Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. 5 (2023) 994

Journal of the Nigerian Society of Physical Sciences

Modeling Extreme Stochastic Variations using the Maximum Order Statistics of Convoluted Distributions

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Abstract

Modeling extreme stochastic phenomena associated with catastrophic temperatures, heat waves, earthquakes and destructive floods is an aspect of proactive mitigation of risk. Hydrologists, reliability engineers, meteorologist and researchers among other stakeholders are faced with the challenges of providing adequate model for fitting real life datasets from the extreme natural hazardous occurrences in our environment. Convoluted distributions (CD) and generalized extreme value (GEV) distribution for r- largest order statistics (r-LOS) have been some of the prominent existing techniques for modeling the extreme events. This study explored the properties of order statistics from the convoluted distribution as alternative procedure for analyzing the extreme maximum with the aim of obtaining superior modeling fit compared to some other existing techniques. The new procedure called MAXOS-G employed the potential properties of the Maximum Order Statistics (MAXOS) and the flexibilities of convoluted distributions where G is taken to be Weibull-Exponential Pareto (WEP) and the New Kumaraswamy-Weibull (NKwei) distributions. The maximum order statistics of the WEP distribution (MAXOS-WEP) and NKwei distribution (MAXOS-NKwei) was derived and applied to three datasets consisting of annual maximum flood discharges, annual maximum precipitation and annual maximum one-day rainfall. Some properties of the MAXOS-WEP was investigated including the moment, mean, variance, skewness, and kurtosis. Characterization of WEP distribution by the L-moment of maximum order statistics was presented and coefficient of L-variation, L-skewness and L-kurtosis were derived. The results from the application to three datasets using R-software justified the importance of this new procedure for modeling the maximum events. The MAXOS-NKwei and MAXOS-WEP models provide superior goodness-of-fit to the datasets than the WEP and NKwei distributions and better than some previously proposed convoluted distributions for modeling the datasets.

DOI:10.46481/jnsps.2023.994

Keywords: Extreme convoluted distributions, Maximum order statistics, MAXOS-G, MAXOS-NKwei, MAXOS-WEP, Annual maximum precipitation

Article History :
Received: 17 August 2022
Received in revised form: 09 January 2023
Accepted for publication: 09 January 2023
Published: 24 February 2023

Communicated by: O. Adeyeye

1. Introduction

Modelling of extreme value datasets from stochastic random phenomena in our environments becomes an imperative and important area of studies due to the adverse effects from natural hazards associated with global warming as a result of changes in climatic conditions. Humongous losses associated with the extreme catastrophic phenomena is unquantifiable as revealed by several works, some of which include [1] - [3]. [1] has showed that the traditional method used by many researchers for analyzing extreme value data is based on the gen-

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eralized extreme value (GEV) distribution. Modeling of exceedances over a high threshold using generalized Pareto distribution (GPD) by [4] most especially in hydrology is considered to be an alternative approach by the author. Some other techniques for investigating extreme phenomena and the modeling of real life datasets from the extreme scenario includes

- The use of generalized extreme value distribution for r-largest order statistics
- The application of lifetime distributions derived either by convolution of two baseline distributions or by some extension of existing classical distributions.

1.1. Convoluted Distributions

[5] revealed some areas of applications of heavy-tailed distributions which includes finance, hydrology, earthquake and engineering. [4] established a domain of attraction relating to the extreme distribution called the generalized Pareto distribution (GPD). [6] harnessed the long tail feature of the GPD distribution for modeling extreme value data [5] developed the Beta-Pareto (BP) distribution which the authors used for modeling two flood datasets. The properties of the beta generalized Pareto (BGPD) distribution was explored by [7] for modeling extreme value data. The Weibull distribution has wide application for modeling survival and reliability data. Generalizations from the distribution include the exponentiated-Weibull distribution applied to extreme value data by [8]. The Weibull distribution was used as a generator to propose the Weibull-G family of distribution by [9] from which the authors developed Weibul−Weibull, the Weibull-Normal and [10] developed a New Weibull-Pareto (WP) distributions. In another dimension, [11] introduced the Weibull-X family as a special case of the T-X family of distributions. Many authors defined several distributions from the Weibull-X family including, the Weibull-Pareto by [11] and Weibull-Rayleigh by [12] and [13].

[14] proposed a new generalized distribution denoted the Kumaraswamy-G (Kw-G). Convoluted distributions from the family include Kumaraswamy-Weibull (Kw-W) distribution by [15], the Kumaraswamy-Exponentiated Frechet (KeFr) by [16], [17] introduced the Kumaraswamy-Transmuted Pareto (KTP) distribution. The improved version of (Kw-G) denoted (NKw-G) was discovered by [10] , the author developed the New Kumaraswamy Weibull distribution (NKw-W) which was applied for modeling two datasets relating to extreme phenomena representing annual maximum flood discharge and annual maximum precipitation.

The convolution of exponential and Pareto distributions by [18] produced the Exponential Pareto (EP) distribution which [19, 20, 21, 22, 23, 24] further generalized and studied with diverse applications using different approaches. [25] proposed Transmuted Topp-Leone extended Frechet distribution for modeling extreme value of dataset with some application.

1.2. Order Statistics

The study by [26], faced with the problem of estimating the mean value of the difference between two successive values $X_{(k+1)}$ and $X_{(k)}$ of order statistics in a sample of size n from a population whose probability density has a continuous function confirmed that order statistics is not a new area of study. [27] extended on the work of Karl Pearson by estimating the mean of the sample range in a sample of size n. [28] used order statistics for estimation and test of hypothesis and applied the study to flood related problems; [29] employed the property of linear functions of order statistics to characterize distributions by their L-moments. [30] among other authors described order statistics as an area of statistical study that deals with properties and applications of ordered random variables and their associated functions. Order statistics has its wide applications in such areas that include actuarial modeling, finance, reliability engineering, hydrology, meteorology and climatology, signal processing, auction and sports.

Definition 1: Let $X_1, X_2, ..., X_n$ be a random sample of size n from distribution with CDF, $F(x)$, the order statistics corresponding to the random sample is the rearrangement of the sample in order of magnitude denoted by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$

[31] discussed the importance of order statistics and some techniques for deriving statistical and asymptotic distributions. [32] established the recurrence relations for the moment of order statistics for beta distribution and the generalized beta distribution. [33] obtained recurrence relations for the moment of order statistics from generalized beta distribution and [34] derived expressions for moments and recurrence relation of order statistics from the Power Lomax distribution and tabulated some numerical results with application.

Definition 2: The largest and smallest order statistics of a random sample $X_1, X_2, ..., X_n$ is the maximum and minimum observation from $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ denoted by $X_{(n:n)}$ and $X_{(1:n)}$ respectively.

The minimum and maximum are sometimes the centre of attraction in many areas of application because of some beneficial statistical measures which are of great interest to the researchers. The study of $r-out-of-n$ system of size n identical components characterized with independent life-lengths requires the statistical tools of order statistics; when $r = 1$ and $r = n$; it is called the parallel and series system respectively which is the maximum $X_{(n:n)}$ and minimum $X_{(1:n)}$ order statistics called the extreme order statistics. The extreme order statistics of two parameter Lomax distribution was studied by [35].

1.3. Generalized Extreme Value Distribution for r-Largest Order Statistics

[36] obtained the limiting form of the frequency of extreme values and subsequently, the generalized extreme value (GEV) distributions from [37] and then [38] became popular in finance, hydrology, actuarial and insurance among other field for modeling extreme events. The impacts of extreme random phenomena associated with unfavorable climatic conditions described in [39] have been investigated using the GEV distribution including the changes in Australian temperatures due to extremes modelled by [40]. Depending on the shape parameter, the limiting forms of the GEV distribution are Gumbel, Frechet and Weibull distributions.

There was a breakthorugh by some researchers who later discovered the importance of statistical tools of order statistics for analyzing some extreme events against the popular GEV block maxima; for instance, [1] opined that the use of extreme value analysis is a wasteful approach for modeling the block maxima and from [41], the $r-$largest order statistics (r-LOS) has the tendency to capture more useful information from extremes dataset and is a useful tool for analyzing extreme events. The use of limiting distribution of the (r-LOS) from GEV distribution for estimating return values of significant wave height was investigated by [42]. The challenge with this approach according to [43] is that as the value of r increases, there is decrease in the rate of convergence to the limiting joint distribution and [41] observed that the variance of the estimator will be high for small r and there is bias when the value of r is too large.

[44] estimated extreme wind speed using the method of (r-LOS) and concluded that Gumbel distribution is a suitable model for the dataset. When there are few numbers of observations, [45] suggested the use of (r-LOS) instead of the extreme value distributions for analyzing maximum values of a given dataset consisting of few observations. In modeling the average maximum daily temperature, the (r-LOS) when $r = 4$ was fitted to data by [46]. Recently [46] used the (r-LOS) from extreme value distributions to model daily maximum temperatures from some meteorological stations in Thailand.

This study is motivated by some useful applications of potential benefits from the properties of order statistics as [47] has also revealed that distributions of order statistics can generate some families of distribution. The research is aimed at exploring the distributional properties of maximum order statistics (MAXOS) of the Weibull Exponential Pareto (WEP) and the New Kumaraswamy Weibull (NKwei) distributions with application to extreme value dataset. The remaining parts of the paper are structured as follows; Section 2 contains the design and formulation of the proposed distributions using the new procedure. Section 3 is used for investigating some order statistical properties of the WEP distribution. Section 4 provides characterization of WEP distribution by the L-moment of maximum order statistics and estimation of parameters. Results from applications to real life datasets are presented in section 5. Section 6 and 7 is for discussions and conclusions respectively.

2. Materials and Methods

2.1. T-X Families of Distribution

[48] defined the cumulative distribution function (CDF) and the probability density function (PDF) for a random variable X from the T-X family of distributions respectively as follows;

$$\mathcal{G}(x) = \int_0^{-log(1-F(x))} r(t)dt = \mathcal{R}(x)\{-log(1-F(x))\} \quad (1)$$

$$g(x) = \frac{f(x)}{1-F(x)} r\{-log(1-F(x))\} \quad (2)$$

where $r(t)$ is the pdf of non negative continuous random variable T defined on $[0,\infty)$ The CDF for a random variable T from Weibull distribution with parameter α and γ is given as,

$$\mathcal{R}(x)(t) = 1 - exp\Big(-\Big(\frac{t}{\gamma}\Big)\Big)^{\alpha} \quad (3)$$

The corresponding probability density function is the derivative of the CDF presented as,

$$r(t) = \frac{\alpha}{\gamma}\Big(\frac{t}{\gamma}\Big)^{\alpha-1} exp\Big(-\Big(\frac{t}{\gamma}\Big)\Big)^{\alpha}; \alpha,\gamma > 0; x > 0 \quad (4)$$

The $T - X$ family of distributions has Weibull-X family by [48] as special case when random variable T follows a Weibull distribution and is given as,

$$\mathcal{G}(x) = 1 - exp\Big(-\Big(-\frac{log(1-F(x))}{\gamma}\Big)^{\alpha}\Big) \quad (5)$$

$$g(x) = \frac{\alpha}{\gamma}\frac{f(x)}{1-F(x)}\Big(-\frac{log(1-F(x))}{\gamma}\Big)^{\alpha-1} exp\Big(-\Big(-\frac{log(1-F(x))}{\gamma}\Big)^{\alpha}\Big) \quad (6)$$

2.2. Exponential Pareto Distribution

Let X be a Pareto random variable, the CDF and PDF of exponential Pareto distribution defined respectively by [18] is given by

$$\mathcal{F}(x) = 1 - exp\Big(-\lambda\Big(\frac{x}{k}\Big)\Big)^{\theta}; \lambda,\theta,k > 0; x > 0 \quad (7)$$

$$f(x) = \frac{\theta\lambda}{k}\Big(\frac{x}{k}\Big)^{\theta-1} exp\Big(-\lambda\Big(\frac{x}{k}\Big)\Big)^{\theta}; \lambda,\theta,k > 0; x > 0 \quad (8)$$

2.3. Weibull Exponential Pareto (WEP) Distribution

Let T be a Weibull random variable having the cdf with scale parameter γ and shape α. Substitute (7) into the cdf of Weibull-X family of distribution in (5) to get;

$$\mathcal{G}(x) = 1 - exp\Big(-\Big\{\frac{\lambda(\frac{x}{k})^{\theta}}{\gamma}\Big\}^{\alpha}\Big) \quad (9)$$

The derivative of (9) is the associated density function of WEP distribution given by;

$$g(x) = \frac{\alpha\lambda\theta}{k\gamma}\Big(\frac{x}{k}\Big)^{\theta-1}\Big\{\frac{\lambda(\frac{x}{k})^{\theta}}{\gamma}\Big\}^{\alpha-1} exp\Big(-\Big\{\frac{\lambda(\frac{x}{k})^{\theta}}{\gamma}\Big\}^{\alpha}\Big) \quad (10)$$

2.4. The Maximum Order Statistics Generalized Distribution

The behavior of the maximum order statistics $X_{(n:n)}$ is a special study of the extreme value theory (EVT). The limiting distribution of $X_{(n:n)}$ for a non-degenerate distribution function $G(x)$ belong to the family of extreme value distributions with the mathematical expressions give by

$$Pr\Big\{\frac{X_{(n:n)} - b_n}{a_n} \le x\Big\} \implies \mathcal{G}(x) \quad as \quad n \implies \infty \quad (11)$$

where $a_n > 0$ and b_n are sequences of constants

The CDF of MAXOS-G distribution for a continuous baseline distribution $G(x)$ is derived using beta generalized framework as follows;

$$Pr\Big\{\frac{X_{(n:n)} - b_n}{a_n} \le x\Big\} = \frac{1}{B(\alpha,1)}\int_0^{G(x)_{wep}} t^{\alpha-1}dt = \Big(G(x)\Big)^n \quad (12)$$

where $B(n,m) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}$ and $B(\alpha,1) = B(n,1)$

2.4.1. The Maximum Order Statistics of the Weibull Exponential Pareto (MAXOS-WEP) Distribution

The CDF of MAXOS-WEP distribution when the continuous baseline distribution $G(x)$ is WEP distribution is derived by substituting (9) into (12) to obtain the CDF of MAXOS-WEP distribution for $\beta = \lambda/\gamma$ defined as;

$$\mathcal{F}_{(n:n)}(x) = Pr\left\{\frac{X_{(n:n)} - b_n}{a_n} \leq x\right\} = \left[1 - exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^n \quad (13)$$

The corresponding PDF of MAXOS-WEP distribution is obtained as derivative of $\mathcal{F}_{(n:n)}(x)$ given by

$$f_{(n:n)}(x) = \left[1 - exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n-1} \frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)$$
$$= \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left[exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{i+1} \frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \quad (14)$$

2.4.2. The Maximum Order Statistics of the new Kumaraswamy Weibull (MAXOS-NKwW) Distribution

The CDF of New Kumaraswamy (NKwW) distribution introduced by [10] is given by

$$\mathcal{F}(x) = 1 - \left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^b \quad (15)$$

The PDF of New Kumaraswamy Weibull distribution is obtained as

$$f(x) = AB\left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^{b-1}\left(\frac{[1-exp(-\lambda x^{\theta})]}{[exp(-\lambda x^{\theta})]} + (\lambda x^{\theta})\right) \quad (16)$$

where

$$A = ab\lambda\theta x^{\theta-1} exp(-\lambda x^{\theta}) exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}$$
$$B = \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^{a-1}$$

The CDF of MAXOS-NKwW distribution is derived by substituting CDF of New Kumaraswamy Weibull distribution in (15) into (12) and is given by

$$\mathcal{F}_{(n:n)}(x) = \left[1 - \left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^b\right]^n \quad (17)$$

Theorem 2.1: *Let the CDF and PDF of the New Kumaraswamy distribution be $F(x)$ and $f(x)$ defined in (15) and (16) respectively. Then the density function of the MAXOS-NKwW distribution is given by;*

$$f_{(n:n)}(x) = A^* \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\sum_{j=0}^{2b-1}(-1)^j\binom{2b-1}{j}\sum_{k=0}^{2a-1}(-1)^k\binom{2a-1}{k} \quad (18)$$

where

$$A^* = nab\lambda\theta x^{\theta-1} exp(-\lambda x^{\theta})\left(exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^{k+1}\left(\frac{[1-exp(-\lambda x^{\theta})]}{[exp(-\lambda x^{\theta})]} + (\lambda x^{\theta})\right)$$

Proof: The derivative corresponding to the CDF of MAXOS-NKwW distribution in (17) is given by;

$$f_{(n:n)}(x) = ABC\left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^{b-1} \quad (19)$$

where

$$A = nab\lambda\theta x^{\theta-1} exp(-\lambda x^{\theta}) exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\left(\frac{[1-exp(-\lambda x^{\theta})]}{[exp(-\lambda x^{\theta})]} + (\lambda x^{\theta})\right)$$
$$B = \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^{a-1}$$
$$C = \left[1 - \left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^b\right]^{n-1}$$

using binomial expansion,

$$f_{(n:n)}(x) = AB\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(1 - \left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^a\right)^{2b-1}$$
$$= A\sum_{i=0}^{n-1}(-1)^{i+j}\binom{n-1}{i}\sum_{j=0}^{2b-1}\binom{2b-1}{j}\left(1 - exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^{2a-1}$$
$$= A^*\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\sum_{j=0}^{2b-1}(-1)^j\binom{2b-1}{j}\sum_{k=0}^{2a-1}(-1)^k\binom{2a-1}{k} \quad (20)$$

where

$$A^* = nab\lambda\theta x^{\theta-1} exp(-\lambda x^{\theta})\left(exp(-\lambda x^{\theta})^{[1-exp(-\lambda x^{\theta})]}\right)^{k+1}\left(\frac{[1-exp(-\lambda x^{\theta})]}{[exp(-\lambda x^{\theta})]} + (\lambda x^{\theta})\right)$$

3. Some Properties of the MAXOS-Weibull Exponential Pareto Distribution

The central moments, mean order statistics and the variance is derived here

3.1. Moments of MAXOS-WEP distribution

Theorem 3.1 *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with cdf and pdf denoted by $\mathcal{F}(x)$ and $f(x)$ respectively and let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be corresponding order statistics. Then the t^{th} moments of the $X_{(n:n)}$ order statistics for $t = 1, 2, ...$ denoted by $\mu_{n:n}^{(t)}$ is given by;*

$$\mu_{n:n}^{(t)} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i} nk^t\left(\frac{1}{\beta}\right)^{\frac{t}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{t}{\alpha\theta}+1\right) \quad (21)$$

Proof

$$\mu_{n:n}^{(t)} = \int_0^{\infty} x^t f_{n:n}(x)dx \quad (22)$$

Then substituting the pdf of MAXOS-WEP in (14) into (22) to get

$$\mu_{n:n}^{(t)} \int_0^{\infty} x^t \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left[exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{i+1} \frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} dx \quad (23)$$

$$\mu_{n:n}^{(t)} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\int_0^{\infty} x^t\left[exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{i+1} \frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} dx \quad (24)$$

Let $y = (i+1)\left(\beta\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}$, by transformation of variable we have the following quantities;

$x = \frac{ky^{\frac{1}{\alpha\theta}}}{\beta^{\frac{1}{\theta}}(i+1)^{\frac{1}{\alpha\theta}}}$; $\frac{dy}{dx} = \frac{(i+1)\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\beta\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}$

$$\mu_{n:n}^{(t)} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{n}{(i+1)}\int_0^{\infty}\left(\frac{ky^{\frac{1}{\alpha\theta}}}{\beta^{\frac{1}{\theta}}(i+1)^{\frac{1}{\alpha\theta}}}\right)^t e^{-y}dy (25)$$

$$\mu_{n:n}^{(t)} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{n}{(i+1)}\int_0^{\infty}\left(\frac{k}{\beta^{\frac{1}{\theta}}(i+1)^{\frac{1}{\alpha\theta}}}\right)^t y^{\frac{a}{\alpha\theta}} e^{-y}dy (26)$$

Using the gamma function $\int_0^{\infty} y^r e^{-y}dy = \Gamma\left(r+1\right)$

$$\mu_{n:n}^{(t)} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}nk^t\left(\frac{1}{\beta}\right)^{\frac{t}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{t}{\alpha\theta}+1\right) (27)$$

The **mean order statistics** and the **variance of order statistics** are derived and given respectively as;

$$\mu_{n:n} = nk\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(\frac{1}{\beta}\right)^{\frac{1}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{1}{\alpha\theta}+1)}\Gamma\left(\frac{1}{\alpha\theta}+1\right) (28)$$

and

$\sigma_{n:n}^{(2)} = \mu_{n:n}^{(2)} - \left(\mu_{n:n}\right)^2$

$$\sigma_{n:n}^{(2)} = nk^2\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(\frac{1}{\beta}\right)^{\frac{2}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{2}{\alpha\theta}+1)}\Gamma\left(\frac{2}{\alpha\theta}+1\right)-\left(\mu_{r:n}\right)^2 ($$

The results can be applied for the prediction of expected maximum of future occurrences such as expected maximum flood level in hydrology.

Corollary 3.1 The result in Theorem 3.1 reduces to the explicit expression of the moments and the mean of exponential pareto (EP) distribution studied by [18], if $n = \alpha = 1$ as deduced and respectively given by;

$$\frac{k^t}{(\beta)^{\frac{t}{\theta}}}\Gamma\left(\frac{t}{\theta}+1\right) \quad (30)$$

$$\frac{k}{(\beta)^{\frac{1}{\theta}}}\Gamma\left(\frac{1}{\theta}+1\right) \quad (31)$$

Equation (9) in ([18], p.137)

3.2. Some Properties of MAXOS-WEP Distribution by Graphical Visualization

The following plots in Figure 1 and Figure 2 provide the visualization of some properties of the MAXOS-WEP distributions

- The pdf and cdf of MAXOS-WEP $X_{(j:j)}$ for various sample sizes $j = 2, 3, ..., n$ converges to the maximum $X_{(n:n)}$ which is the supremum of all the sample sizes. Figure 1 reveals that the pdf of $X_{(30:30)}, X_{(32:32)}$ and $X_{(34:34)}$ converges almost to the pdf of $X_{(35:35)}$.

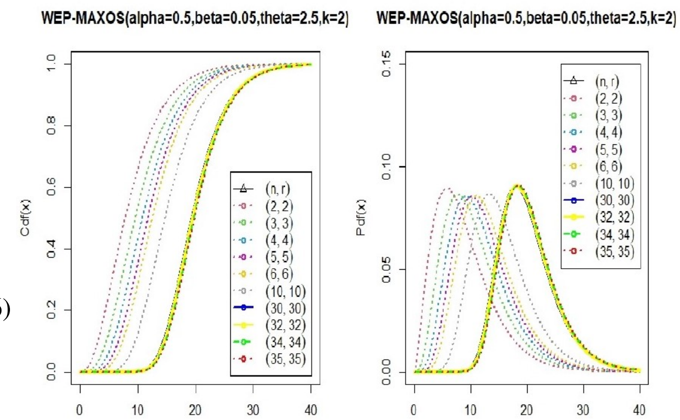

Figure 1. Cdf and Pdf of MAXOS-WEP for arbitrary parameters ($\alpha = 0.5, \beta = 0.05, \theta = 2.5, k = 2$) and various sample sizes

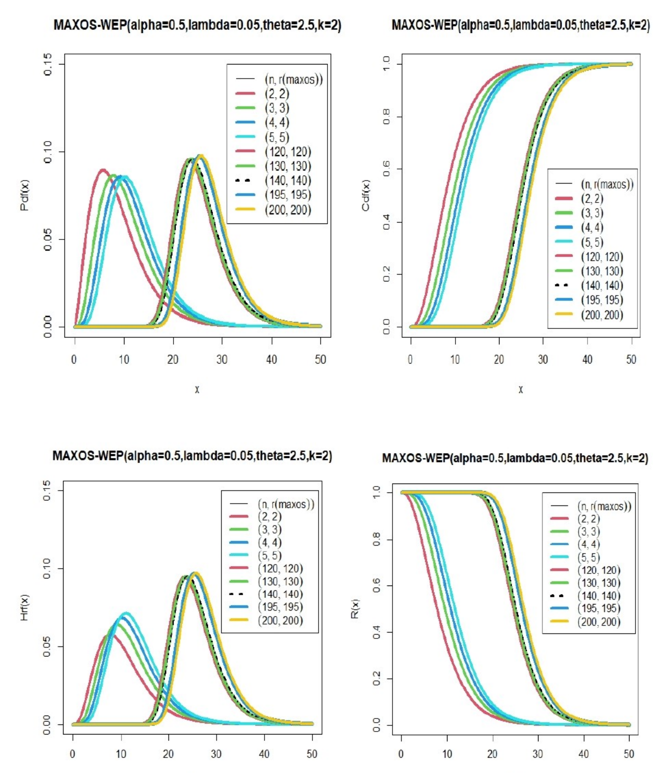

Figure 2. Pdf, Cdf, H(x) and R(x) of MAXOS-WEP for arbitrary parameters

- If the sample size n is large enough, the parallel systems $X_{(n:n)}$ and $X_{(n+i:n+i)}$ for $i = 1, 2, ...$ from the WEP distribution is equivalent in distribution represented by $X_{(n:n)} \stackrel{d}{=} X_{(n+i:n+i)}$. In Figure 2, the maximum order statistics between $X_{(195:195)}$ and $X_{(200:200)}$ which are $X_{(196:196)}, X_{(197:197)}, X_{(198:198)}, X_{(199:199)}$ tends to have identical distribution as it is evident that the plots of their densities will coincide and overlap the pdf plots of $X_{(195:195)}$ and $X_{(200:200)}$ with the blue and yellow colours respectively and can be approximated by a common distributional properties.

- The shapes of MAXOS-WEP distributions is identical for large sample sizes as revealed in figure 2. As the sample sizes increases the pdf of $X_{(120:120)}, X_{(130:130)}$ and $X_{(140:140)}$tends to overlap with no clear difference.

- The dispersive ordering between $X_{(j:j)}$ for small sample sizes $j = 2, 3, 4, 5$ is higher while for large sample sizes $X_{(j:j)}, say from j = 30, 31, 32, ...$ as shown in Figure 1, the dispersive ordering becomes smaller and gradually diminishes at a faster rate with increase in n.

- The pdf and cdf of MAXOS-WEP $X_{(j:j)}$ is less skewed than the pdf and cdf of $X_{(j+1:j+1)}$ for $j = 2, 3, ...$ Figure 1 reveals from the pdf that $X_{(2:2)} \leq X_{(3:3)} \leq X_{(4:4)}$ in skewness.

- The MAXOS-WEP has the same kurtosis for $X_{(j:j)}, j = 2, 3, ...$ which could be leptokurtick or mesokurtic depending on either the parameter values of the distribution or the sample size.

- The shapes of the hazard rate function from bottom left of Figure 2 reveals that the hazard function is identical in shape as the density function posted in the top-left of Figure 2. $h_{(j:j)}(x) \implies f_{(j:j)}(x)$ in shape $\forall j > 2$ for large sample sizes

3.3. Limiting Properties of MAXOS-WEP Distribution

The asymptotic properties of the distributions is investigated by taking limits of the density function and cumulative distribution function as $x \to \infty$ and as $x \to 0$ in this subsection. Using (13) and (14)

$$\lim_{x\to o} F_{(n:n)}(x) = \lim_{x\to o}\left[1 - exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n} = 0 \tag{32}$$

$$\lim_{x\to\infty} F_{(n:n)}(x) = \lim_{x\to\infty}\left[1 - exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n} = 1 \tag{33}$$

$$\implies 0 \leq F_{(n:n)}(x) \leq 1$$

$$\lim_{x\to o} f_{(n:n)}(x) = \sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\left[exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{i+1}$$

$$X\frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} = 0 \tag{34}$$

$$\lim_{x\to\infty} f_{(n:n)}(x) = \sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\left[exp\left(-\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{i+1}$$

$$X\frac{n\alpha\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\beta\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} = 0 \tag{35}$$

$$\implies \lim_{x\to\infty} = 0 = \lim_{x\to 0}$$

4. Characterization of the WEP Distribution by *L*-Moments of the Maxima Order Statistics

The L-moments can be derived as the expectations of extreme order statistics which has been defined in [49] and given by;

$$\lambda_r = \sum_{i=1}^{r}(-1)^{r-i}i^{-1}\binom{r-1}{i-1}\binom{r+i-2}{i-1}E(X_{(i:i)}) \tag{36}$$

Expansion of (36) leads to the following system of equations for the first four L-moments $\lambda_r, r = 1, 2, 3, 4.$ in terms of the maximum order statistics

$$\lambda_1 = E\left(X_{1:1}\right) \tag{37}$$

$$\lambda_2 = E\left(X_{2:2}\right) - E\left(X_{1:1}\right) \tag{38}$$

$$\lambda_3 = 2E\left(X_{3:3}\right) - 3E\left(X_{2:2}\right) + E\left(X_{1:1}\right) \tag{39}$$

$$\lambda_4 = 5E\left(X_{4:4}\right) - 10E\left(X_{3:3}\right) + 6E\left(X_{2:2}\right) - E\left(X_{1:1}\right) \tag{40}$$

By application of moment of MAXOS-WEP result in (21) when $t = 1$

$$\mu_{n:n} = nk\sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{(i+1)}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{41}$$

$$\lambda_{\mathbf{1}} = E(X_{1:1}) = \mu = k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta} + 1\right) \tag{42}$$

$$\begin{aligned} E(X_{2:2}) &= 2k\sum_{i=0}^{1}(-1)^{i}\binom{1}{i}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{(i+1)}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta} + 1\right) \\ &= 2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta} + 1\right) - 2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta} + 1\right) \end{aligned} \tag{43}$$

$$\left\{\begin{aligned} \lambda_{\mathbf{2}} &= E(X_{2:2}) - E(X_{1:1}) \\ &= 2k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) - 2k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\tfrac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) - k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) \\ &= k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta} + 1\right) - 2k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\tfrac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\tfrac{1}{\alpha\theta} + 1\right) \\ &= k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta} + 1\right)\left[1 - 2\left[\tfrac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\right] \\ &= k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta} + 1\right)\left[1 - \left[\tfrac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right] \end{aligned}\right\} \tag{44}$$

$$\left\{\begin{aligned} E(X_{3:3}) &= 3k\textstyle\sum_{i=0}^{2}(-1)^{i}\binom{2}{i}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{(i+1)}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta} + 1\right) \\ &= 3k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) - 6k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\tfrac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) + 3k\left[\tfrac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\tfrac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\tfrac{1}{\alpha\theta}+1\right) \end{aligned}\right\} \tag{45}$$

$$\lambda_{\mathbf{3}} = 2E(X_{3:3}) - 3E(X_{2:2}) + E(X_{1:1})$$

$$= 2\left\{3k\left[\frac{1}{\lambda}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) + 3k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\}$$

$$-3\left\{2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} + k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)$$

$$= 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) + k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)$$

$$= k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left\{6\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1} - 6\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1} + 1\right\}$$

$$= k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left\{1 - 3\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}} + 2\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\right\} \quad (46)$$

$$E(X_{4:4}) = 4k\sum_{i=0}^{3}(-1)^i\binom{3}{i}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{(i+1)}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \quad (47)$$

$$= \left\{\begin{array}{c} 4k\binom{3}{0}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 4k\binom{3}{1}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \\ +4k\binom{3}{2}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 4k\binom{3}{3}\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{array}\right\} \quad (48)$$

$$= \left\{\begin{array}{c} 4k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 12\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \\ +12\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 4k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{array}\right\} \quad (49)$$

$$\lambda_{\mathbf{4}} = 5E(X_{4:4}) - 10E(X_{3:3}) + 6E(X_{2:2}) - E(X_{1:1}) \quad (50)$$

$$\lambda_{\mathbf{4}} = \left\{\begin{array}{c} 5\left\{4k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 12\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ +5\left\{12\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 4k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ -10\left\{3k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} - 10\left\{+3k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ +6\left\{2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 2k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}+1}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} - k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{array}\right\} \quad (51)$$

$$= \left\{\begin{array}{c} \left\{20k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 30\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ +\left\{20\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 5k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ -\left\{30k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 30k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ +\left\{12k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} \\ -\left\{10k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\right\} - k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{array}\right\} \quad (52)$$

$$= \left\{\begin{array}{c} k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 6k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \\ +10\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) - 5k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{array}\right\} \quad (53)$$

$$= k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left\{1 - 6\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}} + 10\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}} - 5\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}}\right\} \quad (54)$$

The L-Moment ratios are derived as

$$\tau_{\mathbf{2}} = \frac{\lambda_2}{\lambda_1} = \frac{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right]}{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)} = \left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right] \quad (55)$$

$$\tau_{\mathbf{3}} = \frac{\lambda_3}{\lambda_2} = \frac{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left\{1-3\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}+2\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\right\}}{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right]} = \frac{\left\{1-3\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}+2\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}\right\}}{\left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right]} \quad (56)$$

$$\tau_{\mathbf{4}} = \frac{\lambda_4}{\lambda_2} = \frac{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left\{1-6\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}+10\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}-5\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}}\right\}}{k\left[\frac{1}{\beta}\right]^{\frac{1}{\theta}}\Gamma\left(\frac{1}{\alpha\theta}+1\right)\left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right]}$$

$$= \frac{\left\{1-6\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}+10\left[\frac{1}{3}\right]^{\frac{1}{\alpha\theta}}-5\left[\frac{1}{4}\right]^{\frac{1}{\alpha\theta}}\right\}}{\left[1-\left[\frac{1}{2}\right]^{\frac{1}{\alpha\theta}}\right]} \quad (57)$$

The L-Moment estimates of parameters α and θ denoted respectively by $\hat{\alpha}$ and $\hat{\theta}$ can be obtained as the solution to nonlinear equations in (56) and (57). Thereafter the computation of L-Moment estimates of parameters β and k denoted respectively by $\hat{\beta}$ and $\hat{k}$ can be derived from (49) and (54).

5. Results from Real-Life Applications

The usefulness of the maximum order statistics generalized (MAXOS-G) family of convoluted distributions where G represent WEP and NKWei distributions from the study is investigated by analyzing three popular extreme value datasets. Analysis of data is carried out using the R-Software to generate numerical values for the goodness-of-fit-statistics. The standard and widely used goodness-of-fit- statistics for model selection criteria by researchers in making decision about competing distributions are -loglikelihood (- LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criteria (HQIC), the Kolmogorov-Smirnov (K-S) statistics and the P-value. They can be found in several related works including [24], [50], [51] and most recently in [56]. The goodness-of-fit-statistics are defined as:

Table 1. MLEs of Parameters for Annual maximum flood data

Models	α	β	θ	k	λ
MAXOS-NKWei	1.0369	1.2478	0.5195	-	0.5034
MAXOS-WEP	2.8914	1.2165	0.1895	8.6417	-
NKWei	5.8540	0.7002	0.8877	-	0.0969
WEP	1.0592	0.0077	1.6407	2.9449	-

$$AIC = -2L + 2m;$$

$$CAIC = -2L + \frac{2mn}{(n+1)};$$

$$AICC = AIC + \frac{2m(m+1)}{(n-m-1)},$$

$$BIC = -2L + m\log(n);$$

$$HQIC = -2L + 2m(\log(n))$$

L is the estimated log-likelihood (LL), m is the number of parameters in the model. The p-value and K-S are statistics associated with the goodness-of-fit criteria.

Decision: The best fitted model is identified with the smallest values of the estimated goodness-of-fit criteria or model with the highest p-value returned for the K-S statistics.

5.1. Application to annual maximum flood discharges hydrological dataset

The hydrological data which was taken from [10] in application to NKWei distribution was previously studied by[52, 53], the data represent the annual maximum flood discharges (in units of 1000 cubic feet per second) of the North Saskachevan River at Edmonton, over a period of 48 years. The data are: (19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560)

The dataset with line plots in Fig 3 received a new analysis in this present study.

Applications to MAXOS-WEP and MAXOS-NKWei distributions has the parameter estimates and the goodness-of-fit statistics presented in Table 1 and Table 2 respectively.

Figure 4 is the plots for the histogram with density functions and estimated cdf to assess the performance of the distributions.

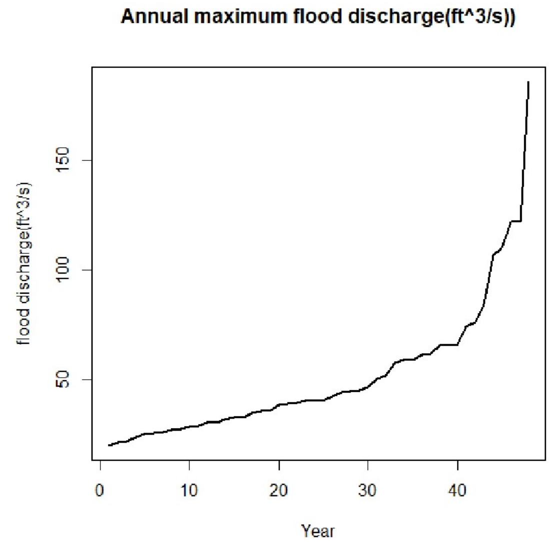

Figure 3. Annual maximum flood discharge (in units of 1000 cubic feet per second)

Table 2. Estimated Goodness-of-fit Statistics for Annual maximum flood data

Model	-LL	AIC	BIC	K-S	p-value
MAXOS-NKWei	216.01	440.08	447.51	0.0694	0.9749
MAXOS-WEP	216.65	441.30	448.78	0.0712	0.9681
NKWei	216.91	441.82	449.31	0.0767	0.9405
WEP	225.74	459.47	466.95	0.1465	0.2540

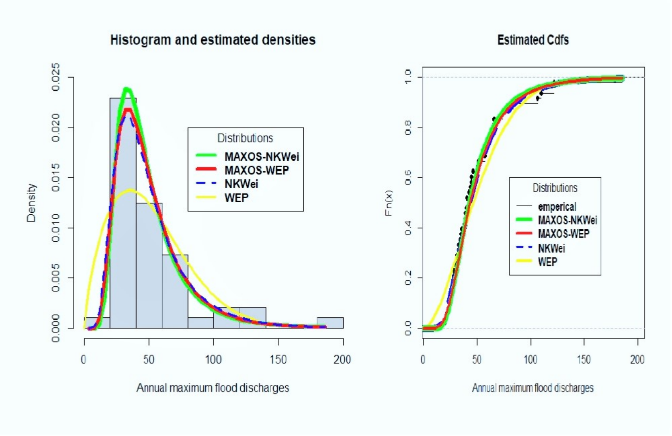

Figure 4. Histogram and Fitted Distribution for annual maximum flood discharges of the North Saskachevan River at Edmonton

5.2. Application to Annual Maximum Precipitation Dataset

The data with line plots in Figure 4 is available in extreme package from the R-Software, it has been studied by [52] and recently applied to the NKw-W distribution by [10]. The data

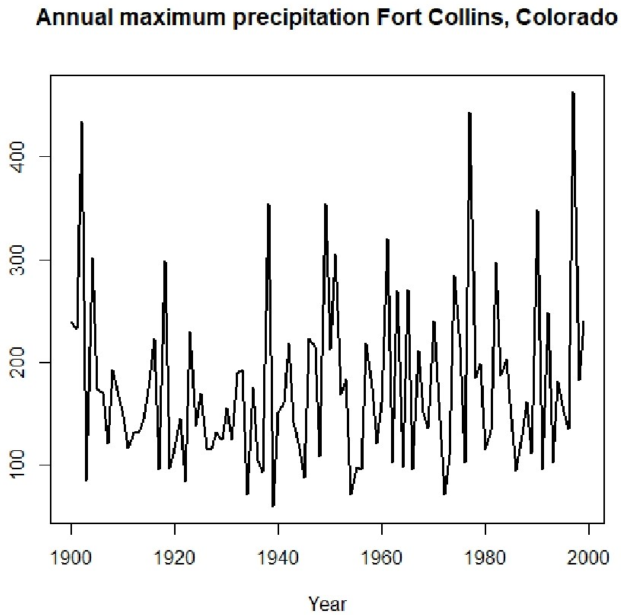

Figure 5. Annual maximum precipitation amount at Fort Collins, USA, 1900 – 1999.

Table 3. MLEs of Parameters for Precipitation data.

Models	α	β	θ	k	λ
MAXOS-NKWei	0.7610	0.5318	0.5454	-	0.5816
MAXOS-WEP	0.3029	0.5263	1.7755	5.4395	-
NKWei	11.228	0.7698	0.8310	-	0.0481
WEP	3.0488	0.1055	0.7732	10.8781	-

represent the annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado from 1900 through 1999. The data are:
(239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241).
The dataset has the time series plots displayed in Figure 5,

The results presented in Table 3 and Table 4 established the flexibility of the proposed MAXOS-G distributions from real-life application of MAXOS-WEP and MAXOS-NKWei distributions to the annual maximum precipitation with comparisons to their baseline components when G is a WEP and NKWei distributions.

The histogram and fitted distributions posted in Figure 6 supported the results in Table 4 about the superior performance of the proposed technique.

Table 4. Estimated Goodness-of-fit Statistics for Precipitation data.

Model	*-LL*	*AIC*	*BIC*	K-S	p-value
MAXOS-NKWei	565.11	1138.23	1148.65	0.037	0.999
MAXOS-WEP	565.12	1138.25	1148.67	0.042	0.995
NKWei	565.22	1138.44	1148.86	0.045	0.988
WEP	576.33	1160.67	1171.09	0.096	0.314

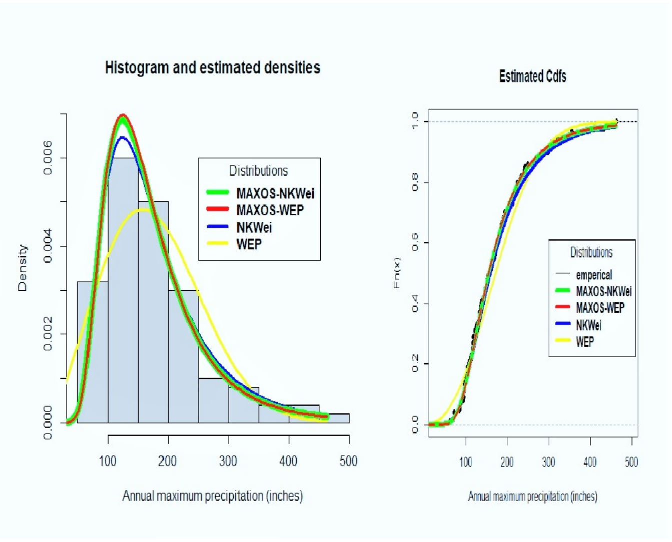

Figure 6. Histogram and Fitted Distribution for Annual Maximum Precipitation for one rain gauge in Fort Collins, Colorado from 1900 through 1999.

Table 5. MLEs of Parameters for annual maximum one-day rainfall data.

Model	α	β	θ	k	λ
MAXOS-NKWei	0.3817	0.2642	0.6303	-	0.7125
MAXOS-WEP	0.6541	0.5647	1.0089	5.4882	-
NKWei	8.9151	0.7163	0.0555	-	0.9217
WEP	1.6118	0.0072	1.4442	3.2738	-
TC	61.538	–	–	19.959	-

5.3. Application to Annual Maximum one-day Rainfall Data

The data from [54] was analyzed using the transmuted-Cauchy (TC) distribution developed by [55] . This present study analysed the data and performances of NKwei, WEP and TC convoluted distributions were compared with the performances of models from the MAXOS-G and presented in Table 6, estimated values of parameters are displayed in Table 5.

The results presented in Table 6 revealed improved estimated goodness-of-fit statistics from the new procedure which indicates the superiority of the MAXOS-G approach over the convoluted distributions. Application of MAXOS-WEP and MAXOS-NKWei distributions to the annual maximum one-day rainfall justified the need to explore for new technique.

Table 6. Goodness-of-fit Statistics of annual maximum one-day rainfall data.

Model	-LL	AIC	BIC	K-S	p-value
MAXOS-NKWei	194.02	396.04	402.69	0.067	0.9948
MAXOS-WEP	194.04	396.08	402.73	0.066	0.9948
NKWei	194.10	396.19	402.37	0.073	0.9846
TC	197.83	401.66	–	0.085	0.9423
WEP	197.86	402.92	409.57	0.113	0.7020

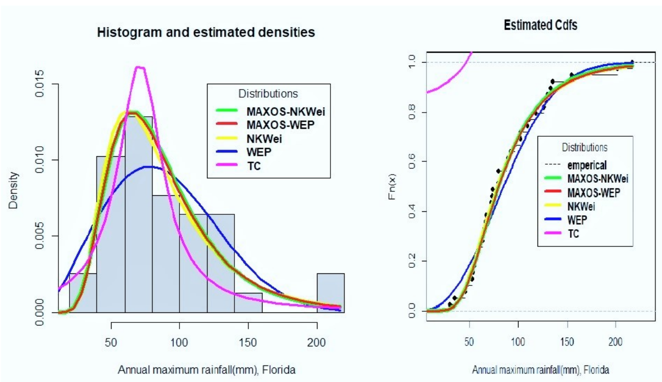

Figure 7. Histogram and Fitted Distribution for Annual Maximum One-Day Rainfall at Florida Atlantic University

6. Discussion

Statistical modeling of extreme stochastic inevitable phenomena in our environments is at the heart of many disciplines including meteorologist, hydrologist, climatologist, reliability engineering, medical sciences, actuarial and insurance among others. This research explored for some novel approach applicable for modeling extreme value datasets by developing two new models called the MAXOS-NKwei and MAXOS-WEP distributions using the MAXOS-G framework obtained as a special case of the beta-G family of distribution. Some Statistical properties of the MAXOS-WEP investigated includes the moment, mean, variance, and asymptotic behaviours. The skewness, kurtosis and some ordering properties were investigated using the graphical visualization from plots of the density, hazard and cumulative distribution functions. The importance of the study and the usefulness of the models was tested by way of application to three extreme value datasets representing the annual maximum flood discharges of the North Saskachevan River at Edmonton; the annual maximum precipitation in Fort Collins, Colorado from 1900 through 1999 and the annual maximum one-day rainfall data from PRISM Climate Group, Oregon State University. The results revealed tremendous improvement over the use of convoluted distributions proposed from the existing literatures. The blue and yellow lines from (Figure 4 and 6) and the pink line from (Figure 7) represent the fitted convoluted distributions to the data, the red and green lines represent the fitted MAXOS-G models. The two proposed models from the MAXOS-G provides superior modeling performance established by the smallest goodness-of-fit statistics and highest p-values compared to the results for NKwei by [10] and TC distribution proposed by [55]. The new technique explored in this research revealed superior modeling capacity of the approach over the continuing reliance on convoluted distributions for extreme value modelling which hitherto has dominated the literatures.

7. Conclusion

The Weibull Exponential Pareto (WEP) distribution has enormous potential for analyzing random phenomena that is right skewed and approximately symmetric associated with hydrology, reliability engineering, public health and some other area of applications that are characterized with heavy tail or large kurtosis. The New Kumaraswamy Weibull (NKWei) distribution provides better goodness-of-fit estimates than many established generalized Weibull model existing in the literatures as revealed in [10] . However, exploring some potential properties of order statistics from the two convoluted distributions using the MAXOS-G framework revealed an improved performance in the modeling capacity of the convoluted distributions. The MAXOS-WEP and the MAXOS-NKWei has superior performance for modeling the hydrological extreme value datasets. The results proved that the two distributions from MAXOS-G are the best model for the data when compared to the baseline (G) distributions. The graphical plots in (Figure 4, 6 and 7) further corroborated the superiority of the proposed method for modeling the three annual maximum datasets over the existing methods. The MAXOS-G framework will be an important modeling tool in extreme value analysis and a good choice in several fields of applications such as hydrology, actuarial, survival analysis, economics, quality control, medical sciences and meteorology.

Acknowledgments

The authors are grateful for the comments and suggestions of the anonymous reviewers and the editor towards production of the final manuscript.

There is no funding from any source towards the success of this project. No funds, grants, or other support was received

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