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Odd Order Integrator with two Complex Functions Control Parameters for Solving Systems of Initial Value Problems

Opeyemi O. Enoch*, Cathrine O. Alakofa, Lukman O. Salaudeen

Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria

Abstract

In this study, a numerical integrator that is based on a nonlinear interpolant, for the local representation of the theoretical solution is presented. The resulting integrator aims to solve second and higher-order initial value problems as systems of first-order initial value problems. The method is designed to have two complex functions as control parameters. The control parameters may become real, depending on the nature of the second-order initial value problems to be solved. The generalization and properties of the scheme are also presented.

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1. Introduction

In recent years, numerous numerical solution methods have been employed and utilized for solving initial value problems. Initial value problems in ODEs of the form

$$y'' = f(x, y, y');\ y(x) = a,\ y'(x) = b \tag{1}$$

where f is continuous within the interval of integration, has received attention from many authors.

Some of the developed methods are Euler's method, Runge-Kutta's method, etc. Some researchers, in the likes of [1-3] have discussed in detail the possibility of solving higher-order IVPs by reducing them to the systems of lower-order initial value problems, commonly first-order. Researchers such as [4, 5] modelled a series of algorithms in their implementations needed to solve first-order initial value problems. [5] created a self-adjusting numerical integrator with an initial value switch inbuilt problems with points of discontinuity, the methodology employed did not require complex computational methods for navigating the points of discontinuity. Also, capable of restarting the integration after jumping a point of singularity. Many authors did not consider the possibilities of adopting their schemes in the case of many differential equations. Since we frequently encounter several differential equations in applications, we can represent such a scenario as a system of m simultaneous first-order equations in m dependent variables, such as $1y, 2y...my$.

We have an initial value problem of a first-order system if each of these variables meets a predetermined condition at the exact value of x, which we can denote as:

*Corresponding author tel. no: +234 7066466859
Email address: opeyemi.enoch@fuoye.edu.ng (Opeyemi O. Enoch)

$$\begin{aligned}
{}^{1}y' &= {}^{1}f(x,{}^{1}y,{}^{2}y,\ldots,{}^{m}y), \quad {}^{1}y(a) = {}^{1}\eta \\
{}^{2}y' &= {}^{2}f(x,{}^{1}y,{}^{2}y,\ldots,{}^{m}y), \quad {}^{2}y(a) = {}^{2}\eta \\
\vdots &= \quad \vdots \qquad\qquad \vdots \\
{}^{m}y' &= {}^{m}f(x,{}^{1}y,{}^{2}y,\ldots,{}^{m}y), \quad {}^{m}y(a) = {}^{m}\eta. \qquad (2)
\end{aligned}$$

If the ${}^{i}y, i = 1, 2, \ldots, m$ We have a multi-point boundary value problem if the following conditions hold for multiple initial values of x; if there are only two different initial values of x, we have a two-point boundary value problem..
We may write the initial value problem (2) in the form:

$$y' = f(x, y); \quad y(a) = \eta \qquad (3)$$

A need for Numerical Method
It is important to remember that some ordinary differential equations, like those used to simulate real-world issues, cannot be resolved analytically. However, numerical methods for solving ordinary differential equations were developed to provide approximate solutions to the ordinary differential equations to address this issue. We examine the prospect of closing this deficit in light of those mentioned above.

Solving Ordinary Differential Equations
The problem of solving ordinary differential equations can be divided into initial value, and boundary value problems—these two categories depend on the conditions specified at the domain's endpoints. We must be quick to note that there are several acceptable numerical approaches for addressing the issues of initial value problems in ordinary differential equations. Still, most of them lack control parameters that can handle points of discontinuity.

Literature has it that Leonhard Euler developed the ancestor of all numerical methods today between 1768 and 1770. Carl Runge and Martin Kutta described enhanced Euler's and Runge Kutta's methods in 1895 and 1905, respectively. We have excellent and exhaustive books on this which can be consulted, such as [4, 6-8].

As good as the methods developed by the authors, as mentioned earlier, are, most are still limited in themselves because they lack control parameters that can handle points of singularity nor permit flexible control when needed. Because of this, many researchers have sought to address initial value problems to obtain a more accurate solution by employing various methods, including the Euler and Runge Kutta methods.

In [9], the authors presented the Eulers method for solving initial value problems in ordinary differential equations. [10] discussed accurate solutions of initial value problems for ordinary differential equations with the fourth-order Runge Kutta method, while [11] discussed accuracy analysis of numerical solutions of initial value problems for ordinary differential equations. [12-14] also studied numerical solutions to initial value problems in ordinary differential equations using different numerical methods.

In [14,15] the authors have worked on a trigonometrically fitted scheme for approximating solutions of second-order ordinary differential equations with oscillatory solutions. [16] also derived a new numerical method of third-order Accuracy via the transcendental function of exponential type for the resolution of IVPs in ODEs and also analyzed the properties of the derived form.

Many researchers in the likes of [4, 6, 8, 17] have extensively discussed the approach of reducing higher-order ODEs to a system of lower-order, specifically, order one, and then applying various methods available for solving the resulting system of first-order IVPs. Results from different numerical simulations with stiff differential systems under the adaptive step-size approach are compared to an adaptive step-size version of the well-known Lobatto-IIIA methods. Two three-step Simpson's-type block methods were used based on the second derivative having sixth and eighth order [18]. The self-starting [19] method is studied using a variable step-size approach, which improves performance while requiring fewer steps and acceptable local errors. The approach is evaluated against a few fifth-order techniques already in use and has a logical structure.

The [20] approach is L-stable and accurate to the third order. It is demonstrated that the performance of the suggested approach is significantly enhanced when compared to other methods of the same nature and order when a variable step-size strategy is used. To solve initial value problems of second-order ordinary differential equations, [21] provides new two-step explicit symmetric P-stable methods, incorporating Obrechkoff and hybrid terms, of orders four and six. [22] examined the symplectic multilayer structures of the closed Newton-Cotes formulas. The developed trigonometrically-fitted symplectic approach. [23] Three-step hybrid block methods are derived for the solutions of linear. The derivation is completed with the collocation and interpolation approach and the use of power series as the basis function. The first and second three-step hybrid block approaches are developed by adding one or more off-grid points to the three-step integration interval. [24] demonstrates that defining a new integral using the Generalized Integral Transform is simple.

Although this approach has met with many contentions because of the increase in the number of equations resulting from this approach and has resolved to block methods as better methods, the computational resources required for implementing block methods are usually on the high side. And because of this, the time needed to run some block method algorithms is usually longer. These are some of the limitations of block methods. Other sources that motivated this work are in [26-33].

The remaining part of this paper is organized as follows; Section Two is the derivation of the new scheme, determination of the undetermined coefficients, and statement on some valuable assumptions. The computing of control parameters is presented in Section 2.1.

The analysis of the properties of the integrator is presented in Section Three. Section 3.1 considers the Accuracy of the scheme, while Sections 3.2-3.4 deal with the consistency, stability and convergency of the method. Section Four presents some illustrative examples, a discussion of results and the conclusion is presented in Section Five.

In this work, a one-step method is considered to solve systems of first-order ordinary differential equations.

2. Derivation of the New Scheme

We desire to derive a one-step integrator by assuming a perturbed interpolant of the form:

$$F(x) = P_m(x) + b\cos(Nx + A), \quad m > 0 \tag{4}$$

with

$$P_m(x) = \sum_{j=0}^{m} a_j x^j \tag{5}$$

Thus, written (4)

$$F(x) = \sum_{j=0}^{m} a_j x^j + b\cos(Nx + A), \quad m > 0 \tag{6}$$

where A and N are complex functions.

If we now define the function θ_x as $\theta(x) = Nx + A$, it implies that

$$\theta_n = \theta(x_n) = Nx_n + A \tag{7}$$

If we consider even order polynomials, then m=2 will give equation (8), such that:

$$F(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + b\cos\theta_n \tag{8}$$

$$b = \frac{f''(x,y)}{N^3 \sin\theta_n} = f''(x,y)N^{-3}\csc\theta_n \tag{9}$$

$$a_2 = \frac{1}{2}\left\{f'(x,y) + f''(x,y)N^{-1}\cot\theta_n\right\} \tag{10}$$

$$a_1 = f(x,y) - x_n\left\{f'(x,y) + f''(x,y)N^{-1}\cot\theta_n\right\} + f''(x,y)N^{-2} \tag{11}$$

If we choose m to be equal to 4, then equation (6) will give equation (12), such that:

$$F(x_n) = a_0 + a_1 x_n + a_2 x^2 + a_3 x_n^3 + a_4 x_n^4 + bcos\theta_n \tag{12}$$

By differentiating equation (12) five times, we determined the undetermined coefficients as follows:

$$b = \frac{-f^{(iv)}(x,y)}{N^5 \sin\theta_n} = -f^{(iv)}(x,y)N^{-5}\csc\theta_n \tag{13}$$

$$a_4 = \frac{1}{24}\left\{f'''(x,y) + f^{(iv)}(x,y)N^{-1}\cot\theta_n\right\} \tag{14}$$

$$a_3 = \frac{1}{6}\left\{f''(x,y) - x_n\left(f'''(x,y) + f^{(iv)}(x,y)N^{-1}\cot\theta_n\right) + f^{(iv)}(x,y)N^{-2}\right\} \tag{15}$$

$$a_2 = \frac{1}{2}\left\{f'(x,y) - x_n\left(f''(x,y) + f^{(iv)}(x,y)N^{-2}\right) + \frac{x_n^2}{2}\left(f'''(x,y) + f^{(iv)}(x,y)N^{-1}\cot\theta_n\right) - f^{(iv)}(x,y)N^{-3}\cot\theta_n\right\} \tag{16}$$

$$a_1 = f(x,y) - x_n\left(f'(x,y) - f^{(iv)}(x,y)N^{-3}\cot\theta_n\right) + \frac{x_n^2}{2}\left(f''(x,y) + f^{(iv)}(x,y)N^{-2}\right) - \frac{x_n^3}{6}\left(f'''(x,y) + f^{(iv)}(x,y)N^{-1}\cot\theta_n\right) - f^{(iv)}(x,y)N^{-4} \tag{17}$$

In the same vain, choosing m=6 in equation (6) will give:

$$F(x_n) = a_0 + a_1 x_n + a_2 x^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + bcos\theta_n \tag{18}$$

By following the same steps as before, we determined the undetermined coefficients in equation (18) as follows:

$$b = \frac{f^{(vi)}(x,y)}{N^7 \sin\theta_n} = f^{(vi)}(x,y)N^{-7}\csc\theta_n \tag{19}$$

$$a_6 = \frac{1}{720}\left\{f^{(v)}(x,y) + f^{(vi)}(x,y)N^{-1}\cot\theta_n\right\} \tag{20}$$

$$a_5 = \frac{1}{120}\left\{f^{(iv)}(x,y) - x_n\left(f^{(v)}(x,y) + f^{(vi)}(x,y)N^{-1}\cot\theta_n\right) + f^{(vi)}(x,y)N^{-2}\right\} \tag{21}$$

$$a_4 = \frac{1}{24}\left\{f'''(x,y) - x_n\left(f^{(iv)}(x,y) + f^{(vi)}(x,y)N^{-2}\right) + \frac{x_n^2}{2}\left(f^{(v)}(x,y) + f^{(vi)}(x,y)N^{-1}\cot\theta_n\right) - f^{(vi)}(x,y)N^{-3}\cot\theta_n\right\} \tag{22}$$

$$a_3 = \frac{1}{6}\left\{f''(x,y) - x_n\left(f'''(x,y) - f^{(vi)}(x,y)N^{-3}\cot\theta_n\right) + \frac{x_n^2}{2}\left(f^{(iv)}(x,y) + f^{(vi)}(x,y)N^{-2}\right) - \frac{x_n^3}{6}\left(f^{(v)}(x,y) + f^{(vi)}(x,y)N^{-1}\cot\theta_n\right) - f^{(vi)}(x,y)N^{-4}\right\} \tag{23}$$

$$a_2 = \frac{1}{2}\left\{f'(x,y) - x_n\left(f''(x,y) - f^{(vi)}(x,y)N^{-4}\right) + \frac{x_n^2}{2}\left(f'''(x,y) - f^{(vi)}(x,y)N^{3}\cot\theta_n\right) - \frac{x_n^3}{6}\left(f^{(iv)}(x,y) + f^{(vi)}(x,y)N^{2}\right) + \frac{x_n^4}{24}\left(f^{(v)}(x,y) + f^{(vi)}(x,y)N^{1}\cot\theta_n\right) + f^{(vi)}(x,y)N^{5}\cot\theta_n\right\} \tag{24}$$

$$a_1 = f(x,y) - x_n\left(f'(x,y) + f^{(vi)}(x,y)N^{-5}\cot\theta_n\right) + \frac{x_n^2}{2}\left(f''(x,y) - f^{(vi)}(x,y)N^{-4}\right) - \frac{x_n^3}{6}\left(f'''(x,y) - f^{(vi)}(x,y)N^{-3}\cot\theta_n\right) + \frac{x_n^4}{24}\left(f^{(iv)}(x,y) + f^{(vi)}(x,y)N^{-2}\right) - \frac{x_n^5}{120}\left(f^{(v)}(x,y) + f^{(vi)}(x,y)N^{-1}\cot\theta_n\right) + f^{(vi)}(x,y)N^{-2} \quad (25)$$

Assumptions to be used in the Derivation

The followings are the assumptions:

(i) The interpolating function is made to coincide with the theoretical solution at $x = x_n$ and $x = x_{n+1}$.

$$F(x_n) = y(x_n)$$
$$F(x_{n+1}) = y(x_{n+1})$$

The condition above implies that

$$y(x_n) \cong y_n$$
$$y(x_{n+1}) \cong y_{n+1}$$
$$F(x_{n+1}) - F(x_n) \cong y_{n+1} - y_n \quad (26)$$

(ii) The derivatives of the interpolating function are also required to coincide with the differential equation as well as its first and second, up to the ith derivatives with respect to x at $x = x_n$.

(iii) We denote the ith total derivatives of $f(x,y)$ with respect to x as f^i, such that

$$F'(x_n) = f_n, F''(x_n) = f_n^{(1)}, F'''(x_n) = f_n'', \cdots$$

2.1. Implementation of the Assumptions

When the degree of the polynomial is 2, as obtained in equation (12), we have the one-step method:

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \left[\frac{h^3}{3!} + \frac{h^4}{4!}N\cot\theta_n - \frac{h^5}{5!}N^2\right]f_n^{(2)} \quad (27)$$

when the degree of the polynomial is 4, equation (28) is obtained as the one-step method

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \frac{h^3}{3!}f_n^{(2)} + \frac{h^4}{4!}f_n^{(3)} + \left[\frac{h^5}{5!} + \frac{h^6}{6!}N\cot\theta_n - \frac{h^7}{7!}N^2\right]f_n^{(4)} \quad (28)$$

We again derived equation (29) as the integrator when the degree of the polynomial is 6

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \frac{h^3}{3!}f_n^{(2)} + \frac{h^4}{4!}f_n^{(3)} + \frac{h^5}{5!}f_n^{(4)} + \frac{h^6}{6!}f_n^{(5)} + \left[\frac{h^7}{7!} + \frac{h^8}{8!}N\cot\theta_n - \frac{h^9}{9!}N^{(2)}\right]f_n^{(6)} \quad (29)$$

2.2. Generalization of the Derived Schemes

We observed a recursive trend among equations (27) to (29). This reclusiveness can thus be generalized for cases when the orders of the polynomials are even. Thus the generalization of the schemes is given as follows:

$$y_{n+1} = y_n + hf_n + \sum_{i=2}^{m}\frac{h^i}{i!}f^{i-1} + \left[\frac{h^{m+1}}{(m+1)!} + \frac{h^{m+2}}{(m+2)!}N\cot\theta_n - \frac{h^{m+3}}{(m+3)!}N^2\right]f^m \quad (30)$$

2.3. Computing for A and N

We desire to derive expressions for the complex control parameters by obtaining the sum difference between the numerical and theoretical solutions: the truncation error. On a general note, Taylor's series representation of the theoretical solution is used such that; when the degree of the polynomial is 2, $m = 2$, we have the truncation error as:

$$T_{n+1} = y(x_n + h) - y_{n+1}$$

By substituting the Taylor's series expression for $y(x_n + h)$, we obtained

$$T_{n+1} = \frac{h^4}{4!}f_n^{(3)} + \frac{h^5}{5!}f_n^{(4)} - \frac{h^4}{4!}f_n^{(2)}N\cot\theta_n + \frac{h^5}{5!}f_n^{(2)}N^2$$

For further simplification, we obtained

$$\theta_n = \cot^{-1}\left(\frac{-f_n^{(3)}}{Nf_n^{(2)}}\right) \quad (31)$$

This leads to:

$$A = \cot^{-1}\left(\frac{-f_n^{(3)}}{Nf_n^{(2)}}\right) - Nx_n \quad (32)$$

and:

$$N = \left[\frac{-f_n^{(4)}}{f_n^{(2)}}\right]^{\frac{1}{2}} \quad (33)$$

The generalization of the above components for all integrators with even order polynomials is given as:

$$A = \cot^{-1}\left(\frac{f_n^{(m+1)}}{Nf_n^{(m)}}\right) - Nx_n \quad (34)$$

$$N = \left[\frac{-f_n^{(m+2)}}{f_n^{(m)}}\right]^{\frac{1}{2}} \quad (35)$$

3. Analysis of the properties of the Scheme

3.1. Order of the Scheme

One way by which we can determine the accuracy of the schemes is by computing the order of the scheme [22]. We worked with the following assumptions:

(i) Consider the Taylor's series expansion for the exact solution $y(x)$ given by

$$y(x_n + h) = y(x_n) + \sum_{i=1}^{\infty} h^i y^i(x_n)$$

such that

$$y(x_n + h) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!}f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!}f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!}f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!}f^{(4)}(x_n, y(x_n)) + \frac{h^6}{6!}f^{(5)}(x_n, y(x_n)) + O(h^7) \quad (36)$$

(ii) Let the local truncation error be given as:

$$T_{n+1} = y(x_n + h) - y_{n+1} \quad (37)$$

The intention is to compute the order of the scheme for when $m = 2, 4$and 6, and then generalize. By substituting (27) and (36) into (37), we have

$$T_{n+1} = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!}f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!}f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!}f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!}f^{(4)}(x_n, y(x_n)) + O(h^5) - \left[y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \left(\frac{h^3}{3!} + \frac{h^4}{4!}N\cot\theta_n - \frac{h^5}{5!}N^2\right)f_n^{(2)}\right] \quad (38)$$

Resolving (38), we have

$$T_{n+1} = \frac{h^4}{4!}\left[f^{(3)}(x_n, y(x_n)) + \frac{h}{5}f^{(4)}(x_n, y(x_n)) - N\cot\theta_n f^{(2)} + \frac{h}{5}N^2 f^{(2)}\right] + O(h^5)h \quad (39)$$

Substituting (28) and (36) into (37) gives

$$T_{n+1} = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!}f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!}f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!}f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!}f^{(4)}(x_n, y(x_n)) + \frac{h^6}{6!}f^{(5)}(x_n, y(x_n)) + O(h^7) - \left[y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \frac{h^3}{3!}f_n^{(2)} + \frac{h^4}{4!}f_n^{(3)} + \left(\frac{h^5}{5!} + \frac{h^6}{6!}N\cot\theta_n - \frac{h^7}{7!}N^2\right)f_n^{(4)}\right] \quad (40)$$

Resolving, equation (40) gives

$$T_{n+1} = \frac{h^6}{6!}\left[f^{(5)}(x_n, y(x_n)) - N\cot\theta_n f^{(4)} + \frac{h}{7}N^2 f^{(4)}\right] + O(h^7)h \quad (41)$$

By substituting (29) and (36) into (37), we have

$$T_{n+1} = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!}f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!}f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!}f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!}f^{(4)}(x_n, y(x_n)) + \frac{h^6}{6!}f^{(5)}(x_n, y(x_n)) + \frac{h^7}{7!}f^{(6)}(x_n, y(x_n)) + \frac{h^8}{8!}f^{(7)}(x_n, y(x_n)) + O(h^9) - \left[y_n + hf_n + \frac{h^2}{2!}f_n^{(1)} + \left(\frac{h^3}{3!} + \frac{h^4}{4!}N\cot\theta_n - \frac{h^5}{5!}N^2\right)f_n^{(2)}\right] \quad (42)$$

From (39), the scheme is of order 5 when the order of the polynomial in our assumed solution was 4. Thus, it is good to note that when the order of the polynomial of the assumed solution is $2n$, the order of the resulting integrator shall be $2n + 1$, for $n = 1, 2, 3, \ldots$

3.2. Consistency of the Scheme

A one step numerical method is said to be consistent if it has at least order $p = 1$. It obvious from (39) that the scheme is consistent since it has a third order accuracy [8] and that

$$\lim_{h\to 0}\frac{T_{n+1}}{h} = \lim_{h\to 0}\left[\frac{\frac{h^4}{4!}\left[f^{(3)}(x_n, y(x_n)) + \frac{h}{5}f_n^{(4)} - N\cot\theta_n f_n^{(2)} + \frac{h}{5}N^2 f_n^{(2)}\right] + O(h^5)h}{h}\right] = 0$$

3.3. Stability of the Scheme

Numerical methods are said to be numerically stable if they are capable of damping out the small fluctuations carried out in input data. The notion of stability may be taken in different contexts: it may be associated with the specific numerical technique used, or the step size, h used in numerical computation or with the particular problem being solved.
For stability analysis of the proposed numerical integration method (27), one of the popular ways is to apply it to the problem

$$y' = -\beta y, \; y(0) = 1 \quad (43)$$

The exact solution is $y(x) = \exp(-\beta x)$, $\beta > 0$ Where β is in general, a complex constant. For the integration interval $[x_n, x_{n+1}]$, where $h = x_{n+1} - x_n$, the exact solution at the point $x = x_{n+1}$ is

$$y(x_{n+1}) = e^{-\beta x_{n+1}} = e^{-\beta x_n}\cdot e^{-\beta h} = y(x_n)\cdot e^{-\beta h} \quad (44)$$

Where h is defined as $(x_{n+1} - x_n = h)$. The numerical approximation obtained using the proposed method (27) gives

$$y_{n+1} = y_n + h(-\beta y_n) + \frac{h^2}{2!}(-\beta)^2 y_n + \left(1 + \frac{h}{4}N\cot\theta_n - \frac{h^2}{20}N^2\right)\frac{(-\beta)^3 y_n}{3!} = y_n\left[1 - \beta h + \frac{\beta^2 h^2}{2!} - \frac{\beta^3 h^3}{3!}\left(1 + \frac{h}{4}N\cot\theta_n - \frac{h^2}{20}N^2\right)\right] \quad (45)$$

setting $z = \beta h$ and $(1+\frac{h}{4}N\cot\theta_n - \frac{h^2}{20}N^2)$, equation (45) becomes

$$y_{n+1} = y_n\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!}\varphi\right] \quad (46)$$

Let

$$T = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!}\varphi$$

Hence,

$$y_{n+1} = Ty_n \quad (47)$$

Comparing (44) and (47), it is clearly seen that (46) consists of the first-three terms of the series expansion of $e^{-\beta h}$ Hence, the stability function of the scheme requires that

$$\| T \| < 1 \quad (48)$$

Hence, from (48), the proposed numerical integration method (27) is found to be conditionally stable.

3.4. Convergency of the Scheme

The proposed numerical scheme is said to have convergence for any given, well-posed initial value problem since it satisfies both consistency and stability properties as discussed above.

4. Implementation and Numerical Examples

To validate the efficiency and performance of the proposed numerical method, the method is applied to solve some systems of first-order ordinary differential equations. For each example, the absolute error of the approximate solutions is computed.

Problem 1 [17]

$$\begin{aligned} x' &= x - 4y \\ y' &= x + y \end{aligned}$$

$$x(0) = 2,\ y(0) = 3,\ h = 0.1$$

Exact solution:

$$\begin{aligned} x(t) &= -2\exp t + 4\cos^2 t \exp t - 12\sin t\cos t\exp t \\ y(t) &= -3\exp t + 6\cos^2 t\exp t + 2\sin t\cos t\exp t \end{aligned}$$

Problem 1 [25]

$$\begin{aligned} x' &= x - 4y \\ y' &= x + y \end{aligned}$$

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$$\begin{aligned} x(t) &= -2\exp t + 4\cos^2 t \exp t - 12\sin t\cos t\exp t \\ y(t) &= -3\exp t + 6\cos^2 t\exp t + 2\sin t\cos t\exp t \end{aligned}$$

Problem 2 [25]

$$\begin{aligned} x' &= x + 2y \\ y' &= 3x + 2y \end{aligned}$$

$$x(0) = 6,\ y(0) = 4,\ h = 0.1$$

Exact solution:

$$\begin{aligned} x(t) &= 4\exp 4t + 2\exp(-t) \\ y(t) &= 6\exp 4t - 2\exp(-t) \end{aligned}$$

Problem 3 [25]

$$\begin{aligned} x' &= 3x - y \\ y' &= 4x - y \end{aligned}$$

$$x(0) = 0.2,\ y(0) = 0.5,\ h = 0.1$$

Exact solution:

$$\begin{aligned} x(t) &= \frac{1}{5}\exp t - \frac{1}{10}\exp t \\ y(t) &= \frac{1}{2}\exp t - \frac{1}{5}t\exp t \end{aligned}$$

Problem 4 [26]

The Van Der Pol's Oscillator with unknown theoretical solutions this equation has its applications in many areas of human endeavours. It is interesting to see that the system can help to study human-induced autonomous climatic scenarios since it is made of the components of the vertical axis, which do not have mathematical theoretical solutions. However, these results of simultaneous human occurrences can be obtained numerically. And since secrets are kept in numbers, the numerical solutions are the doors to the mysteries.

$$\begin{aligned} y_1' &= y_2,\quad y_1(0) = 0 \\ y_2' &= 0.001(1 - y_1^2)y_2 - y_1,\quad y_2(0) = 1 \end{aligned}$$

5. Discussion of Result

In Tables 1 and 2, the two control parameters were activated for the first equation but were not active for the second equation. This is because of some inherent quantities of the two equations. The values of $A(x(t))$ for the first equation oscillates between 1.0593 and 0.6782, keeping $N(y(t))$ constant as 1.9036, for the first equation.

But not so for the second equation, for which $A(x(t))$ and $N(y(t))$ are zero all through. The order of the polynomial used for deriving the scheme is 2, which may be responsible for its inability to handle such.

Tables 3 and 4 shows that the two control parameters were used

Table 1. Tabular Result of Test Problem 1

t	X(t)	Y(t)	Exact X(t)	Exact Y(t)	A(X(t))	A(Y(t))	N(X(t))	N(Y(t))
0.1	2.0000	3.0000	2.0000	3.0000	1.0593	0.0000	1.9306	0.0000
0.2	0.8490	3.4690	0.8489	3.4690	0.8663	0.0000	1.9306	0.0000
0.3	−0.6038	3.8506	−0.6038	3.8506	0.6732	0.0000	1.9306	0.0000
0.4	−2.3450	4.1044	−2.3450	4.1044	0.48015	0.0000	1.9306	0.0000
0.5	−4.3423	4.1883	−4.3423	4.1883	0.2871	0.0000	1.9306	0.0000
0.6	−6.5425	4.0598	−6.5425	4.0598	0.0940	0.0000	1.9306	0.0000
0.7	−8.8692	3.6791	−8.8692	3.6791	−0.0990	0.0000	1.9306	0.0000
0.8	−11.2222	3.0113	−11.2222	3.0113	−0.2921	0.0000	1.9306	0.0000
0.9	−13.4775	2.0296	−13.4775	2.0296	−0.4852	0.0000	1.9306	0.0000
1.0	−15.4893	0.7188	−15.4893	0.7188	−0.6782	0.0000	1.9306	0.0000

Table 2. Error Table of Test Problem 1

Error(X(t))	Error(Y(t))	Error in [24](X(t))	Error in [24](y(t))
0.0000	0.0000	0.1299(−2)	0.3458(−3)
0.2319(−6)	0.6916(−6)	0.9816(−3)	0.3829(−3)
0.5258(−6)	0.7635(−6)	0.6239(−3)	0.4072(−3)
0.8790(−6)	0.8077(−6)	0.2224(−3)	0.4141(−3)
0.12851(−5)	0.8157(−6)	0.2291(−3)	0.3985(−3)
0.17334(−5)	0.7790(−6)	0.7276(−3)	0.3427(−3)
0.22088(−5)	0.6897(−6)	0.1264(−2)	0.2849(−3)
0.26910(−5)	0.5408(−6)	0.1826(−2)	0.1842(−3)
0.31554(−5)	0.3269(−6)	0.2395(−2)	0.4551(−4)
0.35727(−5)	0.4500(−7)	0.2952(−2)	0.1304(−3)

Table 3. Tabular Result of Test Problem 2

t	X(t)	Y(t)	Exact X(t)	Exact Y(t)	A(X(t))	A(Y(t))	N(X(t))	N(Y(t))
1.0	6.0000	4.0000	6.0000	4.0000	−0.5504	−3.6796	1.0359	3.3620
1.1	219.1283	326.8531	219.1284	326.8531	−0.6540	−4.0158	1.0359	3.3620
1.2	326.4692	488.0394	326.4692	488.0395	−0.7576	−4.3520	1.0359	3.3620
1.3	486.6440	728.4600	486.6441	728.4601	−0.8612	−4.6882	1.0359	3.3620
1.4	725.6339	1087.0882	725.6340	1087.0884	−0.9647	−5.0244	1.0359	3.3620
1.5	1082.1986	1622.0649	1082.1988	1622.0653	−1.0683	−5.3606	1.0359	3.3620
1.6	1614.1611	2420.1260	1614.1614	2420.1265	−1.1719	−5.6968	1.0359	3.3620
1.7	2407.7835	3610.6657	2407.7839	3610.6664	−1.2755	−6.0330	1.0359	3.3620
1.8	3591.7538	5386.7173	3591.7545	5386.7184	−1.3791	−6.3692	1.0359	3.3620
1.9	5358.0526	8036.2524	5358.0537	8036.2540	−1.4827	−6.7054	1.0359	3.3620
2.0	7993.0811	11988.8738	7993.0827	11988.8762	−1.5863	−7.0416	1.0359	3.3620

Table 4. Error Table of Test Problem 2

Error(X(t))	Error(Y(t))	Error in [24](X(t))	Error in [24](y(t))
0.0000	0.0000	0.5978(−2)	0.9146(−2)
0.4382(−4)	0.6531(−4)	0.1566(−1)	0.2370(−1)
0.6528(−4)	0.9747(−4)	0.3084(−1)	0.4648(−1)
0.9730(−4)	0.1455(−3)	0.5416(−1)	0.8146(−1)
0.1451(−3)	0.2171(−3)	0.8955(−1)	0.1345
0.2164(−3)	0.3240(−3)	0.1429	0.2144
0.3227(−3)	0.4836(−3)	0.2228	0.3343
0.4815(−3)	0.7216(−3)	0.3423	0.5135
0.7183(−3)	0.1077(−2)	0.5208	0.7810
0.1072(−2)	0.1606(−2)	0.7868	1.1799
0.1598(−2)	0.2397(−2)	1.1835	1.7746

Table 5. Tabular Result of Test Problem 3

t	X(t)	Y(t)	Exact X(t)	Exact Y(t)	A(X(t))	A(Y(t))	N(X(t))	N(Y(t))
1.0	0.2000	0.5000	0.2000	0.5000	0.0000	−0.6849	0.0000	1.7331
1.1	0.2718	0.8155	0.2718	0.8155	0.0000	−0.8582	0.0000	1.7331
1.2	0.2704	0.8412	0.2704	0.8412	0.0000	−1.0315	0.0000	1.7331
1.3	0.2656	0.8632	0.2656	0.8632	0.0000	−1.2048	0.0000	1.7331
1.4	0.2569	0.8806	0.2569	0.8806	0.0000	−1.3781	0.0000	1.7331
1.5	0.2433	0.8921	0.2433	0.8921	0.0000	−1.5514	0.0000	1.7331
1.6	0.2241	0.8963	0.2241	0.8963	0.0000	−1.7247	0.0000	1.7331
1.7	0.1981	0.8915	0.1981	0.8915	0.0000	−1.8980	0.0000	1.7331
1.8	0.1642	0.8758	0.1642	0.8758	0.0000	−2.0714	0.0000	1.7331
1.9	0.1210	0.8470	0.1210	0.8470	0.0000	−2.2447	0.0000	1.7331
2.0	0.0669	0.8023	0.0669	0.8023	0.0000	−2.4180	0.0000	1.7331

Table 6. Error Table of Test Problem 3

Error(X(t))	Error(Y(t))	Error in [24](X(t))	Error in [24](y(t))
0.0000	0.0000	0.4259(−3)	0.7547(−3)
0.4(−9)	0.38(−8)	0.4580(−3)	0.8164(−3)
0.21(−8)	0.7(−9)	0.4871(−3)	0.8734(−3)
0.49(−8)	0.57(−8)	0.5127(−3)	0.9250(−3)
0.79(−8)	0.111(−7)	0.5346(−3)	0.9702(−3)
0.113(−7)	0.171(−7)	0.5526(−3)	0.1008(−2)
0.149(−7)	0.236(−7)	0.5664(−3)	0.1038(−2)
0.188(−7)	0.306(−7)	0.5762(−3)	0.1058(−2)
0.230(−7)	0.381(−7)	0.5820(−3)	0.1067(−2)
0.275(−7)	0.462(−7)	0.5841(−3)	0.1063(−2)
0.324(−7)	0.549(−7)	0.5830(−3)	0.1045(−2)

Table 7. Tabular Result of Test Problem 4

	y_1			y_2		
x	$Krogh$[25]	$Scheme$	$G-A$[22]	$Krogh$[25]	$Scheme$	$G-A$[22]
0.0	0.00000000	0.00000000	0.00000000	1.00000000	1.00000000	1.00000000
0.6	0.56624448	0.56624448	0.56624449	0.83005702	0.83005702	0.83005701
1.2	0.93663469	0.93663469	0.93663469	0.36677933	0.36677933	0.36677931
1.8	0.98015691	0.98015691	0.98015691	−0.22611777	−0.22611779	−0.22611779
2.4	068113087	068113086	068113085	−0.74094524	−0.74094526	−0.74094526
3.0	0.14270081	0.14270079	0.14270077	−1.00026329	−1.00026329	−1.00026330
3.6	−0.44872180	−0.44872183	−0.44872184	−0.91138071	−0.91138069	−0.91138076
4.2	−0.88581760	−0.88581762	−0.88581763	−0.50098681	−0.50098678	−0.50098676
4.8	−1.01403600	−1.01403600	−1.01403600	0.08658083	0.08658086	0.08658089
5.4	−0.78798033	−0.78798032	−0.78798029	0.64443653	0.64443556	0.64443659
6.0	−0.28565527	−0.28565523	−0.28565519	0.98012921	0.98012923	0.98012924

for the two equations in the first-order initial value problem system. Though $A(x(t))$ and $A(y(t))$ are all negative, they are not static, because their values keep changing. $N(x(t))$ and $N(y(t))$ has non-negative values, constant and static respectively. The order of the polynomial used for deriving the scheme is 4. This might be responsible for its better performance.

In Tables 5 and 6, the two control parameters $A(x(t))$ and $N(y(t))$ are zero in the first equation. The values of $A(x(t))$ for the second equation oscillates between -0.6849 and -2.4180, keeping $N(y(t))$ constant as 1.7331. The order of the polynomial used for the derivation of the used scheme is 6.

In Table 7, the result perform favorably well with the existing results. Moreover, the results perform better than the existing method compared with.

6. Conclusion

A numerical method capable of solving equation systems has been developed by using an interpolating function, a combination of trigonometric and polynomial functions. Clearly, the novel plan outperforms both exact solutions and existing methods. The method has proven to be effective in its use in the models of IVP in climate change. The possibility of using the scheme to solve second and higher order directly is being considered and shall be reported soon.

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