoid expressions that involve hypergeometric functions, Mittag-Leffler functions or infinite series [10], to mention a few.

As good as their works are, seeking a clearer insight into these possible polynomials is expedient, knowing fully that solutions to most difficult problems may not necessarily be complex. Significantly, it is known that the Riemann Zeta function

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is a type of L function whose solutions are connected to some algebraic functions, and polynomials are also types of solutions of algebraic functions. A polynomial approach can be anticipated to prove the Riemann Hypothesis.

This study aims to present the link that connects Euler's Quadratic Equation (EQE) and the Analytic Continuation Formula of the Riemann Zeta equation (ACF) by using the partial summation on a generalized polynomial function of Euler's Quadratic Equation. (EQE).

The remaining part of this paper is organized as follows: Section Two is the Materials and Methods. The Derivation of the Analytic Continuation Formula of the Riemann Zeta equation (ACF) from Euler's Quadratic Equation (EQE) is presented in Section Three. Section Four considers the obtained Expression in Terms of Prime Numbers and Their Products. While Section Five is for obtaining a quadratic function; $G(t_n)$, in terms of the Analytic Continuation Formula of the Riemann Zeta equation (ACF). Section six presents a new way of defining the Analytic Continuation Formula of the Riemann Zeta equation (ACF) via an equivalent equation. The Concluding remark is presented in section seven.

2. Materials and Methods

The following equations and method shall be used in the derivation of the Analytic Continuation Formula of the Riemann Zeta equation (ACF) from the Euler's quadratic equation (EQE) [11]:

$$\prod\left(\frac{z}{2}\right) = -\frac{z}{2}\Gamma\left(\frac{z}{2}\right) = \sum_{n\geq 1}^{\infty}\left(\frac{2n+z}{2n}\right) \tag{1}$$

$$n^z = \prod_{i=1}^{m} P_i^{\propto_i z} \tag{2}$$

From the expression [14-15],

$$\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{\mathrm{nz}}} = -1 \tag{3}$$

The method of partial summation as found in literature [14-15] is given as:

$$\sum_{P\leq x}\log^2 p + \sum_{\mathrm{pq}\leq x}\log p\log q = 2x\log x + \vartheta(x) \tag{4}$$

By partial summation, we get from Equation (5)

$$\sum_{p\leq x}\log p + \sum_{\mathrm{pq}\leq x}\frac{\log p\log q}{\log \mathrm{pq}} = 2x + \vartheta\left(\frac{x}{\log x}\right), \tag{5}$$

the above Equation (5) gives Equation (6)

$$\sum_{\mathrm{pq}\leq x}\log p\log q = \sum_{p\leq x}\log p\sum_{q\leq x/p}\log q \tag{6}$$

Or

$$=2x\sum_{p\leq x}\frac{\log p}{p} - \sum_{p\leq x}\log p\sum_{\mathrm{qr}\leq x/p}\frac{\log q\log r}{\log \mathrm{qr}} + \vartheta\left(x\sum_{p\leq x}\frac{\log p}{p\left(1+\log\frac{x}{p}\right)}\right) \tag{7}$$

$$= 2x\log x - \sum_{\mathrm{qr}\leq x/p}\frac{\log q\log r}{\log \mathrm{qr}}\vartheta\left(\frac{x}{\mathrm{pq}}\right) + o\left(x\log\log x\right) \tag{8}$$

These equations above shall be used to derive the analytic continuation formula of the Riemann Zeta Function from Euler's Quadratic Equation. By this derivation, it will be readily obvious to see through into the generation of the zeros of the Riemann Zeta Function.

3. Derivation of the ACF from EQE

The polynomial function in (1) was always discovered to be prime for $1 \leq x \leq 40$ [12]. It was from this quadratic equation that Euler obtained the first few prime numbers [13].

$$1 \leq x \leq 40: \qquad x^2 + x + 41 \tag{9}$$

Another known fact is that Equation 10 is also a prime for $x = 1\ldots,39$.

$$x^2 - x + 41 \tag{10}$$

Interestingly, the roots of (9, 10) are $x = -0.5 \pm 6.3836i$ and $x = 0.5 \pm 6.3836i$ respectively, which are the similitude of the non-trivial zeros of the Riemann zeta function. With these similarities [16], the question that readily comes to mind is; what is the possibility of using the structure of this Euler's equation to obtain ACF if the coefficients of x^2 and x are taken as k and for the constant integral, 41, is replaced with $G(t_n)$? To seek an answer to this question, a transformation of the Euler's equation, when it is multiplied by another linear equation whose roots will are $-2n : n = 1, 2, 3, \ldots$, will be

$$\zeta_E(z) = \left(\mathrm{k}z^2 - kz + G(t_n)\right)(z + 2n) \tag{11}$$

The roots of this polynomial will be the same as the trivial and the non-trivial zeros of the Riemann Zeta function under certain conditions that $G(t_n)$ is known. It has been shown that there are Meromorphic functions that are equivalent to the Riemann zeta function [6-7], and they are given as:

$$\zeta_E(z) = \frac{(z+2n)}{(z-1)}\left(kz^2 - \text{ kz} + G(n)\right); k = 4 \tag{12}$$

Or

$$\zeta_E(z) = \frac{(z+2n)}{(e^{z-1}-1)}\left(kz^2 - \text{ kz} + G(n)\right); k = 4 \tag{13}$$

Provided that $G(n)$ is also known.

The Equation (12) and Equation (13) are transformed into matrices whose Eigenvalues are the trivial and non-trivial spectral points of the Riemann zeta function [5-7, 13] provided that;

$$G(n) = 800.162 + 968.548J(n) \tag{14}$$

Or

$$G(n) = 800.162 + 968.548n^{\nu(n)} \tag{15}$$

Or

$$G(t_n) = 1 + kt^2{}_n \text{ where } k = 4 \tag{16}$$

Such that $G(t_n) = G(n)$.

From 11, let

$$\zeta_E(z) = \left(kz^2 - \text{kz} + G(t_n)\right)(z+2n) \tag{17}$$

Riemann gave the following expression in his work [11]:

$$-\prod\left(\frac{z}{2}\right) = \sum_{n\geq 1}^{\infty}\left(1+\frac{z}{2n}\right) \tag{18}$$

which is the same as $\frac{z}{2}\Gamma\left(\frac{z}{2}\right)$, By taking

$$\prod\left(\frac{z}{2}\right) = -\frac{z}{2}\Gamma\left(\frac{z}{2}\right) = \sum_{n\geq 1}^{\infty}\left(\frac{2n+z}{2n}\right), \tag{19}$$

Equation (17) can written as

$$\zeta_E(z) = 2n\left(1+\frac{z}{2n}\right)\left(kz^2 - \text{kz} + G(t_n)\right) \tag{20}$$

Using the method of discretization, Equation (20) becomes;

$$\gamma(z) = \sum_{n\geq 1}^{\infty}(\zeta_E(z)) \tag{21}$$

$$= \sum_{n\geq 1}^{\infty}\left[2n\left(1+\frac{z}{2n}\right)\left(kz^2 - \text{kz} + G(t_n)\right)\right] \tag{22}$$

Thus applying the method of partial summation in [15],as in Equation (6) the resulting equation from Equation (22) shall be

$\sum_{\text{dq}\leq n}\left[2n\left(1+\frac{z}{2n}\right)\left(\text{kz}^2 - \text{kz} + G(t_n)\right)\right]$

$$= \sum_{d\leq n} 2n\left[kz^2 - kz + G(t_n)\right]\sum_{q\leq n/d}(1+\frac{z}{2n}), \tag{23}$$

Where $d = 2n\left[kz^2 - kz + G(t_n)\right]$ and $q = 1 + \frac{z}{2n}$

Eventually, Equation (23) can be written as:

$$\gamma(z) = \sum_{n\geq 1}^{\infty}(\zeta_E(z)) \tag{24}$$

$$= \sum_{d\leq n}\left[2n\left(\text{kz}^2 - \text{kz} + G(t_n)\right)\right]\sum_{q\leq n/d}\left(1+\frac{z}{2n}\right) \tag{25}$$

Interestingly, since

$$-\frac{z}{2}\Gamma(\frac{z}{2}) = \sum_{n\geq 1}^{\infty}\left(\frac{2n+z}{2n}\right) \tag{26}$$

25 becomes:

$$\gamma(z) = \sum_{n\geq 1}^{\infty}(\zeta_E(z))$$

$$= -\frac{z}{2}\Gamma\left(\frac{z}{2}\right)(z-1)\left[\sum_{d\leq n} 2n\left(\text{kz} + \frac{G(t_n)}{(z-1)}\right)\right] \tag{27}$$

With a little rearrangement and the introduction of $\pi^{\frac{-z}{2}}\pi^{\frac{z}{2}} = 1$, Equation (27) is the same as:

$$\gamma(z) = \varnothing(z)\left(\sum_{d\leq n} 2n\left[\left(kz + \frac{G(t_n)}{(z-1)}\right)\pi^{\frac{z}{2}}\right]\right) \tag{28}$$

where $\varnothing(z) = -\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma(\frac{z}{2})$

If the principle of partial summation is again applied on the series in Equation (28), Equation (29) we will obtain:

$$\left(\sum_{rb\leq n} 2n\left[\left(kz + \frac{G(t_n)}{(z-1)}\right)\pi^{\frac{z}{2}}\right]\right)$$

$$= \left(\pi^{\frac{z}{2}}\sum_{r\leq n}\left(\text{kz} + \frac{G(t_n)}{(z-1)}\right)\sum_{b\leq\frac{n}{r}} 2n\right); \text{ rb} = d \tag{29}$$

By this, (28) becomes:

$$\gamma(z) = \varnothing(z)\left(\pi^{\frac{z}{2}}\sum_{r\leq n}\left(\text{kz} + \frac{G(t_n)}{(z-1)}\right)\sum_{b\leq\frac{n}{r}} 2n\right) \tag{30}$$

where $r = (kz + \frac{G(t_n}{(z-1)})\pi^{\frac{z}{2}}$ and $b = 2n$.

(30) shall be used subsequently in this paper.

4. An Expression in Terms of Prime Numbers and Their Products

A non-conventional expression for −1 [12-13] is;

$$\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}=-1, \tag{31}$$

(31) allows us to write (28) as:

$$\gamma(z)=-\varnothing(z)\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\sum_{d\leq n}2nF(t,z) \tag{32}$$

$$F(t,z)=\left[\left(kz+\frac{G(t_n)}{(z-1)}\right)\pi^{\frac{z}{2}}\right]$$

Pleasantly (32) gives:

$$\gamma(z)\left[\left(\frac{1}{p^z}-1\right)\pi^{\frac{z}{2}}\sum_{d\leq n}2n\left[F(t,z)\right]\right]^{-1}$$
$$=\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}. \tag{33}$$

Intuitively, if (33) is multiplied over prime numbers one comes to;

$$\gamma(z)\prod_p\left[\left(\frac{1}{p^z}-1\right)\sum_{d\leq n}\left[2nF(t,z)\right]\right]^{-1}$$
$$=\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\prod_p\sum_{n\geq1}^{\infty}\frac{1}{p^{nz}}. \tag{34}$$

Conclusively, the RHS of (34) is the same as the Analytic Continuation formula of the Riemann Zeta function, while the LHS is an equivalence of the RHS.

5. Obtaining $G(t_n)$ in Terms of the ACF

Riemann defines $\varepsilon(z)$ [11] as ;

$$\varepsilon(z)=\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{35}$$

By this, $\zeta(z)$ can be represented as:

$$\zeta(z)=\frac{2\varepsilon(z)\pi^{\frac{z}{2}}}{z(z-1)\Gamma\left(\frac{z}{2}\right)} \tag{36}$$

For the LHS of (34) to be equal to (34), we represent ε_e in Equation (37) as

$$\varepsilon_e=\gamma(z)\left[\prod_p\left(\frac{1}{p^z}-1\right)\sum_{d\leq n}^{\infty}2nF(t,z)\right]^{-1} \tag{37}$$

Using (29) on (34), we obtain:

$$\varepsilon_e=\gamma(z)\left[J(n)\prod_p\left(\frac{1}{p^z}-1\right)\sum_{r\leq n}\prod_p F(t,z)\right]^{-1} \tag{38}$$

such that

$$J(n)=\sum_{b\leq\frac{n}{r}}\prod_p 2n$$

and

$$\varepsilon_e=\gamma(z)\pi^{-\frac{z}{2}}B(t,z)\prod_p\left(\frac{1}{p^z}-1\right)^{-1} \tag{39}$$

where

$$B(t,z)=\prod_p\left[\sum_{r\leq n}\left(kz+\frac{G(t_n)}{(z-1)}\right)\sum_{b\leq\frac{n}{r}}2n\right]^{-1}$$

Gives:

$$\varepsilon_e=\gamma(z)\pi^{-\frac{z}{2}}\zeta(z)B(t,z). \tag{40}$$

By making the series containing $G(t_n)$ the subject of the expression in (38), we obtain:

$$\sum_{r\leq n}\prod_p\left(kz+\frac{G(t_n)}{(z-1)}\right)$$
$$=\frac{\gamma(z)}{\varepsilon_e}\left[\pi^{\frac{z}{2}}\sum_{b\leq\frac{n}{r}}\prod_p 2n\right]^{-1}\left[\prod_p\left(\frac{1}{p^z}-1\right)\right]^{-1} \tag{41}$$

followed by Further simplification shown that (41) becomes:

$$\sum_{r\leq n}\prod_p\left(kz+\frac{G(t_n)}{(z-1)}\right)$$
$$=\frac{\gamma(z)}{\varepsilon_e}\zeta(z)\left[\pi^{\frac{z}{2}}\sum_{b\leq\frac{n}{r}}\prod_p 2n\right]^{-1} \tag{42}$$

Since

$$\zeta(z)=\left[\prod_p\left(\frac{1}{p^z}-1\right)\right]^{-1}. \tag{43}$$

It can be shown in existing works [1-2] that;

$$n^z=\prod_{i=1}^{m}P_i^{\propto_i z} \tag{44}$$

implies that

$$\sum_{d\leq n} 2n = 2\sum_{d\leq n}\prod_{i=1}^{m} P_i^{\propto_i} \tag{45}$$

With this, (42) can be expressed as (45), in terms of prime numbers such that:

$$\sum_{r\leq n}\prod_{p}\left(kz+\frac{G(t_n)}{(z-1)}\right) = \frac{\gamma(z)}{\varepsilon_e}\zeta(z)\pi^{-\frac{z}{2}}\sum_{b\leq\frac{n}{r}}\prod_{p}\left(\frac{1}{2}\prod_{i=1}^{m}P_i^{-\propto_i}\right) \tag{46}$$

6. Defining the ACF via (34)

If the LHS of (34) is written as:

$$\varepsilon_e = \gamma(z)\prod_{p}\left[\left(\frac{1}{p^z}-1\right)\mathrm{H}(t,z)\right]^{-1} \tag{47}$$

where

$$H(t,z) = \pi^{\frac{z}{2}}\sum_{d\leq n} 2n\left(kz+\frac{G(t_n)}{(z-1)}\right)$$

Then we can write (35) as the RHS of (34) such that;

$$\varepsilon_e = \frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\prod_{p}\sum_{n\geq1}^{\infty}\frac{1}{p^{\mathrm{nz}}} \tag{48}$$

Where $\epsilon(z) = \varepsilon_e$ and From the Nachlass of Riemann [15], $\varepsilon(z)$ is also defined as;

$$\varepsilon(z) = \frac{1}{2}+\frac{z}{2}(z-1)D(x,z) \tag{49}$$

where

$$D(x,z) = \int_1^{\infty}\psi(x)\left(x^{\frac{z}{2}-1}+x^{-\frac{(z+1)}{2}}\right)dx \tag{50}$$

and

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi\mathrm{x}} \tag{51}$$

Such that if $\varepsilon(z) = \varepsilon_e$ then;

$$\frac{1}{2}+\frac{z}{2}(z-1)D(x,z) = \frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\prod_{p}\sum_{n\geq1}^{\infty}\frac{1}{p^{\mathrm{nz}}} \tag{52}$$

Or as;

$$\frac{1}{2}+\frac{z}{2}(z-1)D(x,z) = \gamma(z)\prod_{p}\left[\left(\frac{1}{p^z}-1\right)H(t,z)\right]^{-1} \tag{53}$$

Since

$$H(t,z) = \pi^{\frac{z}{2}}\sum_{d\leq n}\left[2n\left(kz+\frac{G(t_n)}{(z-1)}\right)\right]$$

The evaluation of the intergrades in (52 and 53) will give;

$$D(x,z) = 2N(z) \tag{54}$$

Where

$$N(z) = \sum_{n=1}^{\infty}\left[e^{-n^2\pi}\left(\frac{1^{\left(\frac{z}{2}-1\right)}}{(2n^2\pi+z-2)}+\frac{1^{\left(\frac{z+1}{2}\right)}}{(2n^2\pi-z-1)}\right)\right] \tag{55}$$

With (52), we can write (52) and (53) as follows;

$$\frac{1}{2}+z(z-1)N(z) = \frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\prod_{p}\sum_{n\geq1}^{\infty}\frac{1}{p^{\mathrm{nz}}} \tag{56}$$

and

$$\frac{1}{2}+z(z-1)N(z) = \gamma(z)\prod_{p}\left[\left(\frac{1}{p^z}-1\right)H(t,z)\right]^{-1} \tag{57}$$

(53) and (54) now give new definitions of the $\zeta(z)$ as:

$$\left[\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\right]^{-1}\left[\frac{1}{2}+z(z-1)N(z)\right] = \prod_{p}\sum_{n\geq1}^{\infty}\frac{1}{p^{\mathrm{nz}}} \tag{58}$$

and

$$\gamma(z)^{-1}\prod_{p}H(t,z)\left[\frac{1}{2}+z(z-1)N(z)\right] = \prod_{p}\left[\left(\frac{1}{p^z}-1\right)\right]^{-1} \tag{59}$$

Equating (56) and (57), (60) is obtained:

$$\left[\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\right]^{-1}\left[\frac{1}{2}+z(z-1)N(z)\right]$$
$$=\left[\gamma(z)^{-1}\prod_{p}H(t,z)\right]\left[\frac{1}{2}+z(z-1)N(z)\right] \quad (60)$$

From (60), $\gamma(z)$ is obtained (61) :

$$\gamma(z)=\left[\frac{z}{2}(z-1)\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\right]\prod_{p}H(t,z) \quad (61)$$

such that $H(t,z)$ is represented in (62

$$H(t,z)=\pi^{\frac{z}{2}}\sum_{d\leq n}\left[2n\left(kz+\frac{G(t_n)}{(z-1)}\right)\right] \quad (62)$$

It can be written as Equation (29).

7. Conclusion

With the contents of $G(t_n)$,in (13 and 14): $G(t_n)=1+kt^2{}_n$ *wh*ere $k=4$, (61) and (62) can be implemented to give the same results and zeros of the Analytic Continuation formula and that of the Riemann Zeta function. It has been shown that the Analytic Continuation formula of the Riemann Zeta function can be obtained from Euler's Quadratic Equation. Riemann Zeta Function can be written as (63) provided $G(t_n)$ holds as defined in (58) and (59).

$$\zeta(z)=\prod_{p}H(t,z)$$
$$=\prod_{p}\sum_{d\leq n}\left[2n\pi^{\frac{z}{2}}\left(kz+\frac{G(t_n)}{(z-1)}\right)\right] \quad (63)$$

Enoch obtained the following for the generation of the zeros of the Analytic Continuation formula of the Riemann Zeta function [5-7]:

$$G(t_n)=\frac{\mathrm{kz}(z-1)\sigma}{\tau-\vartheta} \quad (64)$$

Where

$$\sigma=\left(\frac{z}{2}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{\mathrm{nz}}}-1\right) \quad (65)$$

Such that;

$$\vartheta=\frac{z}{2}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{\mathrm{nz}}} \quad (66)$$

and

$$\tau=\left[\frac{1}{2(z-1)}+zN(z)\right]\prod_{p}\left(\frac{1}{p^z}-1\right) \quad (67)$$

He pointed out that (58) and (59) definition of $G(t_n)$ is the same as obtained in Equations (14), (15) and (16). The functions; J_n and v_n are written as polynomials of order two or three for this to be possible by the authors [8-9]. He was able to obtain a general equation for the zeros of the Analytic Continuation formula from Equation (16) as;

$$G(t_n)=1+kt^2{}_n \ ;k=4 \quad (68)$$

By which Equation (69) holds as :

$$t_n=\left(\frac{G(t_n)-1}{k}\right)^{1/2} \quad (69)$$

Again from:

$$1+kt^2{}_n=\frac{\mathrm{kz}(z-1)\sigma}{\tau-\vartheta} \ ;k=4 \quad (70)$$

Such that;

$$t_n=\pm\left(\frac{z(z-1)\sigma}{(\tau-\vartheta)}-\frac{1}{k}\right)^{\frac{1}{2}} \ ;k=4 \quad (71)$$

The k value can hold for any integer, depending on the pattern of choice.

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